A. Adaptive Tuning of the Exploration Rate

In Theorem 2 we have presented a tuning of γ that guarantees a regret of the order of $\tilde{O}(\frac{1}{\eta}\sqrt{T})$. However, this setting requires to upper bound the sum of the quadratic terms with a worst case bound. In this section, we develop an adaptive strategy for the tuning of the exploration rate γ that guarantees an optimal bound w.r.t. to the tightest sum of the quadratic terms.

First, we make rate dependent of the time, i.e. γ_t. Our aim is to choose γ_t in each time step in order to minimize the excess mistake bound $\mathbb{E}\left[\sum_{t=1}^T \gamma_t + \frac{1}{\eta(2-\eta)} \sum_{t=1}^T \frac{k}{\gamma_t} \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t\right]$. The main result is that, adaptively setting γ_t's would result in a bound within (roughly) a constant factor of that obtained by the best fixed γ in hindsight. We start with a technical lemma.

Lemma 4. *Let $c_1, \dots, c_T \in [0, b]$ be a sequence of real numbers, $a > 0$, and define $\gamma_t = \min\left(\sqrt{\frac{b+\sum_{s=1}^{t-1} c_s}{t}}, 1\right)$. We have,*

$$\sum_{t=1}^T \left(\gamma_t + a\frac{c_t}{\gamma_t}\right) \le (2+2a)\sqrt{T}\sqrt{b + \sum_{t=1}^T c_t} + a\sum_{t=1}^T c_t \ .$$

Proof. First, note that

$$\sum_{t=1}^T \gamma_t \le \sum_{t=1}^T \sqrt{\frac{b+\sum_{s=1}^{t-1} c_s}{t}} \le \sqrt{b + \sum_{s=1}^T c_s} \sum_{t=1}^T \sqrt{\frac{1}{t}} \le 2\sqrt{T}\sqrt{b+\sum_{s=1}^T c_s} \ .$$

Second, using the elementary chain of inequalities $\max(a, b) \le a + b, \forall a, b \ge 0$, we have that

$$\begin{aligned}
\sum_{t=1}^T \frac{c_t}{\gamma_t} &= \sum_{t=1}^T \max\left(\frac{c_t\sqrt{t}}{\sqrt{b+\sum_{s=1}^{t-1} c_s}}, c_t\right) \\
&\le \sum_{t=1}^T \sqrt{T}\frac{c_t}{\sqrt{b+\sum_{s=1}^{t-1} c_s}} + \sum_{t=1}^T c_t \\
&\le \sqrt{T}\sum_{t=1}^T \frac{c_t}{\sqrt{\sum_{s=1}^{t} c_s}} + \sum_{t=1}^T c_t \\
&\le 2\sqrt{T}\sqrt{b+\sum_{s=1}^T c_s} + \sum_{t=1}^T c_t,
\end{aligned}$$

where the last inequality uses Lemma 3.5 of (Auer et al., 2002). Combining the two inequalities, we get the desired result. □

Built upon the lemma above, we show that, tailored to our setting, the adaptive tuning would result in a bound within a constant factor of that achieved by the best fixed γ in hindsight.

Theorem 5. *Running SOBA with the adaptive setting of $\gamma_t = \min\left(\sqrt{\frac{k(1+\sum_{s=1}^{t-1} \boldsymbol{z}_s^T \boldsymbol{A}_s^{-1} \boldsymbol{z}_s)}{t}}, 1\right)$ and $a = X^2$, we have that*

$$\mathbb{E}[M] \le L_\eta(U) + O\left(X^2\|U\|_F^2 + \frac{1}{\eta}(\sqrt{dk^2 T \ln T} + dk^2 \ln T)\right) \ .$$

Proof Sketch. Following the same proof as Theorem 3, we get that

$$\mathbb{E}\left[\hat{M}_T\right] \le L_\eta(\boldsymbol{U}) + \frac{a\eta\|\boldsymbol{U}\|_F^2}{2-\eta} + \frac{1}{\eta(2-\eta)}\mathbb{E}[\sum_{t=1}^T \frac{k}{\gamma_t} \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t]$$

Meanwhile by triangle inequality,

$$\mathbb{E}[M_T] \le \mathbb{E}[\hat{M}_T] + \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}[\tilde{y}_t \neq \hat{y}_t]\right] \le \mathbb{E}[\hat{M}_T] + \mathbb{E}\left[\sum_{t=1}^{T} \gamma_t\right] .$$

Combining the two inequalities above, we get

$$\mathbb{E}\left[M_T\right] \le L_\eta(\boldsymbol{U}) + \frac{a\eta\|\boldsymbol{U}\|_F^2}{2-\eta} + \mathbb{E}\left[\frac{1}{\eta(2-\eta)} \sum_{t=1}^{T} \frac{k\, \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t}{\gamma_t} + \sum_{t=1}^{T} \gamma_t\right].$$

We take a closer look at the last term. Lemma 4 with $c_t = k\boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t \in [0, k]$, $b = k$, $a = \frac{1}{\eta(2-\eta)}$, implies that

$$\begin{aligned}
&\sum_{t=1}^{T} \gamma_t + \sum_{t=1}^{T} \frac{k}{\eta(2-\eta)\gamma_t} \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t \\
&\quad \le \left(2 + \frac{2}{\eta(2-\eta)}\right) \sqrt{T} \sqrt{k(1 + \sum_{t=1}^{T} \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t)} + \frac{1}{\eta(2-\eta)}\, k(1 + \sum_{t=1}^{T} \boldsymbol{z}_t^T \boldsymbol{A}_t^{-1} \boldsymbol{z}_t) .
\end{aligned}$$

Taking the expecation of both sides and using Lemma 3, we get that the last term on the right hand side is at most $\frac{12}{\eta}(\sqrt{dk^2 T \ln T} + dk^2 \ln T)$. This completes the proof. □

B. Deferred Proofs

Proof of Theorem 1. Let $p \ge 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote by b_t the indicator variable that multiclass Perceptron makes an update, i.e. makes a mistake. We have:

$$\begin{aligned}
&\langle \boldsymbol{W}_{T+1}, \boldsymbol{U} \rangle \\
&\quad \le \|\boldsymbol{W}_{T+1}\|_F \, \|\boldsymbol{U}\|_F \\
&\quad = \|\boldsymbol{U}\|_F \sqrt{\|\boldsymbol{W}_T\|^2 + 2b_t \langle \boldsymbol{W}_T, (\boldsymbol{e}_{y_T} - \boldsymbol{e}_{\hat{y}_T}) \otimes \boldsymbol{x}_T \rangle + 2b_t^2 \|\boldsymbol{x}_T\|_2^2} \\
&\quad \le \|\boldsymbol{U}\|_F \sqrt{\|\boldsymbol{W}_T\|_F^2 + 2b_t^2 \|\boldsymbol{x}_T\|_2^2} \\
&\quad \le \cdots \\
&\quad \le \|\boldsymbol{U}\|_F \sqrt{2 \sum_{t=1}^{T} b_t^2 \|\boldsymbol{x}_t\|_2^2} \\
&\quad \le \|\boldsymbol{U}\|_F \, X\sqrt{2} \sqrt{\sum_{t=1}^{T} b_t^2} \\
&\quad = \|\boldsymbol{U}\|_F \, X\sqrt{2} \sqrt{\sum_{t=1}^{T} b_t}
\end{aligned}$$

Also, we have, that

$$\begin{aligned}\langle \boldsymbol{W}_{T+1}, \boldsymbol{U}\rangle &= \sum_{t=1}^{T} b_t \langle \boldsymbol{U}, (\boldsymbol{e}_{y_t} - \boldsymbol{e}_{\hat{y}_t}) \otimes \boldsymbol{x}_t\rangle \\ &= \sum_{t=1}^{T} b_t [1 - (1 - \langle \boldsymbol{U}, (\boldsymbol{e}_{y_t} - \boldsymbol{e}_{\hat{y}_t}) \otimes \boldsymbol{x}_t\rangle)] \\ &\geq \sum_{t=1}^{T} b_t [1 - |1 - \langle \boldsymbol{U}, (\boldsymbol{e}_{y_t} - \boldsymbol{e}_{\hat{y}_t}) \otimes \boldsymbol{x}_t\rangle|_+] \\ &\geq \sum_{t=1}^{T} b_t - \sum_{t=1}^{T} b_t \ell(\boldsymbol{U}, (\boldsymbol{x}_t, y_t)) \\ &\geq \sum_{t=1}^{T} b_t - (\sum_{t=1}^{T} b_t^p)^{\frac{1}{p}} (\sum_{t=1}^{T} \ell(\boldsymbol{U}, (\boldsymbol{x}_t, y_t))^q)^{\frac{1}{q}} \\ &= \sum_{t=1}^{T} b_t - (\sum_{t=1}^{T} b_t)^{\frac{1}{p}} (\sum_{t=1}^{T} \ell(\boldsymbol{U}, (\boldsymbol{x}_t, y_t))^q)^{\frac{1}{q}} \ .\end{aligned}$$

Putting all together we have

$$\|\boldsymbol{U}\|_F \, X\sqrt{2}\sqrt{\sum_{t=1}^{T} b_t} \geq \sum_{t=1}^{T} b_t - \left(\sum_{t=1}^{T} b_t\right)^{\frac{1}{p}} L_{\mathrm{MH},q}(\boldsymbol{U})^{\frac{1}{q}} \ .$$

Noting that $\sum_{t=1}^{T} b_t$ is equal to number of mistake M_T, we get the stated bound. □

Lemma 5. *Suppose we are given positive real numbers L, T, H, U and function $F(\gamma) = \min(T, L + \gamma T + \frac{UH}{\gamma} + \sqrt{\frac{UHL}{\gamma}})$, where $\gamma \in [0, 1]$. Then:*

1. *If $L \leq (U+1)\sqrt{HT}$, then taking $\gamma^* = \min(\sqrt{\frac{H}{T}}, 1)$ gives that $F(\gamma^*) \leq L + 3(U+1)\sqrt{HT}$.*
2. *If $L > (U+1)\sqrt{HT}$, then taking $\gamma^* = \min((\frac{HL}{T^2})^{\frac{1}{3}}, 1)$ gives that $F(\gamma^*) \leq L + 2(\sqrt{U}+1)(HLT)^{\frac{1}{3}}$.*

Proof. We prove the two cases separately.

1. If $T \leq H$, then $\gamma^* = 1$, $F(\gamma^*) \leq T \leq L + 3(U+1)\sqrt{HT}$.

   Otherwise, $T > H$. In this case, $\gamma^* = \sqrt{\frac{H}{T}}$. We have that

$$\begin{aligned} & F(\gamma^*) \\ = \ & L + \gamma^* T + \frac{UH}{\gamma^*} + \sqrt{\frac{UHL}{\gamma^*}} \\ = \ & L + \sqrt{HT} + U\sqrt{HT} + \sqrt{UL\sqrt{HT}} \\ \leq \ & L + (U+1)\sqrt{HT} + L + U\sqrt{HT} \\ \leq \ & L + 3(U+1)\sqrt{HT}. \end{aligned}$$

   where the first inequality is from that arithmetic mean-geometric mean inequality, the second inequality is by the assumption on L.

2. If $HL > T^2$, then $\gamma^* = 1$, $F(\gamma^*) \leq T \leq (HLT)^{\frac{1}{3}}$.

   Otherwise, $HL \leq T^2$. In this case, $\gamma^* = (\frac{HL}{T^2})^{\frac{1}{3}}$. We have that

$$
\begin{aligned}
F(\gamma^*) &= L + \gamma^* T + \frac{UH}{\gamma^*} + \sqrt{\frac{UHL^*}{\gamma}} \\
&= L + (HLT)^{\frac{1}{3}} + UH^{\frac{2}{3}}T^{\frac{2}{3}}L^{-\frac{1}{3}} + \sqrt{U}(HLT)^{\frac{1}{3}} \\
&\leq L + (\sqrt{U} + U^{\frac{1}{3}} + 1)(HLT)^{\frac{1}{3}} \\
&\leq L + 2(\sqrt{U} + 1)(HLT)^{\frac{1}{3}}.
\end{aligned}
$$

   where the first inequality is from algebra and the condition on L, implying $UH^{\frac{2}{3}}T^{\frac{2}{3}}L^{-\frac{1}{3}} \leq (HLT)^{\frac{1}{3}}U(\frac{HT}{L^2})^{\frac{1}{3}} \leq U^{\frac{1}{3}}(HLT)^{\frac{1}{3}}$, the second inequality is from that $U^{\frac{1}{3}} \leq \sqrt{U} + 1$.

□

C. Per-Step Analysis of Online Least Squares

For completeness, we present a technical lemma in online least squares, which has appeared in (e.g., Orabona et al., 2012).

Lemma 6. *Suppose z_t's are vectors, and α_t's are scalars. For all $t \geq 1$, define $\boldsymbol{A}_t = \sum_{s=1}^{t} z_s z_s^T$, $w_t = -\boldsymbol{A}_{t-1}^{-1}\sum_{s=1}^{t-1}\alpha_s z_s$. Then for any vector u, we have:*

$$
\frac{1}{2}(\langle w_t\,,\,z_t\rangle + \alpha_t)^2(1 - z_t^T\boldsymbol{A}_t^{-1}z_t) - \frac{1}{2}(\langle u\,,\,z_t\rangle + \alpha_t)^2 \leq \frac{1}{2}\|u - w_t\|^2_{\boldsymbol{A}_{t-1}} - \frac{1}{2}\|u - w_{t+1}\|^2_{\boldsymbol{A}_t}\,.
$$

Proof. Observe that w_t's have the following recurrence:

$$
w_{t+1} = \boldsymbol{A}_t^{-1}(\boldsymbol{A}_{t-1}w_t - \alpha_t z_t)
$$

Since $\boldsymbol{A}_t = \boldsymbol{A}_{t-1} + z_t z_t^T$, we have

$$
\boldsymbol{A}_t w_{t+1} = \boldsymbol{A}_t w_t - (w_t^T z_t + \alpha_t) z_t
$$

Now, by standard online mirror descent analysis (See e.g. Cesa-Bianchi & Lugosi, 2006, proof of Theorem 11.1), we have

$$
\begin{aligned}
\left\langle w_t - u\,,\,(w_t^T z_t + \alpha_t)z_t\right\rangle &\leq \frac{1}{2}\|u - w_t\|^2_{\boldsymbol{A}_t} - \frac{1}{2}\|u - w_{t+1}\|^2_{\boldsymbol{A}_t} + \frac{1}{2}(w_t^T z_t + \alpha_t)^2 z_t^T\boldsymbol{A}_t^{-1}z_t \\
&\leq \frac{1}{2}\|u - w_t\|^2_{\boldsymbol{A}_{t-1}} - \frac{1}{2}\|u - w_{t+1}\|^2_{\boldsymbol{A}_t} + \frac{1}{2}(w_t^T z_t + \alpha_t)^2 z_t^T\boldsymbol{A}_t^{-1}z_t + \frac{1}{2}(u^T z_t - w_t^T z_t)^2
\end{aligned}
$$

Now, move the last term on the RHS to the LHS, we get

$$
(w_t^T z_t - u^T z_t)\cdot\frac{1}{2}(w_t^T z_t + u^T z_t + 2\alpha_t) \leq \frac{1}{2}\|u - w_t\|^2_{\boldsymbol{A}_{t-1}} - \frac{1}{2}\|u - w_{t+1}\|^2_{\boldsymbol{A}_t} + \frac{1}{2}(w_t^T z_t + \alpha_t)^2 z_t^T\boldsymbol{A}_t^{-1}z_t
$$

i.e.

$$
\frac{1}{2}(\langle w_t\,,\,z_t\rangle + \alpha_t)^2 - \frac{1}{2}(\langle u\,,\,z_t\rangle + \alpha_t)^2 \leq \frac{1}{2}\|u - w_t\|^2_{\boldsymbol{A}_{t-1}} - \frac{1}{2}\|u - w_{t+1}\|^2_{\boldsymbol{A}_t} + \frac{1}{2}(w_t^T z_t + \alpha_t)^2 z_t^T\boldsymbol{A}_t^{-1}z_t\,.
$$

Now moving the last term on the RHS to the LHS, the lemma follows. □