Appendix material

A. Recursive forward computation of the average path matrix

The average alignment under Gibbs distribution p_γ can be computed with the following forward recurrence, which mimics closely Bellman's original recursion. For each $i \in [\![n]\!], j \in [\![m]\!]$, define

$$E_{i+1,j+1} = \begin{bmatrix} e^{-\delta_{i+1,j+1}/\gamma} E_{i,j} & \mathbf{0}_i \\ \mathbf{0}_j^T & e^{-r_{i+1,j+1}/\gamma} \end{bmatrix} + \begin{bmatrix} e^{-\delta_{i+1,j+1}/\gamma} E_{i,j+1} \\ \mathbf{0}_j^T & e^{-r_{i+1,j+1}/\gamma} \end{bmatrix} + \begin{bmatrix} e^{-\delta_{i+1,j+1}/\gamma} E_{i+1,j} & \mathbf{0}_i \\ & e^{-r_{ij}/\gamma} \end{bmatrix}$$

Here terms r_{ij} are computed following the recursion in Algorithm 2. Border matrices are initialized to 0, except for $E_{1,1}$ which is initialized to $[1]$. Upon completion, the average alignment matrix is stored in $E_{n,m}$.

The operation above consists in summing three matrices of size $(i+1, j+1)$. There are exactly (nm) such updates. A careful implementation of this algorithm, that would only store two arrays of matrices, as Algorithm 1 only store two arrays of values, can be carried out in $nm \min(n, m)$ space but it would still require $(nm)^2$ operations.
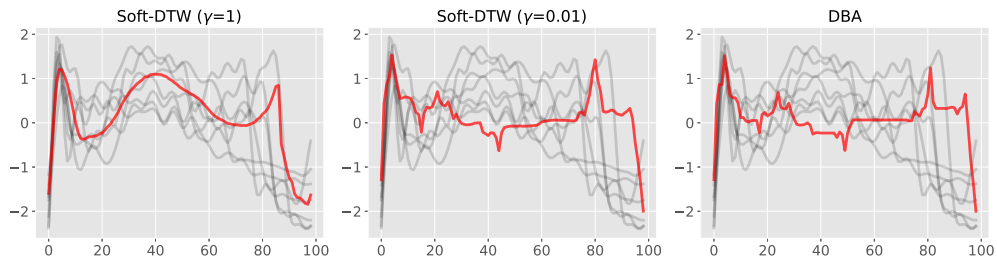
B. Barycenters obtained with random initialization
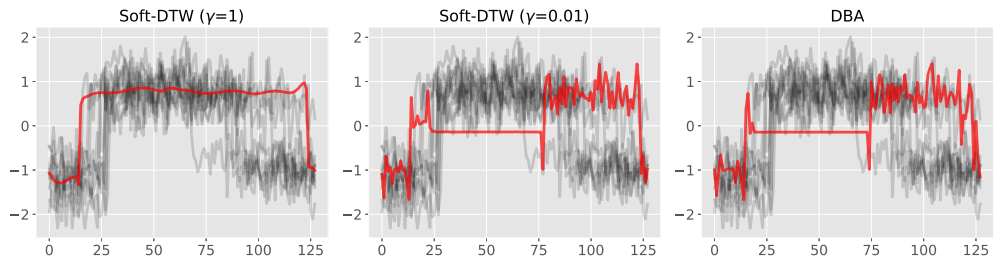
(a) CBF
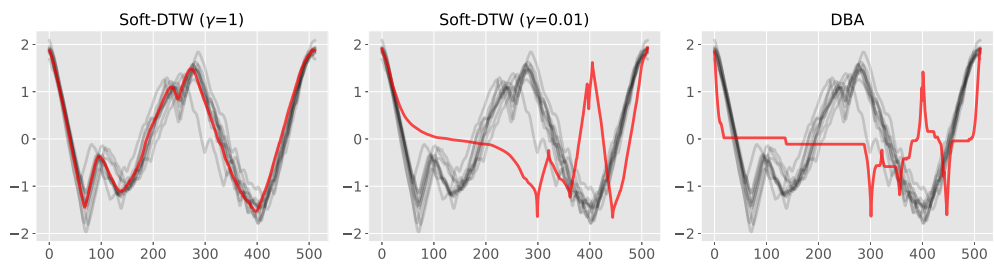
(b) Herring

(c) Medical Images

(d) Synthetic Control

(e) Wave Gesture Library Y

C. Barycenters obtained with Euclidean mean initialization
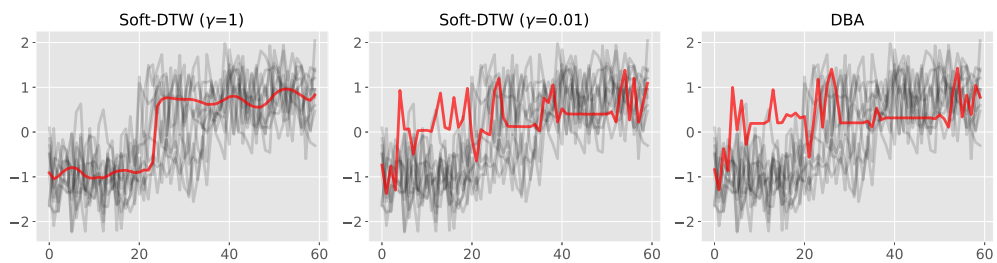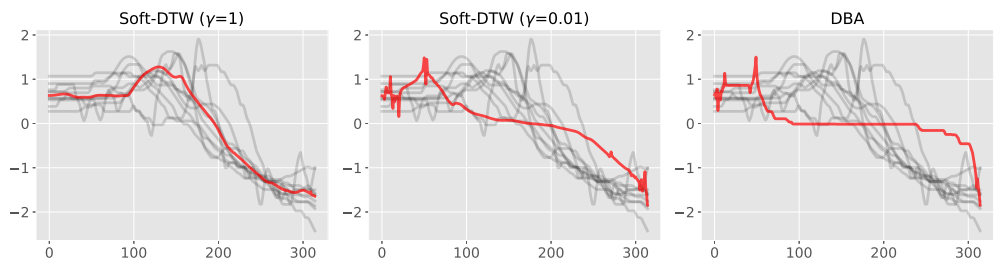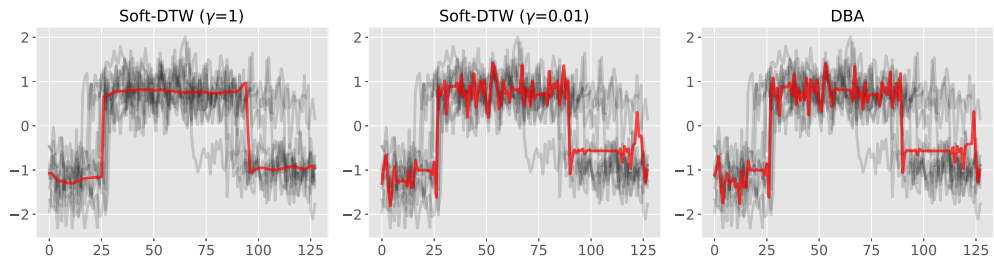

(a) CBF
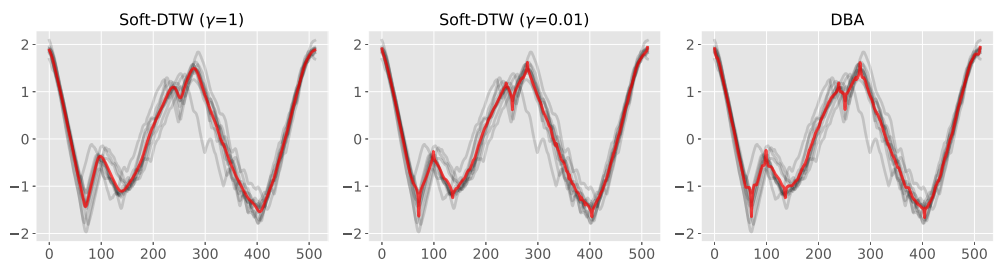
(b) Herring
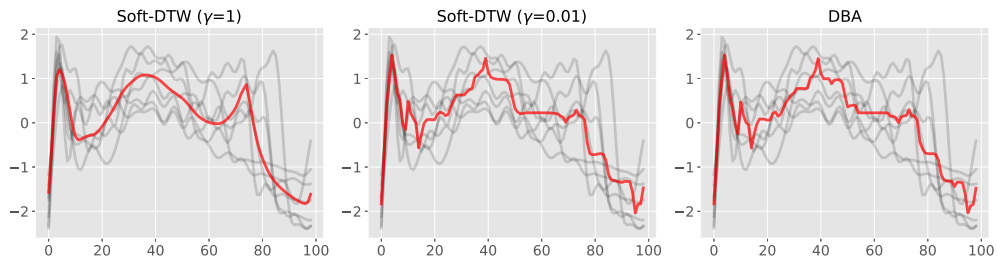
(c) Medical Images
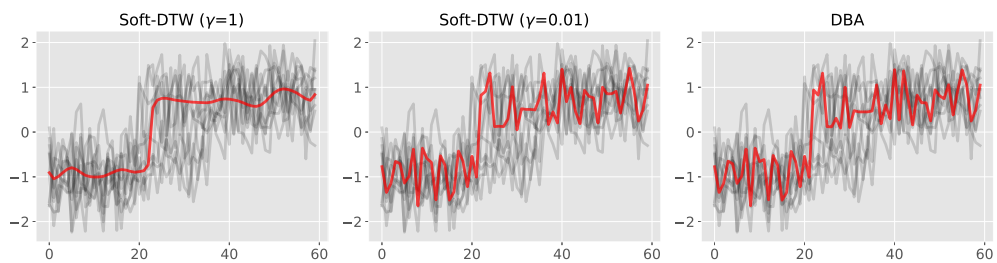
(d) Synthetic Control
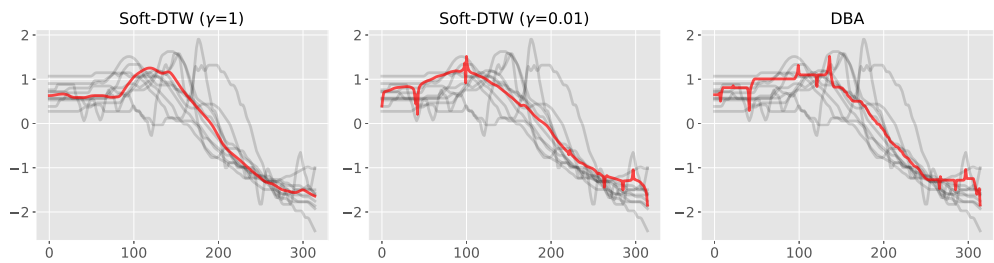
(e) Wave Gesture Library Y

D. More interpolation results

Left: results obtained under Euclidean loss. Right: results obtained under soft-DTW ($\gamma = 1$) loss.
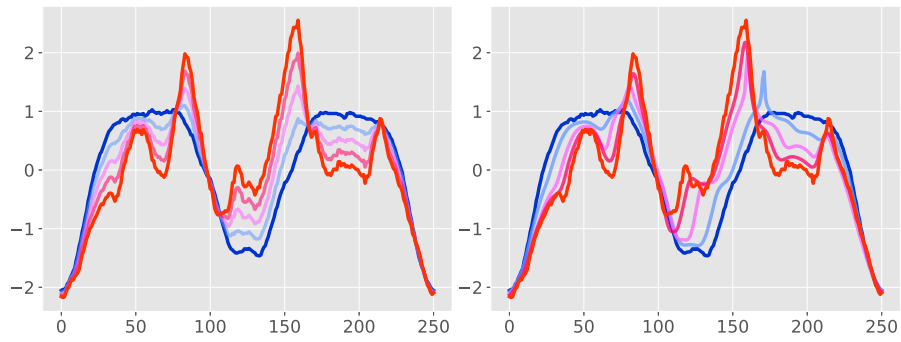
(a) ArrowHead
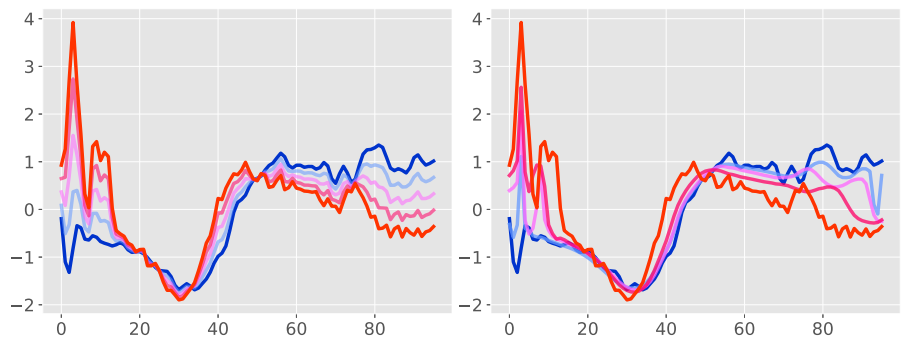
(b) ECG200
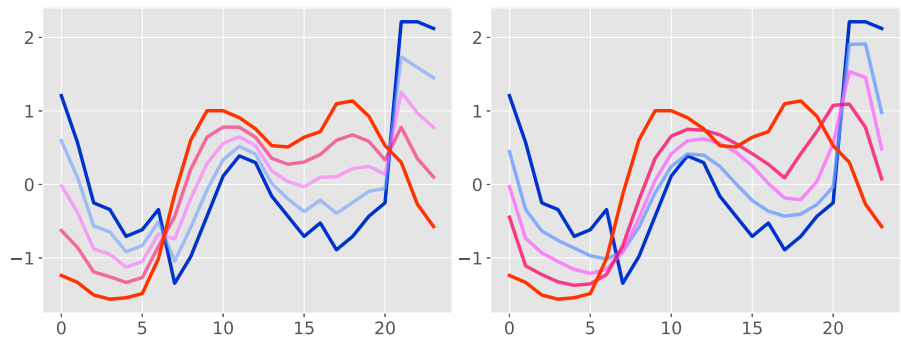
(c) ItalyPowerDemand
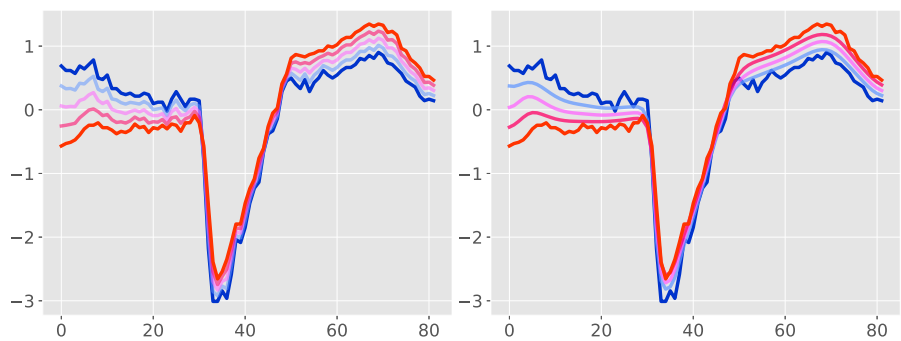
(d) TwoLeadECG

E. Clusters obtained by k-means under DTW or soft-DTW geometry

CBF dataset
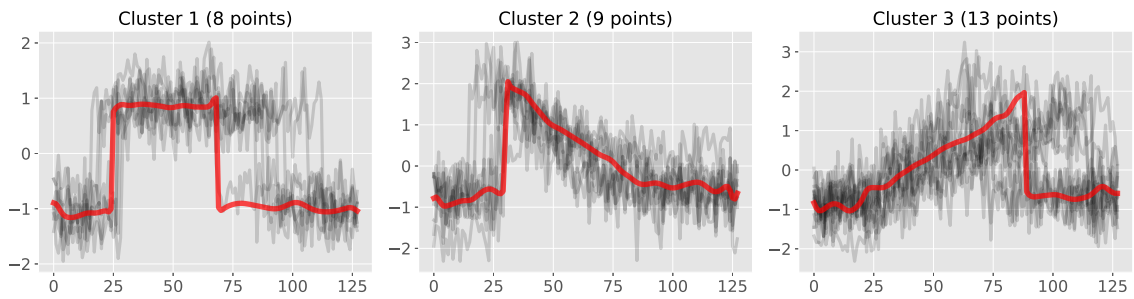
(a) Soft-DTW ($\gamma = 1$, random initialization)
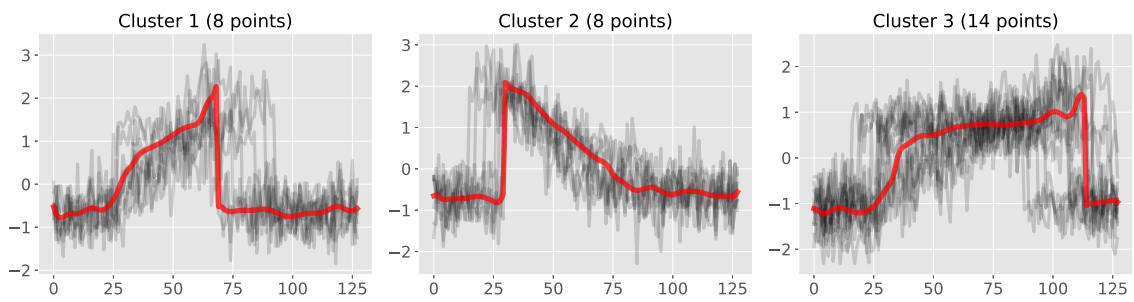
(b) Soft-DTW ($\gamma = 1$, Euclidean mean initialization)
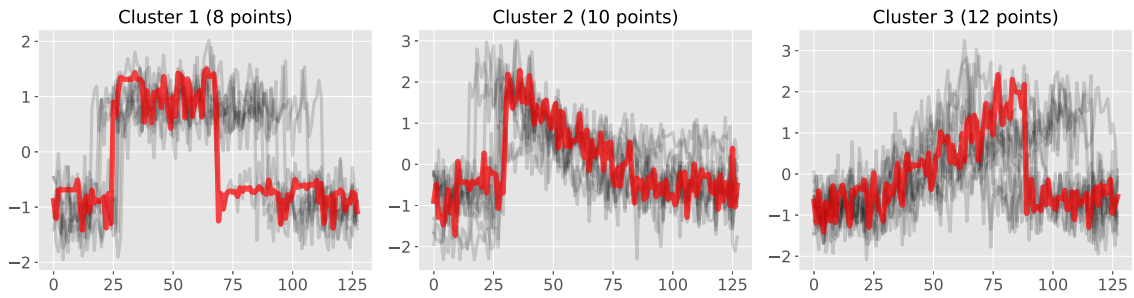
(c) DBA (random initialization)
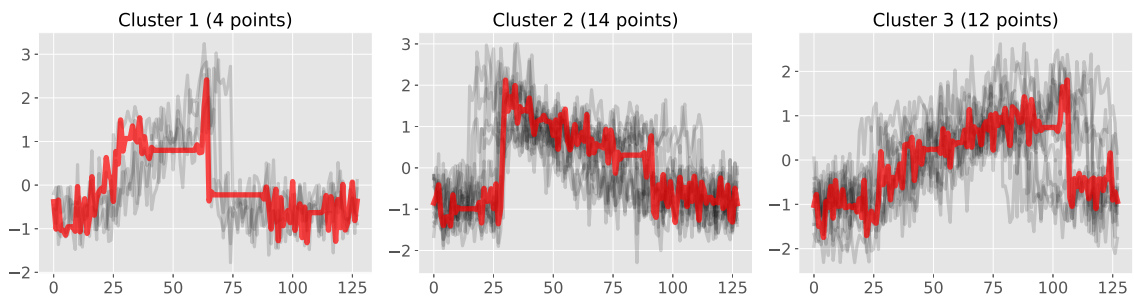
(d) DBA (Euclidean mean initialization)

ECG200 dataset

(a) Soft-DTW ($\gamma = 1$, random initialization)

(b) Soft-DTW ($\gamma = 1$, Euclidean mean initialization)

(c) DBA (random initialization)

(d) DBA (Euclidean mean initialization)

F. More visualizations of time-series prediction

(a) CBF

(b) ECG200

(c) ECG5000

(d) ShapesAll

(e) uWaveGestureLibrary_Y

G. Barycenters: DTW loss (Eq. 4 with $\gamma = 0$) achieved with random init

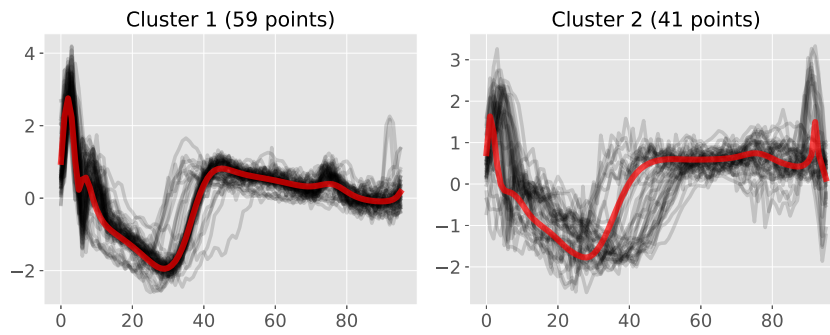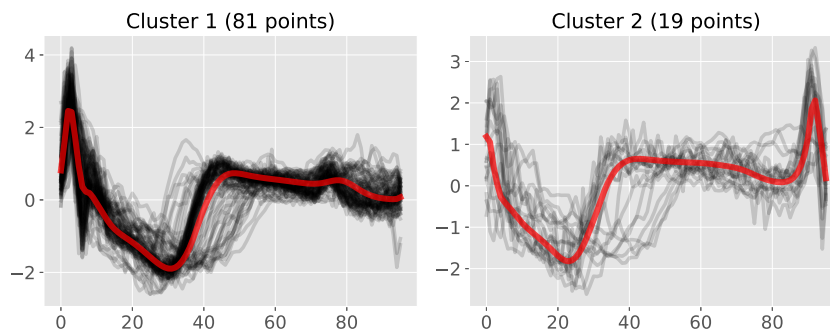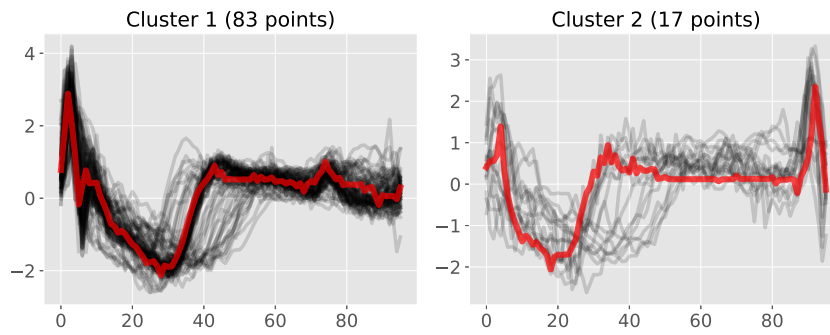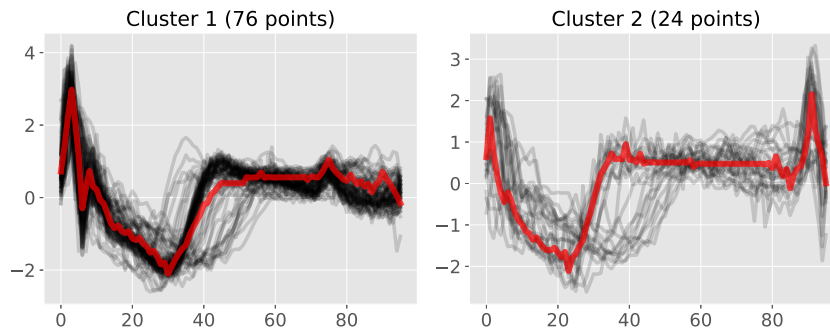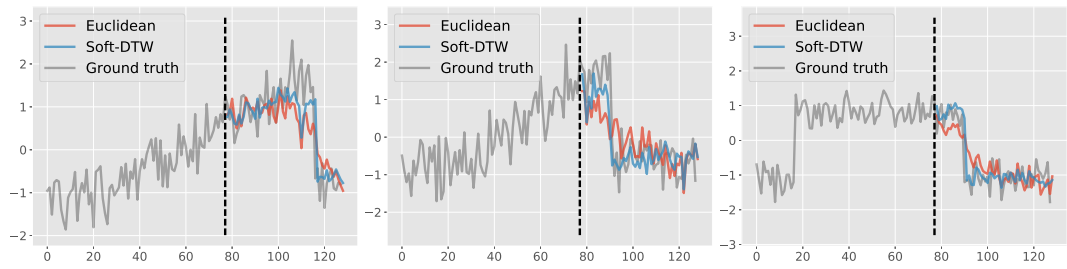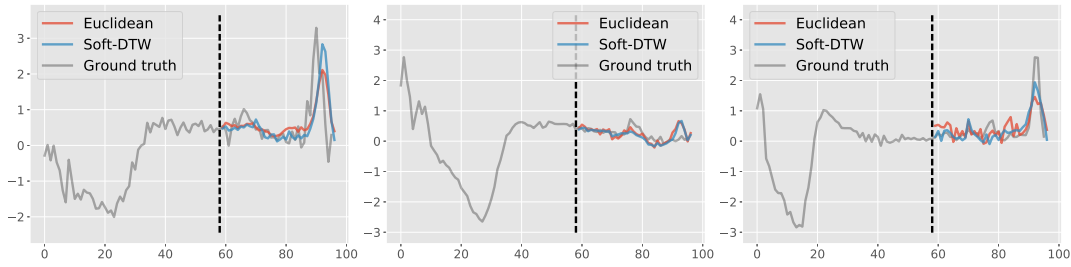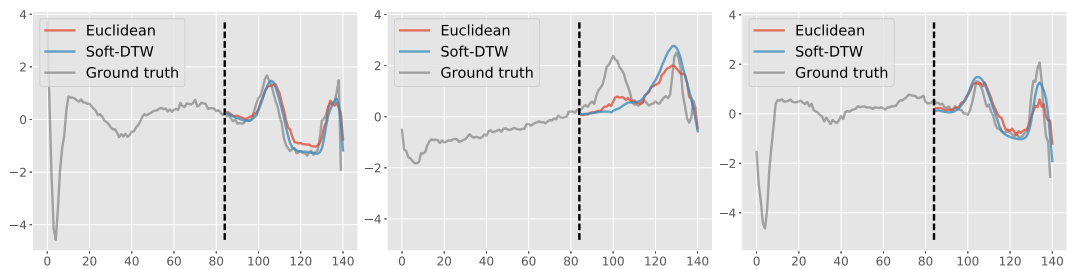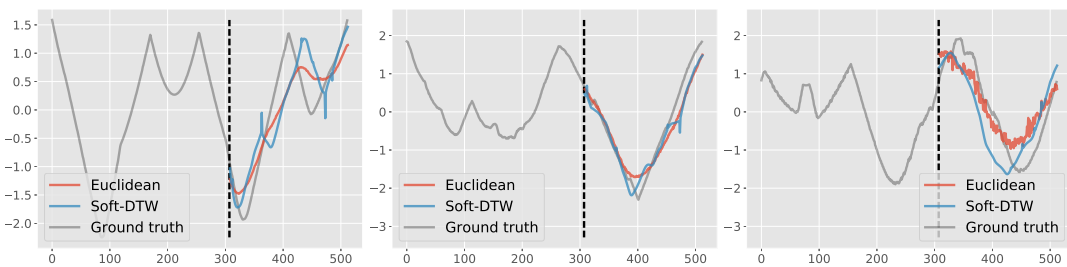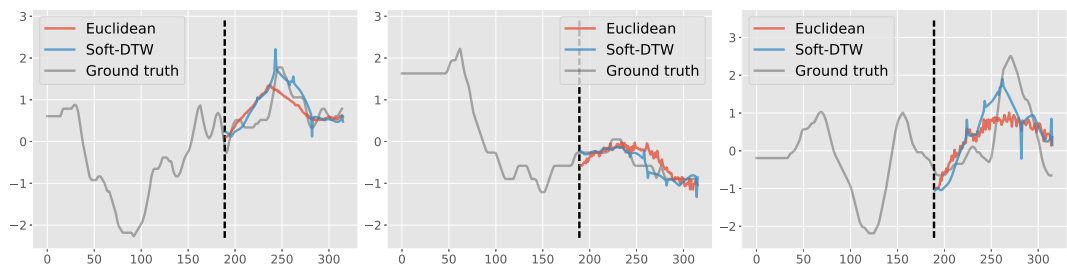| Dataset	Soft-DTW $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Subgradient method	DBA	Euclidean mean
50words	5.000	2.785	**2.513**	2.721	44.399	4.554	25.388
Adiac	0.235	0.207	0.257	0.428	25.533	0.754	**0.177**
ArrowHead	2.390	1.598	**1.487**	1.664	36.125	2.512	2.743
Beef	10.471	6.541	**6.200**	6.238	88.100	7.780	25.347
BeetleFly	35.790	23.655	**22.559**	23.105	77.993	25.122	191.574
BirdChicken	23.300	12.542	**11.164**	11.954	45.777	12.820	92.061
CBF	21.098	**11.949**	12.564	12.667	30.281	14.836	28.236
Car	2.639	1.750	**1.611**	1.914	80.437	2.609	5.106
ChlorineConcentration	22.260	**13.932**	14.818	15.044	32.134	16.168	15.411
CinC_ECG_torso	118.872	80.248	**76.536**	76.812	262.221	90.663	761.238
Coffee	1.036	0.871	1.262	1.630	41.741	2.380	**0.591**
Computers	231.421	182.380	**178.184**	179.886	∞	183.388	391.830
Cricket_X	41.514	29.290	**28.424**	28.851	70.128	28.955	104.699
Cricket_Y	51.858	**30.321**	30.337	31.041	70.989	33.098	107.712
Cricket_Z	43.458	30.264	**28.668**	29.373	68.382	32.182	128.129
DiatomSizeReduction	0.055	0.054	0.064	0.132	49.308	0.418	**0.033**
DistalPhalanxOutlineAgeGroup	1.380	1.074	1.407	1.509	11.539	1.761	**0.981**
DistalPhalanxOutlineCorrect	2.501	**1.968**	2.267	2.634	13.169	2.991	2.374
DistalPhalanxTW	1.148	0.906	1.015	1.159	10.957	1.228	**0.867**
ECG200	7.374	**6.400**	6.871	7.047	19.514	8.257	10.107
ECG5000	**8.951**	8.961	10.601	10.265	34.558	12.098	14.517
ECGFiveDays	9.816	**9.019**	9.364	9.407	17.898	9.837	23.975
Earthquakes	148.959	**85.219**	85.470	85.515	∞	85.788	330.776
ElectricDevices	27.852	**23.769**	23.783	24.009	∞	24.869	56.470
FISH	0.978	**0.641**	0.662	0.957	63.566	1.680	1.543
FaceAll	15.068	**12.373**	13.248	13.494	24.582	15.980	29.982
FaceFour	15.500	**14.519**	15.002	14.849	∞	16.339	25.410
FacesUCR	15.033	**13.077**	13.405	14.174	30.621	14.796	27.026
FordA	56.936	**45.492**	46.170	46.038	96.087	49.723	218.482
FordB	59.117	47.812	**47.058**	47.642	102.279	50.262	250.595
Gun_Point	7.204	2.507	**2.037**	2.211	22.590	2.374	7.286
Ham	24.833	**19.101**	20.397	20.713	55.769	22.807	30.685
HandOutlines	3.400	**2.690**	2.814	28.759	353.235	3.422	7.838
Haptics	16.424	14.351	**14.129**	14.320	172.988	16.464	39.559
Herring	1.212	**0.946**	1.022	1.349	71.388	2.097	1.884
InlineSkate	83.107	29.672	**22.819**	N/A	N/A	N/A	N/A
ItalyPowerDemand	2.442	**2.124**	2.316	2.372	5.434	2.355	2.329
MedicalImages	6.934	**5.809**	5.980	6.089	22.777	6.252	10.911
MiddlePhalanxOutlineAgeGroup	0.858	0.753	1.305	1.375	11.474	1.605	**0.624**
MiddlePhalanxOutlineCorrect	0.832	0.714	0.985	1.030	11.643	1.678	**0.611**
MiddlePhalanxTW	0.755	0.581	0.963	1.206	10.684	1.274	**0.447**
MoteStrain	24.177	21.639	21.616	**21.554**	32.007	22.437	26.646
NonInvasiveFatalECG_Thorax1	6.671	**3.324**	4.738	5.031	59.162	6.378	3.568
NonInvasiveFatalECG_Thorax2	**2.559**	3.159	3.587	4.097	40339.200	6.494	2.864
OSULeaf	30.041	20.692	20.034	**19.950**	76.057	23.915	136.512
OliveOil	0.657	0.959	1.494	1.804	95.499	3.420	**0.008**
PhalangesOutlinesCorrect	1.383	**1.114**	1.405	1.516	11.070	1.743	1.210
Phoneme	99.205	**72.412**	73.666	73.767	157.124	78.664	138.157
Plane	1.079	**0.849**	1.220	1.629	20.328	2.111	1.209
ProximalPhalanxOutlineAgeGroup	0.618	0.511	0.691	0.878	10.437	1.177	**0.322**
ProximalPhalanxOutlineCorrect	0.749	0.654	0.833	0.882	10.767	1.111	**0.615**
ProximalPhalanxTW	0.653	0.536	0.672	0.778	10.377	1.133	**0.462**
RefrigerationDevices	159.745	146.601	**140.634**	141.200	∞	148.931	363.732
ScreenType	156.442	123.746	**121.432**	122.247	∞	132.748	334.379
ShapeletSim	236.039	**123.605**	125.657	126.062	154.480	130.428	283.458
ShapesAll	15.267	7.108	**6.466**	6.818	58.734	8.236	67.790
SmallKitchenAppliances	176.073	164.419	**162.009**	162.585	∞	168.170	524.542
SonyAIBORobotSurface	5.735	**4.916**	5.425	5.337	63.040	5.813	5.430
SonyAIBORobotSurfaceII	11.994	**11.048**	11.278	11.405	6637415.793	11.481	13.783
StarLightCurves	22.342	11.757	7.934	**7.654**	102.255	11.600	47.353
Strawberry	1.392	**1.191**	1.477	1.627	28.696	2.347	1.300
SwedishLeaf	2.409	**1.968**	2.476	2.904	19.638	3.283	6.274
Symbols	0.845	0.488	**0.439**	0.460	∞	0.828	4.953
ToeSegmentation1	35.904	27.461	**26.554**	26.988	63.040	29.838	129.858
ToeSegmentation2	34.177	24.476	**23.003**	23.194	∞	24.944	170.222
Trace	2.686	1.453	**0.870**	1.031	43.017	2.233	26.037
TwoLeadECG	1.811	**1.514**	1.641	1.701	7.961	1.802	2.216
Two_Patterns	12.048	9.294	**7.764**	8.143	22.489	8.937	60.963
UWaveGestureLibraryAll	68.276	42.692	**38.327**	40.320	∞	49.486	181.901
Wine	0.728	0.500	0.746	1.147	32.463	1.812	**0.094**
WordsSynonyms	9.305	4.917	**4.491**	4.740	48.605	7.209	29.713
Worms	100.683	64.029	61.527	61.296	**35.906**	68.282	421.381
WormsTwoClass	110.292	68.932	66.258	65.964	**37.047**	72.387	430.774
synthetic_control	14.366	**7.115**	7.506	7.516	15.931	8.123	12.187
uWaveGestureLibrary_X	27.610	16.618	14.902	**14.442**	∞	18.269	75.119
uWaveGestureLibrary_Y	29.964	16.106	14.556	**14.450**	∞	15.961	74.405
uWaveGestureLibrary_Z	40.154	24.001	**22.462**	22.656	∞	25.040	107.540
wafer	25.831	**23.595**	25.828	25.195	∞	27.323	65.100
yoga	27.418	13.524	**11.828**	12.051	40.171	15.319	111.236

H. Barycenters: DTW loss (Eq. (4) with $\gamma = 0$) achieved with Euclidean init

Dataset	Soft-DTW $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Subgradient method	DBA	Euclidean mean
50words	5.400	2.895	**2.355**	2.439	4.064	2.595	22.294
Adiac	0.124	0.103	0.089	**0.069**	0.081	0.071	0.103
ArrowHead	2.677	1.759	**1.282**	1.327	1.587	1.411	2.965
Beef	14.814	6.412	**5.252**	5.694	11.112	5.528	31.486
BeetleFly	33.082	20.819	**20.781**	22.127	25.554	21.960	191.285
BirdChicken	21.646	9.445	**7.807**	8.026	473.653	8.243	70.614
CBF	22.498	11.844	**11.433**	11.597	15.321	12.291	28.228
Car	1.556	0.932	**0.693**	0.901	1.171	1.079	2.439
ChlorineConcentration	19.239	10.663	**10.434**	10.468	11.370	10.638	13.549
CinC_ECG_torso	112.562	78.292	69.415	70.383	76.693	**68.641**	751.445
Coffee	1.078	0.657	0.460	**0.393**	0.435	0.399	0.571
Computers	172.590	**138.605**	144.576	146.409	∞	154.956	381.271
Cricket_X	48.334	35.136	**33.103**	33.312	42.018	34.430	125.879
Cricket_Y	41.804	31.395	**31.044**	31.158	35.957	31.749	97.393
Cricket_Z	46.957	**33.453**	34.005	33.708	45.125	36.025	140.474
DiatomSizeReduction	0.039	0.033	0.028	0.021	0.024	**0.019**	0.032
DistalPhalanxOutlineAgeGroup	1.578	0.988	0.784	**0.779**	0.847	0.794	1.075
DistalPhalanxOutlineCorrect	2.878	2.002	**1.751**	1.754	2475.922	1.790	2.780
DistalPhalanxTW	1.377	0.837	0.655	**0.651**	0.773	0.667	0.997
ECG200	7.266	5.608	**5.395**	5.424	5.955	5.494	9.638
ECG5000	12.430	10.377	**10.332**	10.343	12.340	10.595	18.886
ECGFiveDays	8.416	7.452	**7.046**	7.101	145.106	7.145	23.477
Earthquakes	172.035	91.568	**90.684**	91.071	∞	92.126	335.240
ElectricDevices	30.832	**26.480**	27.131	27.076	∞	27.615	57.938
FISH	1.183	0.806	0.541	**0.508**	0.645	0.551	1.933
FaceAll	18.102	13.305	13.104	**13.074**	16.491	13.915	40.404
FaceFour	17.070	13.069	**12.984**	13.091	∞	13.568	28.203
FacesUCR	17.172	**13.081**	13.293	13.394	15.780	13.498	35.942
FordA	53.903	42.199	**41.835**	41.966	53.545	44.259	235.362
FordB	61.168	48.150	**47.327**	47.743	60.120	50.121	246.802
Gun_Point	5.924	2.132	1.695	**1.666**	2.543	1.682	5.906
Ham	25.353	18.841	17.457	**17.294**	∞	17.917	32.456
HandOutlines	2.238	1.718	1.004	0.527	∞	**0.515**	6.452
Haptics	12.554	8.874	**7.785**	8.197	12.193	8.219	35.613
Herring	1.655	1.117	0.809	**0.760**	0.956	0.817	2.564
InlineSkate	100.849	46.460	**27.248**	35.578	N/A	N/A	N/A
ItalyPowerDemand	2.597	1.990	**1.956**	1.985	2.132	1.997	2.449
MedicalImages	5.719	4.319	4.145	**4.070**	4.791	4.371	8.047
MiddlePhalanxOutlineAgeGroup	0.870	0.578	0.427	**0.415**	0.464	0.426	0.552
MiddlePhalanxOutlineCorrect	0.799	0.609	0.460	**0.443**	0.501	0.461	0.577
MiddlePhalanxTW	0.658	0.466	0.335	**0.321**	0.358	0.332	0.434
MoteStrain	24.451	**20.720**	20.829	21.057	∞	21.273	26.694
NonInvasiveFatalECG_Thorax1	1.619	1.384	0.907	0.785	**0.691**	0.814	1.400
NonInvasiveFatalECG_Thorax2	1.624	1.370	0.932	**0.827**	2.163	0.853	1.409
OSULeaf	27.428	18.666	**18.544**	18.595	24.692	20.244	135.980
OliveOil	0.367	0.074	0.022	0.013	0.011	**0.009**	0.011
PhalangesOutlinesCorrect	1.172	0.895	0.699	**0.695**	0.766	0.704	1.002
Phoneme	135.535	**104.971**	105.478	108.031	126.055	108.513	254.392
Plane	0.928	0.600	0.404	**0.399**	0.499	0.430	1.203
ProximalPhalanxOutlineAgeGroup	0.820	0.502	0.361	**0.346**	0.390	0.356	0.512
ProximalPhalanxOutlineCorrect	0.816	0.630	0.463	**0.452**	0.517	0.461	0.669
ProximalPhalanxTW	0.637	0.431	0.313	**0.304**	0.341	0.308	0.471
RefrigerationDevices	154.420	**133.321**	135.721	135.300	∞	142.697	358.823
ScreenType	189.188	143.582	143.894	**141.776**	∞	148.464	325.840
ShapeletSim	231.937	124.443	**122.000**	122.506	154.089	127.977	284.079
ShapesAll	13.416	7.519	**6.420**	6.509	7.317	7.478	80.306
SmallKitchenAppliances	188.030	173.670	169.755	**167.097**	∞	173.004	505.356
SonyAIBORobotSurface	5.715	4.002	3.870	3.896	∞	**3.828**	5.444
SonyAIBORobotSurfaceII	11.300	8.947	**8.853**	8.871	12.651	8.977	14.225
StarLightCurves	13.581	6.619	4.054	**3.765**	7.247	4.517	30.354
Strawberry	2.218	1.413	1.128	**1.070**	1.374	1.156	2.128
SwedishLeaf	2.957	2.068	**2.049**	2.081	2.520	2.163	6.236
Symbols	0.762	0.451	0.412	**0.401**	∞	0.474	4.822
ToeSegmentation1	35.832	26.067	26.337	**25.735**	31.157	27.493	131.683
ToeSegmentation2	34.264	22.238	**20.800**	21.563	∞	23.080	164.101
Trace	1.737	1.744	1.508	**1.378**	4.170	1.969	26.814
TwoLeadECG	1.533	1.172	**1.030**	1.043	1.323	1.093	2.046
Two_Patterns	10.891	7.505	**6.045**	6.079	18.987	6.584	66.027
UWaveGestureLibraryAll	67.549	38.179	**32.894**	33.426	∞	39.241	167.486
Wine	0.707	0.188	0.127	0.111	0.114	**0.110**	0.118
WordsSynonyms	9.804	7.282	**6.711**	6.785	8.884	6.868	39.843
Worms	101.850	61.067	58.725	**56.793**	244.738	63.234	415.674
WormsTwoClass	122.901	68.771	**64.655**	64.898	1297.616	72.011	395.088
synthetic_control	18.147	**9.189**	9.307	9.350	11.520	9.614	19.237
uWaveGestureLibrary_X	34.423	19.787	18.746	**17.807**	∞	24.269	93.839
uWaveGestureLibrary_Y	27.744	14.309	**13.010**	13.607	∞	15.283	51.854
uWaveGestureLibrary_Z	21.927	10.081	8.456	**8.453**	∞	11.040	47.947
wafer	32.561	29.197	28.908	**28.820**	∞	33.379	67.413
yoga	23.698	11.632	9.433	**9.204**	16.239	10.058	93.688

I. k-means clustering: DTW loss achieved (Eq. (5) with $\gamma = 0$, log-scaled) when using random initialization

Dataset	Soft-DTW $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Subgradient method	DBA	Euclidean mean
50words	16.294	16.193	**16.125**	16.135	16.163	16.156	16.205
Adiac	**11.933**	**11.933**	**11.933**	**11.933**	**11.933**	**11.933**	**11.933**
ArrowHead	9.020	8.757	8.699	**8.687**	8.732	8.692	8.958
Beef	11.215	11.095	11.069	**11.061**	**11.061**	11.117	11.215
BeetleFly	9.946	9.618	**9.531**	9.592	9.619	9.591	10.368
BirdChicken	9.996	9.652	**9.374**	9.515	9.870	9.585	10.335
CBF	10.150	10.065	**10.005**	10.006	10.009	10.009	10.150
Car	9.392	9.290	9.067	**9.039**	9.059	9.046	9.276
ChlorineConcentration	15.512	15.214	15.182	**15.175**	15.176	15.176	15.331
CinC_ECG_torso	13.134	**12.837**	12.848	12.868	13.621	12.877	13.621
Coffee	7.150	6.893	6.692	**6.628**	6.693	6.651	6.825
Computers	**16.420**	16.498	16.489	16.502	16.960	16.475	16.960
Cricket_X	16.922	16.696	16.629	**16.628**	16.649	16.653	16.955
Cricket_Y	16.783	16.588	16.545	**16.533**	16.570	16.570	16.803
Cricket_Z	16.874	16.669	16.597	**16.593**	16.620	16.620	16.981
DiatomSizeReduction	5.959	5.907	5.889	**5.739**	5.798	5.758	5.932
DistalPhalanxOutlineAgeGroup	11.193	11.220	11.202	11.198	11.194	11.196	**11.158**
DistalPhalanxOutlineCorrect	12.467	12.373	**12.340**	12.342	12.494	12.350	12.483
DistalPhalanxTW	11.244	11.260	11.263	11.251	11.264	11.261	**11.222**
ECG200	11.395	11.317	11.323	**11.274**	11.300	11.289	11.501
ECG5000	16.169	**16.084**	16.142	16.136	16.137	16.136	16.211
ECGFiveDays	8.734	8.579	8.522	**8.513**	8.713	8.533	8.818
Earthquakes	14.757	14.727	**14.726**	14.728	14.757	**14.726**	14.757
ElectricDevices	22.404	22.428	22.401	22.398	22.630	22.399	**22.332**
FISH	10.841	10.740	10.594	**10.514**	10.560	10.566	10.841
FaceAll	16.272	16.187	16.185	16.183	16.197	**16.182**	16.291
FaceFour	10.422	10.318	**10.302**	10.316	10.575	10.321	10.533
FacesUCR	14.479	14.432	14.426	**14.423**	14.430	14.429	14.431
FordA	18.604	18.390	18.388	18.387	18.977	**18.385**	18.977
FordB	17.620	17.429	17.425	17.426	17.466	**17.416**	17.998
Gun_Point	10.242	10.019	9.843	9.743	9.883	**9.738**	10.130
Ham	12.772	12.545	12.488	**12.473**	13.240	12.506	12.957
MedicalImages	15.081	14.982	14.985	**14.979**	14.986	14.986	15.032
MiddlePhalanxOutlineAgeGroup	9.909	9.919	9.856	**9.818**	9.824	9.822	9.856
MiddlePhalanxOutlineCorrect	11.121	11.088	10.984	10.951	10.962	10.961	**10.923**
MiddlePhalanxTW	**10.514**	**10.514**	**10.514**	**10.514**	**10.514**	**10.514**	**10.514**
MoteStrain	9.560	9.484	9.460	9.451	**9.201**	9.470	9.557
NonInvasiveFatalECG_Thorax1	**17.728**	N/A	N/A	N/A	N/A	N/A	N/A
ProximalPhalanxTW	11.055	10.993	10.978	**10.958**	10.968	10.965	10.968
RefrigerationDevices	17.391	17.351	**17.311**	17.322	17.758	17.324	17.758
ScreenType	17.467	17.388	17.306	17.297	18.126	**17.289**	17.838
ShapeletSim	11.176	**10.896**	10.905	10.906	10.916	10.915	11.176
ShapesAll	17.539	17.405	**17.331**	17.333	17.605	17.357	17.509
SmallKitchenAppliances	17.551	17.611	**17.537**	17.606	18.007	17.561	18.007
SonyAIBORobotSurface	8.181	7.959	**7.934**	7.943	8.247	7.958	8.084
SonyAIBORobotSurfaceII	9.349	**9.265**	9.267	9.267	9.325	9.277	9.338
StarLightCurves	19.435	19.110	**19.012**	N/A	N/A	N/A	N/A
Trace	14.570	14.570	14.556	**14.550**	14.555	14.556	14.553
TwoLeadECG	6.939	6.939	6.892	6.879	6.936	6.892	**6.743**
Two_Patterns	17.416	17.379	17.317	17.325	17.524	**17.307**	17.524
UWaveGestureLibraryAll	18.911	18.641	**18.514**	18.531	19.282	18.537	19.244
Wine	7.527	7.297	6.482	6.358	6.390	6.353	**6.223**
WordsSynonyms	15.209	15.093	**15.024**	15.025	15.053	15.036	15.159
Worms	14.184	14.051	**13.889**	13.896	14.943	13.968	14.648
WormsTwoClass	13.727	13.471	**13.429**	13.462	14.944	13.494	14.944
synthetic_control	15.338	15.303	15.292	15.291	15.303	15.295	**15.278**
uWaveGestureLibrary_X	18.789	**18.568**	N/A	N/A	N/A	N/A	N/A

J. k-means clustering: DTW loss achieved (Eq. (5) with $\gamma = 0$, log-scaled) when using Euclidean mean initialization

Dataset	Soft-DTW $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Subgradient method	DBA	Euclidean mean
50words	16.233	16.145	16.046	**16.035**	16.045	16.233	16.233
Adiac	12.311	12.311	12.264	12.241	12.234	**12.233**	12.311
ArrowHead	9.014	8.963	8.766	**8.746**	8.851	8.809	9.014
Beef	11.225	11.110	11.088	11.079	11.077	**11.070**	11.300
BeetleFly	9.895	9.290	9.268	**9.240**	9.512	10.926	10.926
BirdChicken	10.032	9.542	9.352	**9.338**	9.422	9.414	10.335
CBF	10.246	9.995	9.910	**9.908**	9.933	9.921	10.246
Car	9.276	9.229	8.989	**8.910**	8.936	8.935	9.276
ChlorineConcentration	15.331	15.291	15.254	**15.252**	15.270	**15.252**	15.331
CinC_ECG_torso	13.197	12.803	12.752	**12.728**	13.723	13.723	13.723
Coffee	6.825	6.825	6.668	6.599	6.605	**6.591**	6.825
Computers	16.417	16.346	16.301	**16.289**	17.167	16.342	17.167
Cricket_X	16.895	16.719	16.622	16.623	16.612	**16.600**	16.987
Cricket_Y	16.770	16.651	16.546	**16.514**	16.515	16.527	16.861
Cricket_Z	16.924	16.748	16.670	**16.633**	16.653	16.653	17.028
DiatomSizeReduction	5.963	5.963	5.963	**5.884**	5.897	5.896	5.963
DistalPhalanxOutlineAgeGroup	**11.164**	**11.164**	**11.164**	**11.164**	**11.164**	**11.164**	**11.164**
DistalPhalanxOutlineCorrect	12.544	12.533	12.494	12.475	**12.240**	12.479	12.577
DistalPhalanxTW	**11.242**	11.259	11.256	11.243	11.251	11.245	11.259
ECG200	11.462	11.291	11.239	**11.222**	11.231	11.234	11.503
ECG5000	16.253	16.180	16.171	16.183	**16.170**	16.172	16.262
ECGFiveDays	8.738	8.614	**8.543**	8.549	8.709	8.559	8.818
Earthquakes	15.113	14.625	14.601	14.599	15.952	**14.597**	15.952
ElectricDevices	22.295	22.325	22.291	22.290	22.379	**22.283**	22.379
FISH	10.904	10.843	10.589	10.543	**10.527**	10.555	10.904
FaceAll	16.278	16.162	16.145	16.145	16.152	**16.140**	16.347
FaceFour	10.376	10.273	10.239	**10.226**	10.566	10.241	10.566
FacesUCR	14.472	14.434	14.406	14.407	**14.391**	14.481	14.481
FordA	18.581	**18.354**	**18.354**	18.360	20.038	20.038	20.038
FordB	17.649	17.443	17.429	17.436	17.466	**17.427**	19.143
Gun_Point	10.334	10.027	9.806	**9.751**	9.902	9.833	10.334
Ham	12.805	12.603	12.559	**12.558**	12.974	12.561	12.974
HandOutlines	**13.712**	N/A	N/A	N/A	N/A	N/A	N/A
MedicalImages	15.082	14.963	**14.940**	14.942	14.950	14.941	15.091
MiddlePhalanxOutlineAgeGroup	9.856	9.856	9.855	**9.821**	9.823	**9.821**	9.856
MiddlePhalanxOutlineCorrect	10.962	10.962	10.962	10.959	**10.950**	**10.950**	10.962
MiddlePhalanxTW	**10.558**	10.587	10.587	10.569	10.572	10.570	10.587
MoteStrain	9.551	9.454	**9.413**	9.451	9.557	9.446	9.557
NonInvasiveFatalECG_Thorax1	**17.765**	N/A	N/A	N/A	N/A	N/A	N/A
ProximalPhalanxTW	10.978	**10.973**	10.978	10.978	10.978	10.978	10.978
RefrigerationDevices	17.260	17.202	17.093	**17.073**	18.140	17.095	18.140
ScreenType	17.430	17.359	17.294	**17.292**	17.838	17.323	17.838
ShapeletSim	11.497	10.864	**10.845**	10.853	10.865	11.608	11.608
ShapesAll	17.560	17.431	17.335	**17.328**	17.560	17.560	17.560
SmallKitchenAppliances	17.310	**17.273**	17.357	17.357	18.206	18.206	18.206
SonyAIBORobotSurface	8.084	7.980	**7.941**	7.947	8.084	7.948	8.084
SonyAIBORobotSurfaceII	9.338	9.207	9.196	**9.195**	9.338	**9.195**	9.338
StarLightCurves	19.457	19.178	**19.083**	N/A	N/A	N/A	N/A
Trace	14.553	14.553	14.553	**14.549**	14.553	14.553	14.553
TwoLeadECG	6.743	6.705	6.623	**6.606**	6.666	6.633	6.743
Two_Patterns	**17.084**	17.363	17.242	17.255	17.518	17.316	17.942
UWaveGestureLibraryAll	18.820	18.613	18.539	**18.488**	19.259	18.508	19.259
Wine	6.223	6.223	6.223	6.223	6.223	**6.205**	6.223
WordsSynonyms	15.184	15.036	**14.947**	14.951	14.965	14.959	15.196
Worms	14.043	13.860	13.791	13.777	14.696	**13.772**	14.696
WormsTwoClass	13.699	13.440	**13.322**	13.337	15.076	13.390	15.076
synthetic_control	15.472	15.367	15.337	15.338	**15.330**	15.336	15.472
uWaveGestureLibrary_X	18.844	**18.562**	N/A	N/A	N/A	N/A	N/A

K. Time-series prediction: DTW loss achieved when using random init

Dataset	Soft-DTW loss $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Euclidean loss
50words	6.473	**4.921**	4.999	6.489	18.734
Adiac	0.094	**0.074**	0.078	0.109	0.103
ArrowHead	1.851	**1.708**	1.933	1.909	2.073
Beef	12.229	**8.688**	10.244	9.126	22.228
BeetleFly	35.037	25.439	27.588	**23.494**	50.610
BirdChicken	31.878	19.914	25.100	**14.981**	30.693
CBF	10.802	**9.263**	9.995	10.151	12.868
Car	1.724	2.307	2.202	**1.318**	1.588
ChlorineConcentration	7.876	2.108	2.331	1.735	**0.769**
CinC_ECG_torso	45.675	26.337	**23.567**	24.550	48.171
Coffee	0.914	0.727	1.662	1.883	**0.660**
Computers	92.584	84.723	78.953	**75.435**	235.208
Cricket_X	9.394	8.042	**7.123**	7.226	12.080
Cricket_Y	11.989	9.643	**9.534**	9.545	15.002
Cricket_Z	9.161	6.889	**6.585**	7.200	11.003
DiatomSizeReduction	1.182	0.922	**0.820**	0.897	1.203
DistalPhalanxOutlineAgeGroup	0.426	0.291	0.541	0.496	**0.231**
DistalPhalanxOutlineCorrect	0.494	0.476	0.564	0.591	**0.351**
DistalPhalanxTW	0.441	0.330	0.305	1.214	**0.231**
ECG200	1.874	**1.716**	1.884	1.734	1.905
ECG5000	4.895	4.705	4.543	**4.441**	5.463
ECGFiveDays	1.834	1.944	1.699	**1.642**	2.220
Earthquakes	74.738	59.973	60.877	**57.827**	147.980
ElectricDevices	20.186	**15.125**	15.218	15.287	37.121
FISH	0.464	0.429	**0.354**	0.459	0.462
FaceAll	9.317	7.451	7.902	**7.276**	10.716
FaceFour	**19.564**	20.881	28.150	28.839	46.841
FacesUCR	15.359	**14.643**	16.143	17.428	28.576
Gun_Point	0.896	**0.805**	0.923	0.834	0.858
Ham	20.154	17.931	17.786	**17.413**	24.340
Haptics	**16.174**	17.775	17.142	17.423	23.130
Herring	1.000	0.712	**0.666**	0.762	0.865
InsectWingbeatSound	3.460	2.823	2.458	**2.220**	5.437
ItalyPowerDemand	0.911	0.893	**0.711**	0.798	0.881
LargeKitchenAppliances	63.153	60.739	**60.157**	61.841	266.853
Lighting2	73.293	66.341	**65.335**	66.881	147.668
Lighting7	44.446	42.699	**40.608**	41.502	68.902
Meat	0.162	0.242	0.246	0.650	**0.099**
MedicalImages	1.023	0.853	**0.708**	0.778	1.211
MiddlePhalanxOutlineAgeGroup	0.343	0.347	0.570	0.400	**0.312**
MiddlePhalanxOutlineCorrect	0.278	0.204	0.227	0.202	**0.182**
MiddlePhalanxTW	0.251	0.153	0.445	0.314	**0.132**
MoteStrain	10.188	**9.986**	11.119	10.250	11.183
NonInvasiveFatalECG_Thorax1	1.002	0.920	0.675	**0.634**	1.219
OSULeaf	15.125	11.722	11.086	**10.775**	30.739
OliveOil	0.476	0.683	2.082	2.076	**0.020**
PhalangesOutlinesCorrect	0.352	0.216	0.352	0.338	**0.170**
Phoneme	160.536	150.017	148.175	**145.093**	219.704
Plane	0.619	**0.564**	0.834	0.788	0.630
ProximalPhalanxOutlineAgeGroup	0.134	0.062	0.105	0.118	**0.046**
ProximalPhalanxOutlineCorrect	0.129	0.047	0.089	0.128	**0.044**
ProximalPhalanxTW	0.154	0.077	0.102	0.150	**0.055**
RefrigerationDevices	108.421	93.519	**89.370**	89.873	160.361
ShapeletSim	102.413	**70.455**	71.156	72.094	108.936
ShapesAll	10.391	9.027	7.850	**7.207**	18.348
SonyAIBORobotSurface	4.453	4.494	**4.318**	4.910	4.388
SonyAIBORobotSurfaceII	8.072	8.302	**7.758**	8.669	8.628
Strawberry	0.123	0.088	0.137	0.100	**0.081**
SwedishLeaf	1.486	1.277	1.316	**1.169**	1.633
Symbols	17.963	14.039	15.172	**13.192**	38.268
ToeSegmentation1	23.866	22.987	26.056	**22.401**	35.806
ToeSegmentation2	41.450	33.100	**30.931**	31.106	61.899
Trace	0.563	0.379	0.352	**0.279**	0.582
TwoLeadECG	0.441	0.394	**0.318**	0.320	0.336
Two_Patterns	15.035	10.100	10.588	**8.584**	35.923
UWaveGestureLibraryAll	40.324	28.975	26.193	**25.897**	93.019
Wine	0.164	0.203	1.417	0.958	**0.028**
WordsSynonyms	12.466	10.437	**9.165**	9.219	32.003
Worms	81.236	63.938	60.950	**59.995**	114.528
WormsTwoClass	78.455	66.609	**60.207**	61.685	122.619
synthetic_control	7.709	**5.315**	5.390	5.506	7.690
uWaveGestureLibrary_X	13.096	9.995	10.143	**9.433**	19.995
uWaveGestureLibrary_Y	9.793	7.272	7.327	**7.225**	17.706
uWaveGestureLibrary_Z	11.883	8.909	8.494	**8.416**	20.092
wafer	1.049	0.473	**0.386**	0.496	2.636
yoga	2.932	2.431	**1.995**	3.309	4.305

L. Time-series prediction: DTW loss achieved when using Euclidean init

Dataset	Soft-DTW loss $\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$	Euclidean loss
50words	6.330	5.628	4.885	**4.553**	18.734
Adiac	0.082	0.076	**0.064**	0.079	0.103
ArrowHead	1.823	2.016	**1.762**	2.106	2.073
Beef	7.250	6.940	7.146	**3.757**	22.228
BeetleFly	32.430	**26.600**	27.199	29.003	50.610
BirdChicken	24.952	22.600	**19.914**	20.540	30.693
CBF	10.744	8.978	9.215	**8.398**	12.868
Car	0.906	0.812	**0.709**	0.740	1.588
ChlorineConcentration	6.018	0.979	**0.695**	0.698	0.769
CinC_ECG_torso	29.892	**18.638**	19.635	19.191	48.171
Coffee	0.870	0.582	0.511	**0.496**	0.660
Computers	86.619	**79.250**	82.215	81.417	235.208
Cricket_X	10.954	8.200	**7.932**	8.296	12.080
Cricket_Y	11.901	10.150	10.265	**9.574**	15.002
Cricket_Z	9.714	7.760	**7.544**	8.041	11.003
DiatomSizeReduction	0.964	**0.852**	0.874	0.869	1.203
DistalPhalanxOutlineAgeGroup	0.403	0.206	**0.175**	0.177	0.231
DistalPhalanxOutlineCorrect	0.515	0.310	0.300	**0.262**	0.351
DistalPhalanxTW	0.468	0.228	0.186	**0.178**	0.231
ECG200	1.907	1.541	1.565	**1.536**	1.905
ECG5000	4.737	4.190	4.398	**4.148**	5.463
ECGFiveDays	1.584	1.396	**1.322**	1.335	2.220
Earthquakes	71.461	**55.819**	56.504	57.153	147.980
ElectricDevices	19.499	15.045	**14.999**	15.228	37.121
FISH	0.439	0.353	0.319	**0.318**	0.462
FaceAll	9.309	8.687	**7.803**	7.853	10.716
FaceFour	20.483	**20.411**	21.259	21.444	46.841
FacesUCR	14.984	14.530	**14.403**	14.729	28.576
Gun_Point	0.447	0.368	0.300	**0.297**	0.858
Ham	16.152	14.717	**12.252**	13.424	24.340
Haptics	15.177	14.275	12.394	**11.931**	23.130
Herring	0.310	0.305	0.292	**0.249**	0.865
InsectWingbeatSound	3.104	2.346	2.186	**2.036**	5.437
ItalyPowerDemand	0.802	**0.595**	0.623	0.654	0.881
LargeKitchenAppliances	61.531	63.834	59.116	**57.219**	266.853
Lighting2	65.602	62.240	61.561	**60.826**	147.668
Lighting7	43.930	41.668	40.535	**39.422**	68.902
Meat	0.173	0.140	0.103	**0.077**	0.099
MedicalImages	0.932	0.671	**0.615**	0.651	1.211
MiddlePhalanxOutlineAgeGroup	0.283	0.200	**0.134**	0.154	0.312
MiddlePhalanxOutlineCorrect	0.269	0.169	**0.133**	0.148	0.182
MiddlePhalanxTW	0.225	0.126	0.111	**0.094**	0.132
MoteStrain	9.704	**9.321**	9.385	10.156	11.183
NonInvasiveFatalECG_Thorax1	0.921	0.736	0.540	**0.536**	1.219
OSULeaf	16.168	14.372	13.354	**12.350**	30.739
OliveOil	0.179	0.298	0.034	0.026	**0.020**
PhalangesOutlinesCorrect	0.328	0.231	0.161	**0.153**	0.170
Phoneme	173.501	**158.036**	159.293	158.776	219.704
Plane	0.454	0.343	**0.311**	0.370	0.630
ProximalPhalanxOutlineAgeGroup	0.137	0.056	0.039	**0.036**	0.046
ProximalPhalanxOutlineCorrect	0.118	0.046	0.033	**0.030**	0.044
ProximalPhalanxTW	0.164	0.067	0.054	**0.045**	0.055
RefrigerationDevices	107.693	95.458	**90.328**	91.687	160.361
ShapeletSim	106.805	73.626	**72.985**	73.987	108.936
ShapesAll	11.946	7.724	**7.127**	7.522	18.348
SonyAIBORobotSurface	4.068	3.619	**3.400**	3.737	4.388
SonyAIBORobotSurfaceII	6.954	6.585	**6.410**	6.593	8.628
Strawberry	0.110	0.082	0.071	**0.069**	0.081
SwedishLeaf	1.438	1.151	**1.095**	1.108	1.633
Symbols	14.717	**12.040**	15.229	16.199	38.268
ToeSegmentation1	24.293	**20.905**	22.914	22.566	35.806
ToeSegmentation2	44.439	36.333	**35.804**	41.121	61.899
Trace	0.578	0.331	0.272	**0.250**	0.582
TwoLeadECG	0.157	0.131	**0.129**	0.149	0.336
Two_Patterns	14.843	11.616	11.622	**11.059**	35.923
UWaveGestureLibraryAll	42.336	30.864	27.572	**26.573**	93.019
Wine	0.069	0.263	0.045	0.029	**0.028**
WordsSynonyms	12.654	10.089	9.887	**8.946**	32.003
Worms	74.589	71.946	70.245	**67.669**	114.528
WormsTwoClass	81.311	**61.360**	62.281	74.672	122.619
synthetic_control	7.455	5.509	5.374	**5.369**	7.690
uWaveGestureLibrary_X	14.151	10.065	9.231	**9.197**	19.995
uWaveGestureLibrary_Y	9.852	7.285	7.066	**7.036**	17.706
uWaveGestureLibrary_Z	12.019	8.893	**8.453**	8.513	20.092
wafer	1.125	0.838	**0.765**	0.818	2.636
yoga	2.510	2.209	**1.807**	1.851	4.305