Even Faster Accelerated Coordinate Descent Using Non-Uniform Sampling

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Abstract

Accelerated coordinate descent is widely used in optimization due to its cheap per-iteration cost and scalability to large-scale problems. Up to a primal-dual transformation, it is also the same as accelerated stochastic gradient descent that is one of the central methods used in machine learning.

In this paper, we improve the best known running time of accelerated coordinate descent by a factor up to $\sqrt{n}$. Our improvement is based on a clean, novel non-uniform sampling that selects each coordinate with a probability proportional to the square root of its smoothness parameter. Our proof technique also deviates from the classical estimation sequence technique used in prior work. Our speed-up applies to important problems such as empirical risk minimization and solving linear systems, both in theory and in practice.[1]

1 Introduction

First-order methods have received extensive attention in the past two decades due to their ability to handle large-scale optimization problems. Recently, the development of *coordinate* versions of first-order methods have pushed their running times even faster. As a notable example, the state-of-the-art algorithm for empirical risk minimization (ERM) problems, up to a primal-dual transformation, is precisely accelerated coordinate descent (Lin et al., 2014).

In this paper, we consider the following unconstrained minimization problem[2]

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1.1}$$

where the objective $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and convex. Below, we assume that $f(\cdot)$ is L_i-smooth with respect to its i-th coordinate.

Informally, coordinate smoothness means for each input x, if we add its i-th coordinate by at most δ, the corresponding coordinate gradient $\nabla_i f(x+\delta\mathbf{e}_i)$ differs from $\nabla_i f(x)$ by at most L_i times $|\delta|$. Under this definition, the larger L_i is, the less smooth f is along the $\mathbf{e}_i$ direction and therefore the harder it is to minimize f along the $\mathbf{e}_i$ direction.[3] Intuitively, this implies we should spend more energy (i.e., assign more sampling probability) on coordinates with larger L_i. However, it was unclear what the best design is for such a distribution. In this paper, we present a clean and novel non-uniform sampling method which gives a faster convergence rate. Before going into the details, we first draw a distinction between non-accelerated and accelerated coordinate descent methods.

Non-Accelerated vs. Accelerated Methods. For smooth

[1] The results of this paper first appeared on arXiv in December 2015. In March 2016, Nesterov and Stich independently obtained our same results in a technical report (Nesterov & Stich, 2016).

The full version of this paper can be found on `http://arxiv.org/abs/1512.09103`.

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[2] The results of this paper generalize to the so-called proximal case that is to allow an additional separable term $\psi(x) \stackrel{\text{def}}{=} \sum_{i=1}^n \psi_i(x_i)$ to be added. The proofs require some non-trivial changes so we refrain from doing so in this version of the paper.

[3] For instance, if the i-th coordinate is selected, most coordinate-descent methods are only capable of performing an update $x' \leftarrow x - \frac{1}{L_i}\nabla_i f(x)$ with step length inversely proportional to L_i.

convex minimization, many first-order methods converge at a rate $1/\varepsilon$ to obtain an additive error $\varepsilon > 0$. In 1983, Nesterov demonstrated that a better and optimal rate $1/\sqrt{\varepsilon}$ can be obtained using his seminal accelerated gradient descent method. (Nesterov, 1983)

For this reason, people refer to methods converging at rate $1/\varepsilon$ as *non-accelerated* first-order methods, while those at rate $1/\sqrt{\varepsilon}$ as *accelerated* first-order methods. Similarly, when the objective $f(\cdot)$ is known to be strongly convex with parameter $\sigma > 0$, non-accelerated methods converge at a rate inversely proportional to σ, while accelerated ones converge at a rate inversely proportional to $\sqrt{\sigma}$. Although being much faster, accelerated first-order methods are also much more involved to design, see some recent attempts for designing accelerated methods in conceptually simpler manners (O'Donoghue & Candès, 2013; Su et al., 2014; Allen-Zhu & Orecchia, 2014; Bubeck et al., 2015).

Such a distinction continues to hold on the *coordinate-gradient* setting. A coordinate descent method iteratively selects a coordinate $i \in [n]$ at random, and updates the iterate x according to its coordinate gradient $\nabla_i f(x)$. As we shall see later, designing good sampling probabilities is well-studied for *non-accelerated* coordinate descent. In contrast, less is known in the more challenging accelerated regime, and we hope our work fills this gap.

We begin describing our result and compare it to the literature in the Euclidean norm case.

1.1 The Standard Euclidean Norm Case

In the non-accelerated world, in 2012, Nesterov (Nesterov, 2012) proposed a coordinate descent method called RCDM, which is a simple adaption of the full gradient descent method (see for instance the textbook (Nesterov, 2004)). At each iteration, RCDM selects a coordinate i with probability proportional to L_i, and performs update $x' \leftarrow x - \frac{1}{L_i}\nabla_i f(x)$. The number of iterations required to reach an ε error, denoted by T in this paper, satisfies $T = O(\frac{\sum_i L_i}{\varepsilon}\|x_0 - x^*\|^2)$ for RCDM. Here, we denote by x_0 the starting vector, x^* the minimizer of f, and $\|\cdot\|$ the ℓ_2 Euclidean norm.

This convergence rate is usually compared to that of full gradient descent: if L is the global smoothness parameter of $f(\cdot)$, then full gradient descent converges in $T = O(\frac{L}{\varepsilon}\|x_0 - x^*\|^2)$ iterations. Since L_i is never larger than L, and performing a coordinate descent step is usually n times faster than a full gradient step, RCDM performs faster than gradient descent in most applications.

In the same paper (Nesterov, 2012), Nesterov also demonstrated the possibility of performing *accelerated coordinate gradient descent* via a simple adaption of its full-gradient variant (Nesterov, 1983; 2004; 2005). This has been later analyzed in full by Lee and Sidford (Lee & Sidford, 2013), and they named this method *accelerated coordinate descent method (*ACDM*)*. ACDM converges the following number of iterations:

$$T = \begin{cases} \widetilde{O}\Big(\frac{\sqrt{n\sum_i L_i}}{\sqrt{\varepsilon}}\|x_0 - x^*\|\Big), & \text{when } f \text{ is convex} \\ O\Big(\frac{\sqrt{n\sum_i L_i}}{\sqrt{\sigma}}\log\frac{1}{\varepsilon}\Big), & \text{when } f \text{ is } \sigma\text{-strongly convex} \end{cases}$$

ACDM is built upon the estimation sequence technique of Nesterov (Nesterov, 1983; 2004; 2012), and similar to RCDM, ACDM also selects each coordinate i (essentially) with a probability proportional to L_i.[4] Since the analysis of Lee and Sidford is tight, it has been thought that the iteration bound T is not improvable.

In this paper, with a different non-uniform sampling method, we develop a new accelerated coordinate descent method NU_ACDM that converges in T iterations, where

$$T = \begin{cases} O\Big(\frac{\sum_i \sqrt{L_i}}{\sqrt{\varepsilon}}\|x_0 - x^*\|\Big), & \text{when } f \text{ is convex} \\ O\Big(\frac{\sum_i \sqrt{L_i}}{\sqrt{\sigma}}\log\frac{1}{\varepsilon}\Big), & \text{when } f \text{ is } \sigma\text{-strongly convex} \end{cases}$$

Note that NU_ACDM is always faster than ACDM because $\sum_i \sqrt{L_i} \le \sqrt{n\sum_i L_i}$. In the case when $(L_1, \ldots, L_n)$ is non-uniform, our method runs faster by a factor up to $\sqrt{n}$.[5] In our sampling step, we select each coordinate i with probability exactly proportional to $\sqrt{L_i}$, rather than (roughly) proportional to L_i. Thus, we need a different analysis from ACDM (Lee & Sidford, 2013), and also avoid the more complicated estimation sequence analysis.

1.2 The General L_β-Norm Case

Define the L_β norm $\|y\|_{\mathsf{L}_\beta}^2 \stackrel{\text{def}}{=} \sum_i L_i^\beta \cdot y_i^2$ for $\beta \in [0,1]$. Many accelerated coordinate descent methods provide convergence guarantees with respect to the L_1 norm (Lu & Xiao, 2013; Fercoq & Richtárik, 2015) or the L_β norm (Nesterov, 2012; Lin et al., 2014; Lee & Sidford, 2013).

For instance, RCDM takes β as an input, and converges in $T = O\big(\frac{S_{1-\beta}}{\varepsilon}\|x_0 - x^*\|_{\mathsf{L}_\beta}^2\big)$ iterations if one samples each coordinate i with probability $L_i^{1-\beta}/S_{1-\beta}$, where $S_\alpha \stackrel{\text{def}}{=} \sum_i L_i^\alpha$. In (Lee & Sidford, 2013), Lee and Sidford showed that their ACDM converges in T iterations with the same

[4] More precisely, they select each coordinate i with a probability proportional to $\max\{L_i, \frac{1}{n}\sum_j L_j\}$. As a consequence, each coordinate i is selected with probability at least $\Omega(1/n)$. Lee and Sidford emphasized that using this sampling method, rather than choosing each i directly with probability $L_i/(\sum_j L_j)$, is essential for ACDM to obtain the accelerated convergence rate.

[5] If $L_1 = \cdots = L_n$, we have $\sum_i \sqrt{L_i} = \sqrt{n\sum_i L_i}$. However, if $L_1 = 1$ while $L_2 = \cdots = L_n = 0$, we have $\sum_i \sqrt{L_i} = \frac{1}{\sqrt{n}} \cdot \sqrt{n\sum_i L_i}$.

Paper	Euclidean $\beta = 0$ Case		$\beta \in [0,1]$ Case		$\beta = 1$ Case	
	strongly convex	non strongly convex	strongly convex	non strongly convex	strongly convex	non strongly convex
RCDM ,	$\frac{\sum_i L_i}{\sigma} \log \frac{1}{\varepsilon}$	$\frac{\sum_i L_i}{\varepsilon} \|x_0 - x^*\|^2$	$\frac{S_{1-\beta}}{\sigma_\beta} \log \frac{1}{\varepsilon}$	$\frac{S_{1-\beta}}{\varepsilon} \|x_0 - x^*\|^2_{\mathsf{L}_\beta}$	$\frac{n}{\sigma_1} \log \frac{1}{\varepsilon}$	$\frac{n}{\varepsilon} \|x_0 - x^*\|^2_{\mathsf{L}_1}$
APCG , RBCD , Nesterov , APPROX	-	-	-	-	$\frac{n}{\sqrt{\sigma_1}} \log \frac{1}{\varepsilon}$	$\frac{n}{\sqrt{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_1}$
ACDM	$\frac{\sqrt{n \sum_i L_i}}{\sqrt{\sigma}} \log \frac{1}{\varepsilon}$	$\frac{\sqrt{n \sum_i L_i}}{\sqrt{\varepsilon}/\log \frac{1}{\varepsilon}} \|x_0 - x^*\|$	$\frac{\sqrt{nS_{1-\beta}}}{\sqrt{\sigma_\beta}} \log \frac{1}{\varepsilon}$	$\frac{\sqrt{nS_{1-\beta}}}{\sqrt{\varepsilon}/\log \frac{1}{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_\beta}$	$\frac{\sqrt{n}}{\sqrt{\sigma_1}} \log \frac{1}{\varepsilon}$	$\frac{n}{\sqrt{\varepsilon}/\log \frac{1}{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_1}$
this paper	$\frac{\sum_i \sqrt{L_i}}{\sqrt{\sigma}} \log \frac{1}{\varepsilon}$	$\frac{\sum_i \sqrt{L_i}}{\sqrt{\varepsilon}} \|x_0 - x^*\|$	$\frac{S_{(1-\beta)/2}}{\sqrt{\sigma_\beta}} \log \frac{1}{\varepsilon}$	$\frac{S_{(1-\beta)/2}}{\sqrt{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_\beta}$	$\frac{\sqrt{n}}{\sqrt{\sigma_1}} \log \frac{1}{\varepsilon}$	$\frac{n}{\sqrt{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_1}$

Table 1. Comparisons among coordinate descent methods, where $S_\alpha \stackrel{\text{def}}{=} \sum_i L_i^\alpha$.

sampling probabilities $L_i^{1-\beta}/S_{1-\beta}$, where

$$T = \begin{cases} \widetilde{O}\Big(\frac{\sqrt{nS_{1-\beta}}}{\sqrt{\varepsilon}} \|x_0 - x^*\|^2_{\mathsf{L}_\beta}\Big), & \text{when } f \text{ is convex} \\ O\Big(\frac{\sqrt{nS_{1-\beta}}}{\sqrt{\sigma_\beta}} \log \frac{1}{\varepsilon}\Big), & \text{when } f \text{ is } \sigma_\beta\text{-strongly convex w.r.t. the } \mathsf{L}_\beta \text{ norm} \end{cases}$$

This is always faster than RCDM. Note that, in the special case of $\beta = 1$ (and thus using uniform sampling probabilities), this same convergence result is also obtained by Nesterov (Nesterov, 2012), APCG (Lin et al., 2014), RBCD (Lu & Xiao, 2013), and APPROX (Fercoq & Richtárik, 2015). (See Table 1.)

Our method NU_ACDM improves this convergence to

$$T = \begin{cases} O\Big(\frac{S_{(1-\beta)/2}}{\sqrt{\varepsilon}} \|x_0 - x^*\|_{\mathsf{L}_\beta}\Big), & \text{when } f \text{ is convex} \\ O\Big(\frac{S_{(1-\beta)/2}}{\sqrt{\sigma_\beta}} \log \frac{1}{\varepsilon}\Big), & \text{when } f \text{ is } \sigma_\beta\text{-strongly convex w.r.t. the } \mathsf{L}_\beta \text{ norm} \end{cases}$$

Since $\sqrt{S_{1-\beta}} \le S_{(1-\beta)/2} \le \sqrt{nS_{1-\beta}}$, our method is faster than ACDM by a factor up to $\sqrt{n}$. Our improvement is again due to the new choice of sampling probabilities —we select each coordinate i with probability $L_i^{(1-\beta)/2}/S_{(1-\beta)/2}$ which is different from RCDM or ACDM— as well as our new proof that avoids the use of estimation sequence.

Remark 1.1. For the strongly convex case, convergence results with respect to Euclidean norms are usually more relevant to applications: for instance, the ℓ_2 regularizer is the most common one used in machine learning applications and algorithms designed for the Euclidean norm should be used for a better performance.[6] However, in the non-strongly convex case, results with respect to different β are in general incomparable. We include experiments in Section 7.3 to illustrate this.

[6] In contrast, consider an objective $f(x)$ equipped with a regularizer $\frac{\sigma}{2}\|x\|^2$. Such an objective is also strongly convex with respect to the L_β norm with parameter $\min_i L_i^{-\beta}$. If one applies an algorithm designed for the L_β norm using this parameter, the convergence would be much worse than the first column of Table 1.

2 Applications

Empirical Risk Minimization. A cornerstone problem in machine learning is empirical risk minimization (ERM). Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be the feature vectors of n data samples, $\phi_1, \ldots, \phi_n \colon \mathbb{R} \to \mathbb{R}$ be a sequence of convex loss functions, and $r \colon \mathbb{R}^d \to \mathbb{R}$ be a convex function (often known as a regularizer). The goal of ERM problem is to solve the following *primal* convex problem:

$$\min_{w \in \mathbb{R}^d} P(w) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \phi_i\big(\langle a_i, w \rangle\big) + r(w). \quad (2.1)$$

This includes a family of important problems such as SVM, Lasso, ridge regression, and logistic regression. Lin, Lu and Xiao (Lin et al., 2014) showed that the above minimization problem is equivalent to the following dual one:

$$\min_{y \in \mathbb{R}^n} D(y) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \phi_i^*(y_i) + r^*\left(-\frac{1}{n} \sum_{i=1}^n y_i a_i\right), \quad (2.2)$$

where ϕ_i^* and r^* are respectively the Fenchel conjugate function of ϕ_i and r.[7] Most importantly, if properly preprocessed, $D(y)$ can be shown to be coordinate-wise smooth and therefore accelerated coordinate descent methods can be applied to minimize $D(y)$. This approach leads to algorithm APCG, which matches the best known worst-case running time on solving (2.1) up to a logarithmic factor.[8]

However, by taking a closer look, the coordinate smoothness parameters $L_1, \ldots, L_n$ of $D(y)$ are data dependent. Indeed, L_i is roughly proportional to the Euclidean norm square of the i-th feature vector. Therefore, we can apply NU_ACDM in this paper to improve the running time obtained by APCG or AccSDCA. This is done in Section 7.

Note that each iteration of NU_ACDM selects a feature vector with a probability (roughly) proportional to its Euclidean norm. This is very different from the recent work of Zhao and Zhang (Zhao & Zhang, 2015), where they observed that for SDCA (Shalev-Shwartz & Zhang, 2013), a non-

[7] The conjugate of $r(x)$ is $r^*(y) \stackrel{\text{def}}{=} \max_w \{y^T w - r(w)\}$.

[8] Accelerated algorithms for solving (2.1) were first obtained by AccSDCA (Shalev-Shwartz & Zhang, 2014), and more recently improved by Katyusha (Allen-Zhu, 2016).

accelerated method, feature vectors should be sampled with probabilities proportional to their Euclidean norm *squares*. If one also uses the squared norms in the accelerated setting, he will only get a running time similar to ACDM, and therefore worse than our NU_ACDM.

We also mention one recent result that uses our NU_ACDM to develop faster ERM methods by exploiting the clustering structure of the dataset (Allen-Zhu et al., 2016).

Solving Linear Systems. Consider a linear system $Ax = b$ for some full row rank matrix $A \in \mathbb{R}^{m\times n}$ where $m \geq n$. Denoting $a_i \in \mathbb{R}^n$ as the i-th row vector of matrix A, the celebrated Kaczmarz method (Kaczmarz, 1937) iteratively picks one of the row vectors a_i and computes

$$x_{k+1} \leftarrow x_k + \frac{b_i - \langle a_i, x_k\rangle}{\|a_i\|^2} a_i \ .$$

Although many deterministic schemes have been proposed regarding how to select row vectors, many of them are difficult to analyze or compare. In a breakthrough paper, Strohmer and Vershynin (Strohmer & Vershynin, 2009) analyzed a randomized scheme and proved that:

Theorem 2.1 (Randomized Kaczmarz (Strohmer & Vershynin, 2009))**.** *If one samples row i with probability proportional to $\|a_i\|^2$ in each iteration, then the Kaczmarz method produces an ε-approximate solution of $Ax = b$[9] in $O\big(\|A^{-1}\|_2^2 \cdot \|A\|_F^2 \cdot \log\frac{1}{\varepsilon}\big)$ iterations, and each iteration costs a running time $O(n)$.*

Above, x^* is the solution to $Ax = b$, A^{-1} is the left inverse, $\|A^{-1}\|_2$ is one divided by the smallest non-zero singular value of A, and $\|A\|_F = (\sum_{ij} a_{ij}^2)^{1/2}$ is the Frobenius norm.

Randomized Kaczmarz can be viewed as coordinate descent (Lee & Sidford, 2013; Needell et al., 2014; Gower & Richtárik, 2015), and therefore ACDM applies here and gives a faster running time:

Theorem 2.2 (ACDM on Kaczmarz (Lee & Sidford, 2013))**.** *The ACDM method samples row i with probability proportional to $\max\{\|a_i\|^2, \frac{\|A\|_F^2}{m}\}$ at each iteration, and produces an ε-approximate solution to $Ax = b$ in $O\big(\sqrt{m}\|A^{-1}\|_2 \cdot \|A\|_F \cdot \log\frac{1}{\varepsilon}\big)$ iterations, and each iteration costs a running time $O(n)$.*

To obtain the above result, Lee and Sidford rewrote the problem of solving $Ax = b$ as an m-variate quadratic minimization problem

$$\min_{y\in\mathbb{R}^m} \big\{ f(y) \stackrel{\text{def}}{=} \tfrac{1}{2}\|A^T y\|^2 - \langle b, y\rangle \big\} \ .$$

The coordinate smoothness of f is $L_i = \|a_i\|^2$ for every $i \in [m]$, and the strong convexity of f can be deduced as $\sigma = \|A^{-1}\|_2^{-2}$.[10] For this reason, if we apply NU_ACDM instead of ACDM, we immediately get a faster algorithm:

Theorem 2.3 (NU_ACDM on Kaczmarz)**.** *The NU_ACDM method samples row i with probability proportional to $\|a_i\|$ at each iteration, and produces an ε-approximate solution to $Ax = b$ in $O\big(\|A^{-1}\|_2 \cdot \|A\|_{2,1} \cdot \log\frac{1}{\varepsilon}\big)$ iterations, and each iteration costs a running time $O(n)$.*

Above, $\|A\|_{2,1} \stackrel{\text{def}}{=} \sum_{j=1}^m \big(\sum_{i=1}^n |a_{ij}|^2\big)^{1/2}$ is the matrix $L_{2,1}$ norm. Since it satisfies $\|A\|_F \leq \|A\|_{2,1} \leq \sqrt{m}\|A\|_F$, our method is always faster than ACDM, and can be faster by a factor up to $\sqrt{m}$ that depends on the problem structure. We provide empirical evaluation on this in Section 8.

3 Other Related Work

People have considered selecting coordinates non-uniformly from other perspectives. For example, Nutini et al. (Nutini et al., 2015) compared the random coordinate selection rule with the Gauss-Southwell rule, and proved that except in the extreme cases, Gauss-Southwell rule is faster. Needell et el. (Needell et al., 2014) proposed a non-uniform sampling for stochastic gradient descent, and made a connection to the randomized Kaczmarz algorithm. Qu et al. (Qu et al., 2014) gave an algorithm which supports arbitrary sampling on dual variables. Csiba et al. (Csiba et al., 2015) showed that one can adaptively choose a probability distribution over the dual variables that depends on the "dual residues". All of the works cited above are for non-accelerated settings, while this paper focuses on designing fast accelerated method. Note that Qu and Richtárik (Qu & Richtárik, 2014) provided a unified analysis for both accelerated and non-accelerated coordinate descent methods with what they call "arbitrary sampling" in the non-strongly convex case. Our work can be seen as a continuation of that work, in that we instead focus on a particular class of sampling probabilities, for which we derive provably better convergence complexity bounds than prior results both for strongly-convex and non-strongly convex cases. In the non-strongly convex case, our results can be infered from the general results in (Qu & Richtárik, 2014).

4 Notations

Let x^* be an arbitrary minimizer of $f(x)$ and we are interested in finding a vector x satisfying $f(x) - f(x^*) \leq \varepsilon$ for an accuracy parameter $\varepsilon > 0$. We use $\|\cdot\|$ to denote the Euclidean norm and $\mathbf{e}_i \in \mathbb{R}^n$ the i-th unit vector. We denote by $\nabla f(x)$ the full gradient of f at point $x \in \mathbb{R}^n$, and by $\nabla_i f(x)$ the i-th coordinate gradient. With a slight abuse of notation, we view $\nabla_i f(x)$ both as a scaler in $\mathbb{R}$ and as a singleton vector in $\mathbb{R}^n$.

[9] That is, a vector x satisfying $\mathbb{E}[\|x_k - x^*\|^2] \leq \varepsilon\|x_0 - x^*\|^2$.

[10] One has to in fact consider the strong convexity of f in the space orthogonal to the null space $\{y \in \mathbb{R}^m | A^T y = 0\}$. We recommend interested readers to see Section 5.2 of (Lee & Sidford, 2013) for details.

Algorithm 1 NU_ACDM(β, f, x_0, T)

Input: $\beta \in [0,1]$; f a convex function that is coordinate-wise smooth with parameters $(L_1, \ldots, L_n)$, and σ_β-strongly convex with respect to $\|\cdot\|_{\mathsf{L}_\beta}$ for some $\beta \in [0,1]$; x_0 some initial point; and T the number of iterations.

Output: y_T such that $\mathbb{E}[f(y_T)] - f(x^*) \le O((1-\tau)^T) \cdot (f(x_0) - f(x^*))$.

$\alpha \leftarrow (1-\beta)/2$, $S_\alpha \leftarrow \sum_{i=1}^n L_i^\alpha$.

$p_i \leftarrow \frac{L_i^\alpha}{S_\alpha}$ for each $i \in [n]$. ◇ $\sum_i p_i = 1$ *so* $\{p_i\}_i$ *forms a distribution over* $[n]$

$\tau \leftarrow \frac{2}{1+\sqrt{4S_\alpha^2/\sigma_\beta+1}}$, $\eta \leftarrow \frac{1}{\tau S_\alpha^2}$. ◇ $\tau = O(\frac{\sqrt{\sigma_\beta}}{S_\alpha})$ *and* $\eta = O(\frac{1}{\sqrt{\sigma_\beta} S_\alpha})$

$y_0 \leftarrow x_0, \quad z_0 \leftarrow x_0$.

for $k \leftarrow 0$ **to** $T-1$ **do**

  $x_{k+1} \leftarrow \tau z_k + (1-\tau) y_k$.

  Sample i from $\{1, \cdots, n\}$ based on $p = (p_1, \cdots, p_n)$.

  $y_{k+1} \leftarrow y_{k+1}^{(i)} \stackrel{\text{def}}{=} x_{k+1} - \frac{1}{L_i}\nabla_i f(x_{k+1})$

  $z_{k+1} \leftarrow z_{k+1}^{(i)} \stackrel{\text{def}}{=} \frac{1}{1+\eta\sigma_\beta}\big(z_k + \eta\sigma_\beta x_{k+1} - \frac{\eta}{p_i L_i^\beta}\nabla_i f(x_{k+1})\big)$

end for

return y_T.

Definition 4.1. *We say that f is L-smooth if $\forall x, y \in \mathbb{R}^n$, it satisfies $\|\nabla f(x) - \nabla f(y)\| \le L\|x-y\|$.*

We say that f is σ-strongly convex if $\forall x, y \in \mathbb{R}^n$, it satisfies $f(y) \ge f(x) + \langle \nabla f(x), y-x\rangle + \frac{\sigma}{2}\|x-y\|^2$.

Definition 4.2. *f is coordinate-wise smooth with parameters $(L_1, L_2, \ldots, L_n)$, if for all $x \in \mathbb{R}^n$, $\delta > 0$, $i \in [n]$:*

$$|\nabla_i f(x + \delta \mathbf{e}_i) - \nabla_i f(x)| \le L_i \cdot \delta \ .$$

Following the notations of prior work (Nesterov, 2012; Lee & Sidford, 2013), we make the following definitions

Definition 4.3. *Given $\alpha, \beta \in [0,1]$, define*

$$S_\alpha \stackrel{\text{def}}{=} \sum_{i=1}^n L_i^\alpha, \qquad \|x\|_{\mathsf{L}_\beta} \stackrel{\text{def}}{=} \sum_{i=1}^n x_i^2 \cdot L_i^\beta, \quad \text{and} \quad \langle x, y\rangle_{\mathsf{L}_\beta} \stackrel{\text{def}}{=} \sum_{i=1}^n x_i y_i \cdot L_i^\beta \ .$$

Also, define σ_β to be the strong convexity parameter of $f(\cdot)$ with respect to the $\|\cdot\|_{\mathsf{L}_\beta}$ norm. That is, it satisfies $f(y) \ge f(x) + \langle \nabla f(x), y-x\rangle + \frac{\sigma_\beta}{2}\|x-y\|_{\mathsf{L}_\beta}^2$ for all $x, y \in \mathbb{R}^n$.

Clearly, if f is σ strongly convex then $\sigma_0 = \sigma$.

5 NUACDM in Strongly Convex Case

We now propose our new method NU_ACDM to deal with strongly convex and smooth objectives. Suppose $f(\cdot)$ is coordinate-wise smooth with parameters $(L_1, \ldots, L_n)$ and σ_β-strongly convex with respect to $\|\cdot\|_{\mathsf{L}_\beta}$ for some $\beta \in [0,1]$. At a first reading, one can simply consider $\beta = 0$ so f is σ_0-strongly convex with respect to the traditional Euclidean norm. We choose to analyze the full parameter regime of β to better compare us with known literatures.

As described in Algorithm 1, NU_ACDM begins with $x_0 = y_0 = z_0$ and iteratively computes the tuple $x_{k+1}, y_{k+1}, z_{k+1}$ from x_k, y_k, z_k. In iteration $k = 0, 1, \ldots, T-1$, we first compute $x_{k+1} \leftarrow \tau z_k + (1-\tau) y_k$ for some parameter $\tau \in [0,1]$ (whose value will be specified later), and randomly select a coordinate $i \in [n]$ with probability $p_i = L_i^\alpha / S_\alpha$ where $\alpha \stackrel{\text{def}}{=} (1-\beta)/2$.

Whenever i is selected at iteration k, we perform two updates $y_{k+1} \leftarrow x_{k+1} - \frac{1}{L_i}\nabla_i f(x_{k+1})$ and $z_{k+1} \leftarrow \frac{1}{1+\eta\sigma_\beta}\big(z_k + \eta\sigma_\beta x_{k+1} - \frac{\eta}{p_i L_i^\beta}\nabla_i f(x_{k+1})\big)$, both using the i-th coordinate gradient at point x_{k+1}. Here, $\eta > 0$ is the parameter that determines the step length of the second update; its choice will become clear in the analysis. Our main theorem in this section is as follows:

Theorem 5.1. *If $f(x)$ is coordinate-wise smooth with parameters $(L_1, \ldots, L_n)$, and σ_β-strongly convex with respect to $\|\cdot\|_{\mathsf{L}_\beta}$ for some $\beta \in [0,1]$, then* NU_ACDM(β, f, x_0, T) *produces an output y_T satisfying* $\mathbb{E}[f(y_T)] - f(x^*) \le O(1) \cdot (1-\tau)^T (f(x_0) - f(x^*))$, *where* $\tau = \frac{2}{1+\sqrt{4S_{(1-\beta)/2}^2/\sigma_\beta+1}} = \frac{1}{O\left(S_{(1-\beta)/2}/\sqrt{\sigma_\beta}\right)}$.

In particular, if $\beta = 0$ parameter τ becomes $\tau = \frac{1}{O\left(\sum_i \sqrt{L_i}/\sqrt{\sigma}\right)}$. Note that each iteration of NU_ACDM can be implemented to run in time similar to ACDM and therefore RCDM. We include proofs in the full paper.

6 NUACDM in Non-Strongly Convex Case

We propose algorithm NU_ACDM$^{\text{ns}}$ in the case when $f(\cdot)$ is not necessarily strongly convex. NU_ACDM$^{\text{ns}}$ requires some non-trivial modifications on NU_ACDM: for instance, η and τ are no longer constants (see Algorithm 2). The analysis is also slightly different but in the same structure as Section 5. We include proofs in the full paper.

7 Experiments on ERM

We perform experiments on ERM problems to confirm our theoretical improvements. We consider three datasets in

Algorithm 2 $\texttt{NU\_ACDM}^{\texttt{ns}}(\beta, f, x_0, T)$

Input: $\beta \in [0,1]$;
f a convex function that is coordinate-wise smooth with parameters $(L_1, \ldots, L_n)$;
x_0 some initial point; and
T the number of iterations.

Output: y_T such that $\mathbb{E}[f(y_T)] - f(x^*) \le 2\|x_0 - x^*\|^2_{\mathsf{L}_\beta} \cdot S^2_{(1-\beta)/2}/(T+1)^2$.

$\alpha \leftarrow (1-\beta)/2$, $S_\alpha \leftarrow \sum_{i=1}^n L_i^\alpha$.
$p_i \leftarrow \frac{L_i^\alpha}{S_\alpha}$ for each $i \in [n]$. $\diamond$ $\sum_i p_i = 1$ *so* $\{p_i\}_i$ *forms a distribution over* $[n]$
$y_0 \leftarrow x_0, \quad z_0 \leftarrow x_0$.
for $k \leftarrow 0$ **to** $T-1$ **do**
  $\eta_{k+1} \leftarrow \frac{k+2}{2S_\alpha^2}$, and $\tau_k \leftarrow \frac{1}{\eta_{k+1} S_\alpha^2} = \frac{2}{k+2}$.
  $x_{k+1} \leftarrow \tau_k z_k + (1-\tau_k) y_k$.
  Sample i from $\{1, \cdots, n\}$ based on $p = (p_1, \cdots, p_n)$.
  $y_{k+1} \leftarrow y_{k+1}^{(i)} \stackrel{\text{def}}{=} x_{k+1} - \frac{1}{L_i}\nabla_i f(x_{k+1})$
  $z_{k+1} \leftarrow z_{k+1}^{(i)} \stackrel{\text{def}}{=} z_k - \frac{\eta_{k+1}}{p_i L_i^\beta}\nabla_i f(x_{k+1})$
end forreturn y_T.

this section: (1) class 1 of the news20 dataset ($15,935$ samples and $62,061$ features), (2) the w8a dataset ($49,749$ samples and 300 features), and (3) the covtype dataset ($581,012$ samples and 54 features). All of them can be found on the LibSVM website (Fan & Lin), and contain examples that have non-uniform Euclidean norms (see Figure 3 in the appendix for the distribution).

7.1 Experiments on Strongly Convex Objectives

Consider a regularized least-square problem which is problem (2.1) with $\phi_i(t) \stackrel{\text{def}}{=} \frac{1}{2}(t - l_i)^2$, where l_i is the label for feature vector a_i. In the case when $r(w) = \frac{\lambda}{2}\|w\|_2^2$, this problem becomes *ridge regression*, and in the case when $r(w) = \lambda\|w\|_1$, it is known as *Lasso regression*.

Following (2.2), the equivalent dual formulation of regularized least square can be written as

$$\min_{y \in \mathbb{R}^n} \Big\{ D(y) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n \big(\frac{1}{2}y_i^2 + y_i \cdot l_i\big) + r^*\Big(-\frac{1}{n}\sum_{i=1}^n y_i a_i\Big)\Big\} \quad . \quad (7.1)$$

Furthermore, $D(y)$ is $1/n$-strongly convex.

Ridge Regression. In ridge regression, we have $r(w) = \frac{\lambda}{2}\|w\|_2^2$ and accordingly $r^*(z) = \frac{1}{2\lambda}\|z\|_2^2$ in (7.1). It is not hard to verify that $D(y)$ is $L_i \stackrel{\text{def}}{=} \frac{1}{n} + \frac{1}{\lambda n^2}\|a_i\|_2^2$ smooth with respect to its i-th coordinate (and thus with respect to the i-th example). Therefore, the coordinate smoothness parameters are *non-uniform* if examples $a_1, \ldots, a_n$'s do not have the same Euclidean norms.

We can directly apply RCDM, ACDM and our NU_ACDM with $\beta = 0$ and $\sigma = 1/n$ to minimize (7.1). In principle, one can also apply APCG to minimize $D(y)$. However, since APCG is only designed for $\beta = 1$ and needs an unknown parameter $\sigma_1 > 0$ as input, we have tuned it for the fastest convergence in our experiments; whenever we do so, we denote it as $\texttt{APCG}^*$ in the diagrams.[11]

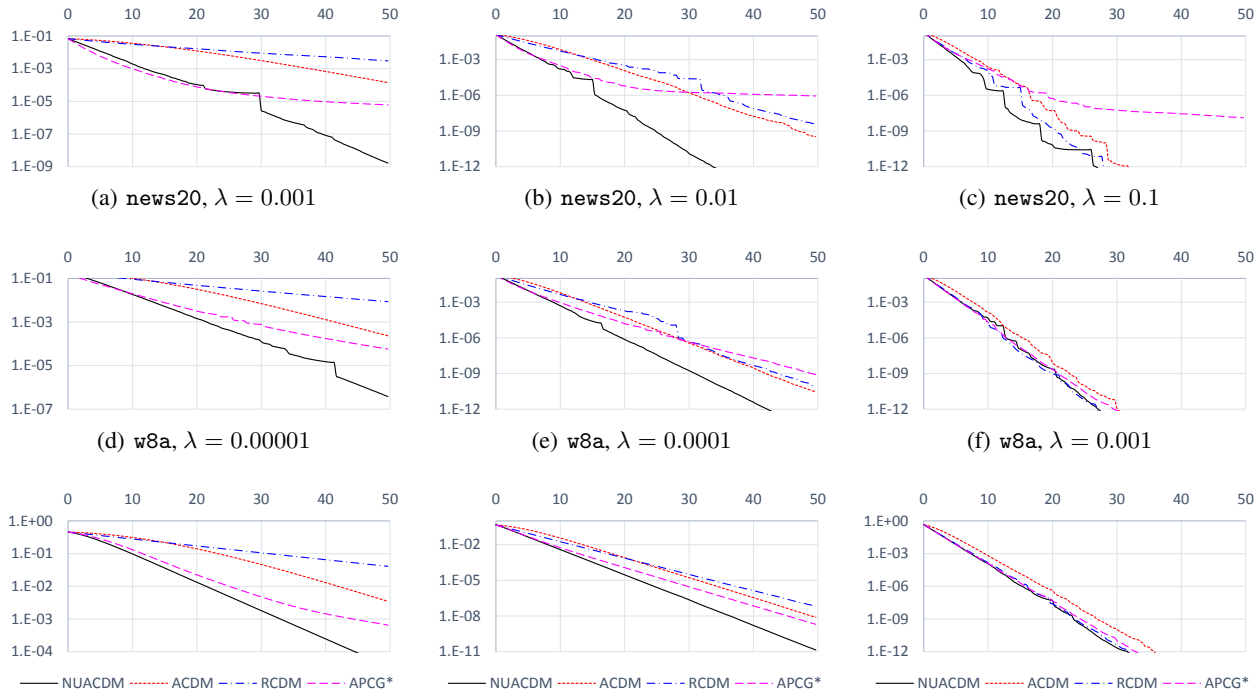
Our experimental results for ridge regression are in Figure 1. Note that theory predicts that NU_ACDM enjoys a speed-up factor of $\frac{\sqrt{n\sum_i L_i}}{\sum_i \sqrt{L_i}} \ge 1$ over ACDM, and we show this factor in Table 2. We make the following observations:

- Since $L_i = \frac{1}{n} + \frac{1}{\lambda n^2}\|a_i\|_2^2$, the smaller the regularization parameter λ is, the more non-uniform the parameters $L_1, \ldots, L_n$ are. This is why the numbers in Table 2 are in decreasing order in each row. Our experiment confirms on this because we obtain the greatest improvements for the left 3 charts in Figure 1.
- news20 has the most non-uniformity on the examples' Euclidean norms among the three datasets. Therefore, the first row Table 2 have the largest speed-up factors. Our experiment confirms on this because we obtain the greatest improvements in the top 3 charts in Figure 1.
- APCG performs quite poorly on dataset news20 because it relies on the L_β norm strong convexity for $\beta = 1$, which is very different from the Euclidean norm strong convexity when the parameters L_i are very non-uniform. We discuss the choice of β in Section 7.3, and would like to point out that APCG performs very well for non strongly convex objectives, see Section 7.2.

Due to strong duality, our convergence speed-up on the dual objective also translates to that on the primal objective. See Figure 5 in the appendix for details.

Lasso. Due to space limitation, we include our experi-

[11] We have chosen 14 values of σ_1 in a reasonable range, where the largest choice of σ_1 is $50,000$ times larger than the smallest choice. Our automated program will then make the final choice of σ_1 based on the convergence speed.



(a) news20, $\lambda = 0.001$ (b) news20, $\lambda = 0.01$ (c) news20, $\lambda = 0.1$

(d) w8a, $\lambda = 0.00001$ (e) w8a, $\lambda = 0.0001$ (f) w8a, $\lambda = 0.001$

(g) covtype, $\lambda = 1$ (h) covtype, $\lambda = 10$ (i) covtype, $\lambda = 100$

Figure 1. Performance Comparison for Ridge Regression. The y axis represents the dual objective distance to minimum, and the x axis represents the number of passes to the dataset. More experiment plots can be found in the full version of this paper.

news20	$\lambda = 0.001$	1.56772	$\lambda = 0.01$	1.30740	$\lambda = 0.1$	1.05110
w8a	$\lambda = 0.00001$	1.11060	$\lambda = 0.0001$	1.04897	$\lambda = 0.001$	1.00373
covtype	$\lambda = 1$	1.04266	$\lambda = 10$	1.02787	$\lambda = 100$	1.00362

Table 2. The theoretical speed-up factor $\sqrt{n\sum_i L_i}/\left(\sum_i \sqrt{L_i}\right)$ of NU_ACDM over ACDM for the three datasets.

	$r = 100\%$	$r = 80\%$,	$r = 60\%$	$r = 40\%$	$r = 20\%$	$r = 10\%$
Speed Up	1	1.0992	1.2464	1.4025	1.6243	1.7379

Table 3. Theoretical Speed-Up Factors $\sqrt{n\sum_i L_i}/\left(\sum_i \sqrt{L_i}\right)$ of NU_ACDM over ACDM for linear systems $Ax = b$.

mental results for Lasso regression in the full paper.

7.2 Experiments on Non-Strongly Convex Objectives

Consider problem (2.1) where $r(w) = \frac{\lambda}{2}\|w\|^2$ is the ℓ_2 regularizer and each $\phi_i(\cdot)$ is some *non-smooth* loss function. In this case, the dual objective (2.2) becomes

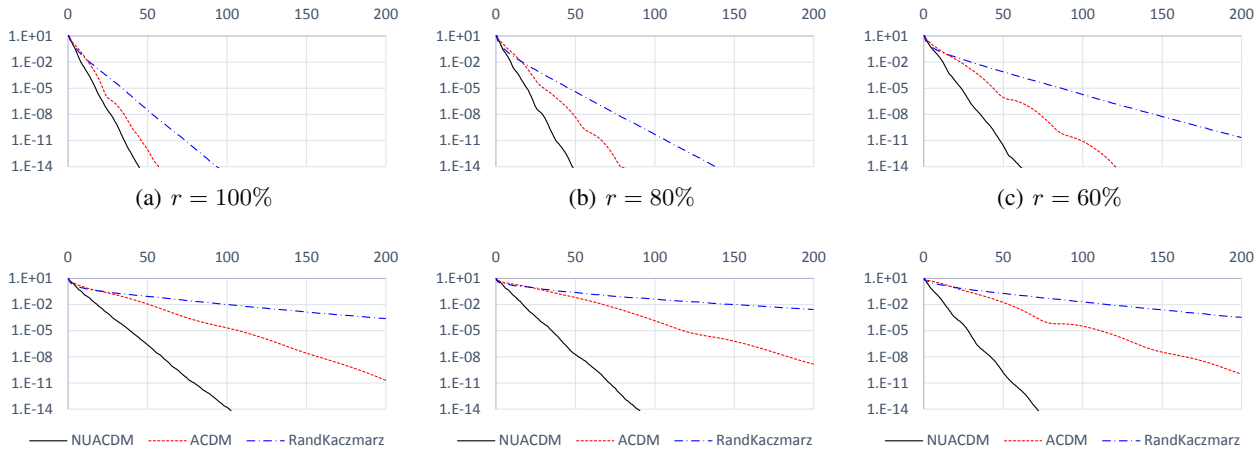
$$\min_{y\in\mathbb{R}^n}\left\{D(y) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n \phi_i^*(y_i) + \frac{1}{2\lambda n^2}\left\|\sum_{i=1}^n y_i a_i\right\|_2^2\right\}. \tag{7.2}$$

This $D(y)$ is not necessarily strongly convex because the penalty functions $\phi_i(\cdot)$ is not smooth. In this section, we conduct an experiment for the case when $\phi_i(\alpha) \stackrel{\text{def}}{=} \frac{1}{2}(\alpha - l_i)^2 + |\alpha - l_i|$ is an $\ell_2 - \ell_1$ penalty function. We call this ERM problem the $\ell_2 - \ell_1$ Penalty Regression.

As before, we know that $D(y)$ is $L_i \stackrel{\text{def}}{=} \frac{1}{n} + \frac{1}{\lambda n^2}\|a_i\|_2^2$ smooth with respect to the i-th coordinate, so we can apply ACDM, RCDM, APCG and our NU_ACDM$^{\text{ns}}$ directly to minimize $D(y)$. We choose $\beta = 0$ for ACDM, RCDM, and NU_ACDM$^{\text{ns}}$ in our experiment, and have to choose $\beta = 1$ for APCG. Our results are shown in Figure 7 in the appendix.

From these experiments, we see that again the theoretical speed-up factors in Table 2 are validated in practice. NU_ACDM$^{\text{ns}}$ has a clear advantage over its close relatives ACDM and RCDM when the coordinate smoothness parameters L_i are very non-uniform (such as dataset news20), and when λ is relatively small.

In contrast to the previous subsection, APCG (which uses



(a) $r = 100\%$ (b) $r = 80\%$ (c) $r = 60\%$

(d) $r = 40\%$ (e) $r = 20\%$ (f) $r = 10\%$

Figure 2. Performance Comparison on Solving $Ax = b$.

$\beta = 1$) performs extremely well and similar to NU_ACDM^{ns} (which uses $\beta = 0$) in Figure 7. As we shall see in the next section, by taking a closer look at different choices of β for non-strongly convex objectives, APCG is in fact analogous to the $\beta = 1$ case of NU_ACDM^{ns}, but is slightly worse than NU_ACDM^{ns} for β being between 0 and 0.8 for all the three datasets we are considering in this paper.

7.3 Dependence on β

As discussed in Remark 1.1, when dealing with a strongly convex objective $f(\cdot)$, we usually work with accelerated coordinate descent methods for Euclidean norm rather than L_β norms. However, the choice becomes less obvious for non-strongly convex objectives.

For instance, in Table 1, by comparing $T = \sum_i \sqrt{L_i/\varepsilon} \cdot \|x_0 - x^*\|$ for $\beta = 0$ and $T = n/\sqrt{\varepsilon} \cdot \|x_0 - x^*\|_{\mathsf{L}_1}$ for $\beta = 1$, it is not immediately clear which one is more preferable to the other. If one works with a standard machine learning boundedness assumption $\|x_0 - x^*\| \leq \Theta$ for some constant Θ, then the convergence for the $\beta = 1$ case reduces to $T = n/\sqrt{\varepsilon} \cdot \|x_0 - x^*\|_{\mathsf{L}_1} \leq n \max_i \sqrt{L_i}/\varepsilon \cdot \Theta$ which is slower than that of the $\beta = 0$ case. However, in general, the best choice of β depends on how the coordinates of the vector $x_0 - x^*$ scale with parameters L_i.

Nevertheless, we can perform a comparison *in practice* between difference choices of β. Focusing on the $\ell_1 - \ell_2$ Penalty Regression dual objective (7.2), we plot the performance of NU_ACDM^{ns} with different β. From Figure 4 in the appendix, we conclude that smaller values of β are perhaps more preferred to larger ones in practice. Not surprisingly, the performance difference becomes less significant for dataset covtype, because it has more uniform smoothness parameters L_i than the other two datasets. Finally, we have included APCG in Figure 4 as well, and it has very similar performance comparing to NU_ACDM^{ns} for $\beta = 1$. This confirms our theoretical finding in Table 1.

8 Experiments on Solving Linear Systems

We generate random linear systems $Ax = b$ and compare randomized Kaczmarz, ACDM, and NU_ACDM.

We choose $m = 300$ and $n = 100$, and generate each entry A_{ij} uniformly at random in $[0, 1]$. We scale a fraction r of A's rows to have Euclidean norm 10, and the rest to have Euclidean norm 1. We generate a random vector x, compute $b = Ax$, and use each of the three algorithms to solve x given A and b.

Since the coordinate smoothness parameters depend on the Euclidean norm squares of A's rows, we expect our NU_ACDM to have a greater speed up comparing to ACDM for small nonzeros values of r. We compute the theoretical speed up factors in Table 3.

In Figure 2, we see that both NU_ACDM and ACDM outperform the non-accelerated randomized Kaczmarz without surprise. Furthermore, NU_ACDM and ACDM are comparable for $r = 100\%$, and the out-performance indeed becomes more significant for smaller values of r.

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