Supplementary Material for "Non-Count Symmetries in Boolean & Multi-Valued Prob. Graphical Models"

1. **Proof of Theorem 5.1:**

   Let the transition probability from s_t to s_{t+1} in NEC-Orbital MCMC be given by $T^{NEC}(s_t \rightarrow s_{t+1})$. And, let $T_2(s'_t \rightarrow s_{t+1})$ denote the transition probability because of the last 2 steps given in Section-5. Then, we can write T^{NEC} as:

$$T^{NEC}(s_t \rightarrow s_{t+1}) = \sum_{s'_t \in \Gamma_\Phi(s_{t+1})} T^M(s_t \rightarrow s'_t) \cdot T_2(s'_t \rightarrow s_{t+1})$$

   where $\Gamma_\Phi(s_{t+1})$ denotes the orbit of s_{t+1} with respect to the original domain

   Obeying the notation in Section 5, $u'_t = rep(s'_t)$ and $u''_t = rep(s_{t+1})$. Also, since the last step (given in Section 5) is equivalent to sampling s_{t+1} uniformly at random from the sub-orbit u''_t, and, if we represent T^{MH} as transition probability for the second step, we get:

$$\begin{aligned} T_2(s'_t \rightarrow s_{t+1}) &= T^{MH}(u'_t \rightarrow u''_t) \cdot P(s_{t+1}|u''_t) \\ &= T^{MH}(u'_t \rightarrow u''_t) \cdot \frac{1}{c(u''_t)} \end{aligned}$$

   Therefore, we get:

$$T_2(s'_t \rightarrow s_{t+1}) = \begin{cases} \left(\frac{1}{|\Gamma_{\Phi^R}(u'_t)|} \cdot A(u'_t \rightarrow u''_t)\right) \cdot \frac{1}{c(u''_t)}, & \text{if } u'_t \neq u''_t \\ \\ \left(\frac{1}{|\Gamma_{\Phi^R}(u'_t)|} + \sum_{z \in \Gamma_{\Phi^R}(u'_t)} \frac{1}{|\Gamma_{\Phi^R}(u'_t)|} \cdot (1 - A(u'_t \rightarrow z))\right) \cdot \frac{1}{c(u''_t)}, & \text{if } u'_t = u''_t \end{cases}$$

   Lemma: $\forall s_1 \forall s_2$ such that $u_1 = rep(s_1), u_2 = rep(s_2)$ and $\Gamma_{\Phi^R}(u_1) = \Gamma_{\Phi^R}(u_2)$, we have:

$$T_2(s_1 \rightarrow s_2) = T_2(s_2 \rightarrow s_1)$$

   This lemma basically shows that T_2 obeys detailed balance with respect to the uniform distribution over states in the (original) orbit of s_1 and s_2.

   Proof of Lemma:
   Case 1: $u_1 \neq u_2$

$$\begin{aligned} T_2(s_1 \rightarrow s_2) &= \left(\frac{1}{|\Gamma_{\Phi^R}(u_1)|} \cdot A(u_1 \rightarrow u_2)\right) \cdot \frac{1}{c(u_2)} \\ &= \frac{1}{|\Gamma_{\Phi^R}(u_1)|} \cdot min\left(1, \frac{c(u_2)}{c(u_1)}\right) \cdot \frac{1}{c(u_2)} \quad = \frac{1}{|\Gamma_{\Phi^R}(u_1)|} \cdot min\left(\frac{1}{c(u_2)}, \frac{1}{c(u_1)}\right) \\ &= \frac{1}{|\Gamma_{\Phi^R}(u_1)|} \cdot min\left(\frac{c(u_1)}{c(u_2)}, 1\right) \cdot \frac{1}{c(u_1)} \\ &= \left(\frac{1}{|\Gamma_{\Phi^R}(u_1)|} \cdot A(u_2 \rightarrow u_1)\right) \cdot \frac{1}{c(u_1)} \\ &= T_2(s_2 \rightarrow s_1) \end{aligned}$$

   Case 2: $u_1 = u_2$
   It trivially holds for this case because the expression of T_2 doesn't depend on the states s_1

and s_2 but just their sub-orbit (which is the same in this case).
Now, let π denote the unique stationary distribution of the Markov Chain $\mathcal{M}$. We show that T^{NEC} also has π as its stationary distribution, i.e. for any state $s \in \mathcal{S}$, we show that:

$$\pi(s) = \sum_{s_0 \in S} \pi(s_0) \cdot T^{NEC}(s_0 \to s)$$

Proof:

$$\begin{aligned}
RHS &= \sum_{s_0 \in \mathcal{S}} \pi(s_0) \cdot T^{NEC}(s_0 \to s) \\
&= \sum_{s_0 \in \mathcal{S}} \pi(s_0) \cdot \left[\sum_{s_0' \in \Gamma_\Phi(s)} T^M(s_0 \to s_0') \cdot T_2(s_0' \to s) \right] \\
&= \sum_{s_0 \in \mathcal{S}} \sum_{s_0' \in \Gamma_\Phi(s)} \pi(s_0) \cdot T^M(s_0 \to s_0') \cdot T_2(s_0' \to s) \\
&= \sum_{s_0 \in \mathcal{S}} \sum_{s_0' \in \Gamma_\Phi(s)} \pi(s_0') \cdot T^M(s_0' \to s_0) \cdot T_2(s_0' \to s); \text{ because of detailed balance of } T^M \\
&= \sum_{s_0 \in \mathcal{S}} \sum_{s_0' \in \Gamma_\Phi(s)} \pi(s) \cdot T^M(s_0' \to s_0) \cdot T_2(s_0' \to s); \text{ because } s_0' \text{ and } s \text{ are in same orbit} \\
&= \pi(s) \cdot \sum_{s_0 \in \mathcal{S}} \sum_{s_0' \in \Gamma_\Phi(s)} T^M(s_0' \to s_0) \cdot T_2(s_0' \to s) \\
&= \pi(s) \cdot \sum_{s_0' \in \Gamma_\Phi(s)} \left[\sum_{s_0 \in \mathcal{S}} T^M(s_0' \to s_0) \right] \cdot T_2(s_0' \to s) \\
&= \pi(s) \cdot \sum_{s_0' \in \Gamma_\Phi(s)} T_2(s_0' \to s) \\
&= \pi(s) \cdot \sum_{s_0' \in \Gamma_\Phi(s)} T_2(s \to s_0'); \text{ because of the lemma} \\
&= \pi(s) \cdot 1 \\
&= LHS
\end{aligned}$$

Hence, proved.