Power of Ordered Hypotheses Testing (Supplementary Material)

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Appendix A: Proof of Theorem 1

Let $\mathcal{F}_k$ be the σ field generated by all non-null pvalues as well as $\{I(p_i \le s), I(p_i > \lambda) : i \ge k\}$. Then $\hat{k}$ is a stopping time with respect to the backward filtration $\mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \subset \mathcal{F}_1$. Recall that $V(s,k) = \sum_{i \le k, i \not\in \mathcal{H}_0} I(p_i \le s)$ and $R(s,k) = \sum_{i \le k} I(p_i \le s)$, it holds that

$$\begin{aligned}\mathrm{FDP} = \frac{V(s,\hat{k})}{R(s,\hat{k}) \vee 1} &= \frac{V(s,\hat{k})}{1 + \sum_{i \le \hat{k}} I(p_i > \lambda)} \cdot \frac{1 + \sum_{i \le \hat{k}} I(p_i > \lambda)}{R(s,\hat{k}) \vee 1} \\ &\le \frac{V(s,\hat{k})}{1 + \sum_{i \in \mathcal{H}_0, i \le \hat{k}} I(p_i > \lambda)} \cdot \frac{1 + \sum_{i \le \hat{k}} I(p_i > \lambda)}{R(s,\hat{k}) \vee 1} \\ &\le \frac{V(s,\hat{k})}{1 + \sum_{i \in \mathcal{H}_0, i \le \hat{k}} I(p_i > \lambda)} \cdot \frac{1-\lambda}{s} q.\end{aligned}$$

Let

$$M(k) = \frac{V(s,k)}{1 + \sum_{i \in \mathcal{H}_0, i \le k} I(p_i > \lambda)}.$$

Now we prove that $M(k)$ is a backward martingale with respect to the filtration $\{\mathcal{F}_k : k = n, n-1, \ldots, 1\}$. In fact, let

$$V^+(k) = V(s,k) = \sum_{i \in \mathcal{H}_0, i \le k} I(p_i \le s), \quad V^-(k) = \sum_{i \in \mathcal{H}_0, i \le k} I(p_i > \lambda).$$

The notations here is comparable to Barber and Candès (2015). Then

$$M(k) = \frac{V^+(k)}{1 + V^-(k)}.$$

If $n \in \mathcal{H}_0^c$ is non-null, then $M(k-1) = M(k)$. If n is null, let $(I_1, I_2) = (I(p_k \le s), I(p_k > \lambda))$. Since $\{p_i : i \in \mathcal{H}_0\}$ are i.i.d., by symmetry,

$$P(I_1 = 1, I_2 = 0 | \mathcal{F}_k) = \frac{V^+(k)}{|\mathcal{H}_0 \cap \{1, \ldots, k\}|}, \quad P(I_1 = 0, I_2 = 1 | \mathcal{F}_k) = \frac{V^-(k)}{|\mathcal{H}_0 \cap \{1, \ldots, k\}|},$$

$$P(I_1 = 0, I_2 = 0 | \mathcal{F}_k) = 1 - \frac{V^+(k) + V^-(k)}{|\mathcal{H}_0 \cap \{1, \ldots, k\}|}.$$

Thus,

$$
\begin{aligned}
&\mathbb{E}(M(k-1)|\mathcal{F}_k)\\
=&\frac{V^+(k)-1}{1+V^-(k)}\cdot\frac{V^+(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}+\frac{V^+(k)}{V^-(k)\vee 1}\cdot\frac{V^-(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}+\frac{V^+(k)}{1+V^-(k)}\cdot\left(1-\frac{V^+(k)+V^-(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}\right)\\
\leq&\frac{V^+(k)-1}{1+V^-(k)}\cdot\frac{V^+(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}+\frac{V^+(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}+\frac{V^+(k)}{1+V^-(k)}\cdot\left(1-\frac{V^+(k)+V^-(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}\right)\\
=&\frac{V^+(k)}{1+V^-(k)}\cdot\frac{V^+(k)+V^-(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}+\frac{V^+(k)}{1+V^-(k)}\cdot\left(1-\frac{V^+(k)+V^-(k)}{|\mathcal{H}_0\cap\{1,\ldots,k\}|}\right)\\
=&\frac{V^+(k)}{1+V^-(k)}=M(k).
\end{aligned}
$$

In summary,

$$\mathbb{E}(M(k-1)|\mathcal{F}_k)\leq M(k)$$

which shows that M_k is a backward super-martingale. Notice that $M(k)\leq n$ is bounded, it follows from optimal stopping time theorem that

$$\mathbb{E}M(\hat{k})\leq\mathbb{E}M(n)=\mathbb{E}\left(\frac{\sum_{i\in\mathcal{H}_0}I(p_i\leq s)}{1+\sum_{i\in\mathcal{H}_0}I(p_i>\lambda)}\right).$$

It is easy to see that

$$\mathcal{L}\left(\sum_{i\in\mathcal{H}_0}I(p_i\leq s)\middle|\sum_{i\in\mathcal{H}_0}I(p_i\leq\lambda)=m\right)=Binom\left(m,\frac{F_1(s)}{F_1(\lambda)}\right)$$

and hence

$$
\begin{aligned}
\mathbb{E}\left(\frac{\sum_{i\in\mathcal{H}_0}I(p_i\leq s)}{1+\sum_{i\in\mathcal{H}_0}I(p_i>\lambda)}\right)&=\mathbb{E}\left(\mathbb{E}\left(\frac{\sum_{i\in\mathcal{H}_0}I(p_i\leq s)}{1+\sum_{i\in\mathcal{H}_0}I(p_i>\lambda)}\middle|\sum_{i\in\mathcal{H}_0}I(p_i\leq\lambda)\right)\right)\\
&=\frac{F_1(s)}{F_1(\lambda)}\cdot\mathbb{E}\left(\frac{\sum_{i\in\mathcal{H}_0}I(p_i\leq\lambda)}{1+\sum_{i\in\mathcal{H}_0}I(p_i>\lambda)}\right)\\
&\leq\frac{F_1(s)}{1-F_1(\lambda)}
\end{aligned}
$$

where the last assertion follows from the fact that for any binomial random variable $X\sim N(r,p)$,

$$\mathbb{E}\frac{X}{r+1-X}\leq\frac{s}{1-\lambda}. \tag{1}$$

In fact,

$$
\begin{aligned}
\mathbb{E}\frac{X}{r+1-X}&=\sum_{i=0}^{n}\frac{i}{r+1-i}\cdot\begin{pmatrix}r\\i\end{pmatrix}p^i(1-p)^{r-i}\\
&=\sum_{i=1}^{r}\frac{r!}{(i-1)!(r+1-i)!}\cdot p^i(1-p)^{r-i}\\
&=\frac{p}{1-p}\cdot\sum_{i=0}^{r-1}\begin{pmatrix}r\\i\end{pmatrix}p^i(1-p)^{r-i}\\
&\leq\frac{p}{1-p}.
\end{aligned}
$$

Since $p_i\succeq U[0,1]$, it holds that $F_1(s)\leq s$ and $F_1(\lambda)\leq\lambda$. Thus, by Optional Stopping Theorem,

$$\mathbb{E}M(\hat{k})\leq\mathbb{E}M(n)=\mathbb{E}\left(\frac{\sum_{i\in\mathcal{H}_0}I(p_i\leq s)}{1+\sum_{i\in\mathcal{H}_0}I(p_i>\lambda)}\right)\leq\frac{s}{1-\lambda}.$$

and hence

$$FDR(\hat{k})=\mathbb{E}\left(\frac{V(s,\hat{k})}{R(s,\hat{k})\vee 1}\right)\leq\frac{s}{1-\lambda}\cdot\frac{1-\lambda}{s}q=q.$$

Appendix B: Proof of Theorem 2

Lemma 1. *Let $B_i \sim Ber(1, p_i)$ are independent Bernoulli random variables. Then for some positive integer r and postive real number ν,*

$$P\left(\sup_{k\geq r}\left|\frac{\sum_{i=1}^{k}(B_i - p_i)}{k}\right| > \nu\right) \leq \left(2 + \frac{4}{\nu^2}\right)e^{-\frac{r\nu^2}{2}}.$$

Proof. Notice that $B_i - p_i$ is subgaussian with parameter 1, we have

$$P\left(\left|\frac{\sum_{i=1}^{k}(B_i - p_i)}{k}\right| > \nu\right) \leq 2e^{-\frac{k\nu^2}{2}}.$$

Then

$$P\left(\sup_{k\geq r}\left|\frac{\sum_{i=1}^{k}(B_i - p_i)}{k}\right| > \nu\right) \leq 2\sum_{k\geq r} e^{-\frac{k\nu^2}{2}} \leq \frac{2e^{-\frac{r\nu^2}{2}}}{1 - e^{-\frac{\nu^2}{2}}} \leq \left(2 + \frac{4}{\nu^2}\right)e^{-\frac{r\nu^2}{2}},$$

where the last step uses the fact that

$$1 - e^{-\frac{\nu^2}{2}} = 1 - \frac{1}{e^{\frac{\nu^2}{2}}} \geq 1 - \frac{1}{1 + \frac{\nu^2}{2}} = \frac{1}{1 + \frac{2}{\nu^2}}.$$

□

Proposition 1. *For a VCT model with instantaneous non-null probability $\pi(t)$,*

$$\max_{k=a_n, a_n+1, \ldots, n}\left|\frac{\#\{i \leq k : i \notin \mathcal{H}_0\}}{k} - \Pi\left(\frac{k}{n}\right)\right| \leq \epsilon_n \tag{2}$$

holds with probability converging to 1 for properly chosen sequences $\{a_n\}$ and $\{\epsilon_n\}$ such that

$$a_n/n \to 0,\ a_n \to \infty,\ \epsilon_n \to 0.$$

In particular we can set $a_n = \lceil(\log n)^2\rceil$ and $\epsilon_n = \frac{1}{\sqrt{\log n}}$.

Remark 1. The condition (12) of Li and Barber (2015) sets $a_n = 0$, which cannot be true. As will be shown later, a growing sequence a_n suffices for our asymptotic analysis.

Proof of Proposition 1. Let $P_n(x)$ be the step function with $P_n(x) = \frac{\lfloor nx \rfloor}{n}$. For any given k,

$$\begin{aligned}
&\left|\frac{\#\{i \leq k : i \notin \mathcal{H}_0\}}{k} - \Pi\left(\frac{k}{n}\right)\right| \\
\leq &\left|\frac{\sum_{i=1}^{k}(I(i \notin \mathcal{H}_0) - \pi(\frac{i}{n}))}{k}\right| + \left|\int_0^{\frac{k}{n}} \pi(x) d(P_n(x) - x)\right| \\
\leq &\left|\frac{\sum_{i=1}^{k}(I(i \notin \mathcal{H}_0) - \pi(\frac{i}{n}))}{k}\right| + \frac{1}{n}\cdot\int_0^{\frac{k}{n}} \pi(x) dx \\
\leq &\left|\frac{\sum_{i=1}^{k}(I(i \notin \mathcal{H}_0) - \pi(\frac{i}{n}))}{k}\right| + \frac{1}{n}.
\end{aligned}$$

Let $\nu_n = \epsilon_n - \frac{1}{n}$. By Lemma 1,

$$P\left(\sup_{k\geq a_n}\left|\frac{\sum_{i=1}^{k}(I(i \notin \mathcal{H}_0) - \pi(\frac{i}{n}))}{k}\right| > \nu_n\right) \leq \left(2 + \frac{4}{\nu_n^2}e^{-\frac{a_n\nu_n^2}{2}}\right) \triangleq \delta_n \to 0.$$

Thus with probability $1 - \delta_n$,

$$\sup_{k\geq a_n}\left|\frac{\#\{i \leq k : i \notin \mathcal{H}_0\}}{k} - \Pi\left(\frac{k}{n}\right)\right| \leq \epsilon_n.$$

□

Proof of Theorem 2. Note that when F_1 is strictly concave,

$$\frac{1-F_1(\lambda)}{1-\lambda} = \frac{F_1(1)-F_1(\lambda)}{1-\lambda} \le \frac{F_1(1)-F_1(0)}{1-0} = 1 \le \frac{F_1(s)-F_1(0)}{s-0} = \frac{F_1(s)}{s}.$$

We will use this result throughout the proof. Select $a_n = \lceil (\log n)^2 \rceil$ and $\epsilon_n = \frac{1}{\sqrt{\log n}}$ and let

$$\delta_n \triangleq \left(2 + \frac{4}{\nu_n^2}\right) \cdot e^{-\frac{a_n \nu_n^2}{2}}.$$

Then $\delta_n \to 0$ and by Lemma 1, with probability $1 - 2\delta_n$,

$$\sup_{k \ge a_n} \left| \frac{\sum_{i=1}^k I(p_i > \lambda)}{k} - \frac{\sum_{i=1}^k \mathbb{E} I(p_i > \lambda)}{k} \right| \le \nu_n,$$

and

$$\sup_{k \ge a_n} \left| \frac{\sum_{i=1}^k I(p_i \le s)}{k} - \frac{\sum_{i=1}^k \mathbb{E} I(p_i \le s)}{k} \right| \le \nu_n.$$

On the other hand, by Proposition 1,

$$\begin{aligned} &\left| \frac{\sum_{i=1}^k \mathbb{E} I(p_i > \lambda)}{k} - \left(\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda)) \right) \right| \\ =& \left| \frac{\#\{i \le k : i \notin \mathcal{H}_0\}}{k} - \Pi\left(\frac{k}{n}\right) \right| \cdot |F_1(\lambda) - \lambda| \le \epsilon_n. \end{aligned}$$

Similarly,

$$\left| \frac{\sum_{i=1}^k \mathbb{E} I(p_i \le s)}{k} - \left(\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s) \right) \right| \le \epsilon_n.$$

These imply that with probability $1 - 2\delta_n$, it holds uniformly for $k \ge a_n$ that

$$\begin{aligned} &\widehat{\mathrm{FDP}}_{AS}(k) \le \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda)) + \epsilon_n + \nu_n + \frac{1}{a_n}}{\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s) - \epsilon_n - \nu_n} \\ \Longrightarrow & \frac{1 + \sum_{i=1}^k I(p_i > \lambda)}{1 \vee \sum_{i=1}^k I(p_i \le s)} - \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda))}{\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)} \\ &\le \left(\epsilon_n + \nu_n + \frac{1}{a_n}\right) \cdot \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda)) + \left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)}{\left(\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s) - \epsilon_n - \nu_n\right)^2} \\ &\le \left(\epsilon_n + \nu_n + \frac{1}{a_n}\right) \cdot \frac{1 - F_1(\lambda) + F_1(s)}{(s - \epsilon_n - \nu_n)^2} \end{aligned}$$

and

$$\begin{aligned} &\widehat{\mathrm{FDP}}_{AS}(k) \ge \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda)) - \epsilon_n - \nu_n + \frac{1}{a_n}}{\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s) + \epsilon_n + \nu_n} \\ \Longrightarrow & \frac{1 + \sum_{i=1}^k I(p_i > \lambda)}{1 \vee \sum_{i=1}^k I(p_i \le s)} - \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda))}{\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)} \\ &\ge -(\epsilon_n + \nu_n) \cdot \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda)) + \left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)}{\left(\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)\right)^2} \\ &\ge -(\epsilon_n + \nu_n) \cdot \frac{1 - F_1(\lambda) + F_1(s)}{s^2} \end{aligned}$$

Since $\epsilon_n, \nu_n \to 0$, we have

$$\lim_{n \to \infty} \sup_{k \ge a_n} \left| \widehat{\mathrm{FDP}}_{AS}(k) - \frac{\left[1 - \Pi\left(\frac{k}{n}\right)\right](1-\lambda) + \Pi\left(\frac{k}{n}\right)(1 - F_1(\lambda))}{\left[1 - \Pi\left(\frac{k}{n}\right)\right] s + \Pi\left(\frac{k}{n}\right) F_1(s)} \right| = 0 \quad a.s. \qquad (3)$$

Recall that

$$\mathrm{FDR}_{AS}^*(t)=\frac{1-\Pi(t)+\Pi(t)\frac{1-F_1(\lambda)}{1-\lambda}}{1-\Pi(t)+\Pi(t)\frac{F_1(s)}{s}},$$

then

$$\frac{d\mathrm{FDR}_{AS}^*(t)}{d\Pi(t)}=\frac{\frac{F_1(s)}{s}-\frac{1-F_1(\lambda)}{1-\lambda}}{\left(\frac{F_1(s)}{s}-1\right)\left(1+(\frac{F_1(s)}{s}-1)\Pi(t)\right)}\le\frac{\frac{F_1(s)}{s}-\frac{1-F_1(\lambda)}{1-\lambda}}{\frac{F_1(s)}{s}-1}\triangleq L$$

and hence $\mathrm{FDR}_{AS}^*(t)$ is LL_ΠLipschitz where L_Π is the Lipschitz constant of Π. This entails that

$$\sup_{k\le n}\sup_{|t-\frac{k}{n}|<\frac{1}{n}}\left|\mathrm{FDR}_{AS}^*(t)-\mathrm{FDR}_{AS}^*\left(\frac{k}{n}\right)\right|\le\frac{LL_\Pi}{n}. \tag{4}$$

(4) together with (3) implies that

$$\lim_{n\to\infty}\sup_{t\ge a_n/n}|\widehat{\mathrm{FDP}}_{AS}(\lfloor nt\rfloor)-\mathrm{FDR}_{AS}^*(t)|=0\quad a.s. \tag{5}$$

Since $a_n/n\to 0$, for any $c>0$,

$$\lim_{n\to\infty}\sup_{t\ge c}|\widehat{\mathrm{FDP}}_{AS}(\lfloor nt\rfloor)-\mathrm{FDR}_{AS}^*(t)|=0\quad a.s. \tag{6}$$

If $\mathrm{FDR}_{AS}^*(0)\ge q$, then for any $c>0$ and $x\ge c$, $\mathrm{FDR}_{AS}^*(t)\ge\mathrm{FDR}_{AS}^*(c)>\mathrm{FDR}_{AS}^*(0)\ge q$. (6) implies that

$$\liminf_{n\to\infty}\inf_{t\ge c}\widehat{\mathrm{FDP}}_{AS}(\lfloor nt\rfloor)\ge\mathrm{FDR}_{AS}^*(c)>q\quad a.s.$$

By definition,

$$\widehat{\mathrm{FDP}}_{AS}\left(\hat{k}_{AS}\right)\le q$$

and hence $\hat{k}/n\le c$ almost surely. This holds for arbitrary $c>0$, therefore, $\hat{k}/n\overset{a.s.}{\to}0=t_{AS}^*$. In this case,

$$\mathrm{Pow}_{AS}=\frac{\#\{i\le\hat{k}:i\notin\mathcal{H}_0,p_i\le s\}}{\#\{i\le n:i\notin\mathcal{H}_0\}}\le\frac{\hat{k}}{n}\cdot\frac{n}{\#\{i\le n:i\notin\mathcal{H}_0\}}\overset{a.s.}{\to}0$$

since $\hat{k}/n\ ^{a.s.}\ b\to 0$ and

$$\frac{n}{\#\{i\le n:i\notin\mathcal{H}_0\}}\overset{a.s.}{\to}\frac{1}{\Pi(1)}<\infty.$$

If $\mathrm{FDR}_{AS}^*(1)\le q$, similar to the above argument, we have

$$\limsup_{n\to\infty}\widehat{\mathrm{FDP}}_{AS}(\lfloor n(1-c)\rfloor)\le\mathrm{FDR}_{AS}^*(1-c)<\mathrm{FDR}_{AS}^*(1)\le q$$

for arbitrary $c>0$. This implies that $\hat{k}/n\ge 1-c$ almost surely. Thus, $\hat{k}/n\overset{a.s.}{\to}1=t_{AS}^*$. In this case,

$$\begin{aligned}\mathrm{Pow}_{AS}&=\frac{\#\{i\le\hat{k}:i\notin\mathcal{H}_0,p_i\le s\}}{\#\{i\le n:i\notin\mathcal{H}_0\}}\\&=\frac{\#\{i\le\hat{k}:i\notin\mathcal{H}_0,p_i\le s\}}{\#\{i\le\hat{k}:i\notin\mathcal{H}_0\}}\cdot\frac{\#\{i\le\hat{k}:i\notin\mathcal{H}_0\}}{\hat{k}}\cdot\frac{\hat{k}}{n}\cdot\frac{n}{\#\{i\le n:i\notin\mathcal{H}_0\}}\end{aligned} \tag{7}$$

Since $\hat{k}/n\overset{a.s.}{\to}1>0$, $\hat{k}\ge a_n$ almost surely and hence

$$\Pi\left(\frac{\hat{k}}{n}\right)-\epsilon_n\ge\frac{\#\{i\le\hat{k}:i\notin\mathcal{H}_0\}}{\hat{k}}\ge\Pi\left(\frac{\hat{k}}{n}\right)-\epsilon_n.$$

This implies that

$$\frac{\#\{i \le \hat{k} : i \notin \mathcal{H}_0\}}{\hat{k}} \to \Pi(1) \quad a.s.$$

and as a byproduct, we know $\#\{i \le \hat{k} : i \notin \mathcal{H}_0\} \overset{a.s.}{\to} \infty$. By Law of Large Number,

$$\frac{\#\{i \le \hat{k} : i \notin \mathcal{H}_0, p_i \le s\}}{\#\{i \le \hat{k} : i \notin \mathcal{H}_0\}} \overset{a.s.}{\to} F_1(s).$$

Therefore,

$$\mathrm{Pow}_{AS} \to \Pi(1) \cdot F_1(s) \cdot 1 \cdot \frac{1}{\Pi(1)} = F_1(s) \quad a.s.$$

If $\mathrm{FDR}^*_{AS}(0) < q < \mathrm{FDR}^*_{AS}(1)$, then $t^*_{AS} = \mathrm{FDR}^{*-1}_{AS}(q)$. For any $c > 0$, (6) implies that

$$\limsup_{n\to\infty} \widehat{\mathrm{FDP}}_{AS}(\lfloor n(t^*_{AS} - c)\rfloor) \le \mathrm{FDR}^*_{AS}(t^*_{AS} - c) < \mathrm{FDR}^*_{AS}(t^*_{AS}) = q,$$

and

$$\liminf_{n\to\infty} \sup_{t \ge t^*_{AS}+c} \widehat{\mathrm{FDP}}_{AS}(\lfloor nt\rfloor) \ge \mathrm{FDR}^*_{AS}(t^*_{AS} + c) > \mathrm{FDR}^*_{AS}(t^*_{AS}) = q.$$

Thus,

$$t^*_{AS} - c \le \frac{\hat{k}}{n} \le t^*_{AS} + c \quad a.s.$$

Since c is arbitrary, we have $\hat{k}/n \overset{a.s.}{\to} t^*_{AS}$. In this case, notice that $\liminf_{n\to\infty} \hat{k}/n > 0$, we can apply the same argument as above and it follows from (7) that

$$\mathrm{Pow}_{AS} \to \Pi(t^*_{AS}) \cdot F_1(s) \cdot t^*_{AS} \cdot \frac{1}{\Pi(1)} = \frac{t^*_{AS}\Pi(t^*_{AS})F_1(s)}{\Pi(1)}.$$

□

References

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