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# Post-Inference Prior Swapping

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## Abstract

While Bayesian methods are praised for their ability to incorporate useful prior knowledge, in practice, convenient priors that allow for computationally cheap or tractable inference are commonly used. In this paper, we investigate the following question: *for a given model, is it possible to compute an inference result with any convenient false prior, and afterwards, given any target prior of interest, quickly transform this result into the target posterior?* A potential solution is to use importance sampling (IS). However, we demonstrate that IS will fail for many choices of the target prior, depending on its parametric form and similarity to the false prior. Instead, we propose prior swapping, a method that leverages the pre-inferred false posterior to efficiently generate accurate posterior samples under arbitrary target priors. Prior swapping lets us apply less-costly inference algorithms to certain models, and incorporate new or updated prior information “post-inference”. We give theoretical guarantees about our method, and demonstrate it empirically on a number of models and priors.

## 1. Introduction

There are many cases in Bayesian modeling where a certain choice of prior distribution allows for computationally simple or tractable inference. For example,

- Conjugate priors yield posteriors with a known parametric form and therefore allow for non-iterative, exact inference (Diaconis et al., 1979).
- Certain priors yield models with tractable conditional or marginal distributions, which allows efficient approximate inference algorithms to be applied (e.g. Gibbs sampling (Smith & Roberts, 1993), sampling

in collapsed models (Teh et al., 2006), or mean-field variational methods (Wang & Blei, 2013)).

- Simple parametric priors allow for computationally cheap density queries, maximization, and sampling, which can reduce costs in iterative inference algorithms (e.g. Metropolis-Hastings (Metropolis et al., 1953), gradient-based MCMC (Neal, 2011), or sequential Monte Carlo (Doucet et al., 2000)).

For these reasons, one might hope to infer a result under a convenient-but-unrealistic prior, and afterwards, attempt to correct the result. More generally, given an inference result (under a convenient prior or otherwise), one might wish to incorporate updated prior information, or see a result under different prior assumptions, without having to re-run a costly inference algorithm.

This leads to the main question of this paper: for a given model, is it possible to use any convenient false prior to infer a false posterior, and afterwards, given any target prior of interest, efficiently and accurately infer the associated target posterior?

One potential strategy involves sampling from the false posterior and reweighting these samples via importance sampling (IS). However, depending on the chosen target prior—both its parametric form and similarity to the false prior—the resulting inference can be inaccurate due to high or infinite variance IS estimates (demonstrated in Sec. 2.1).

We instead aim to devise a method that yields accurate inferences for arbitrary target priors. Furthermore, like IS, we want to make use of the pre-inferred false posterior, without simply running standard inference algorithms on the target posterior. Note that most standard inference algorithms are iterative and *data-dependent*: parameter updates at each iteration involve data, and the computational cost or quality of each update depends on the amount of data used. Hence, running inference algorithms directly on the target posterior can be costly (especially given a large amount of data or many target priors of interest) and defeats the purpose of using a convenient false prior.

In this paper, we propose *prior swapping*, an iterative, *data-independent* method for generating accurate posterior samples under arbitrary target priors. Prior swapping uses the pre-inferred false posterior to perform efficient updates that

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do not depend on the data, and thus proceeds very quickly. We therefore advocate breaking difficult inference problems into two easier steps: first, do inference using the most computationally convenient prior for a given model, and then, for all future priors of interest, use prior swapping.

In the following sections, we demonstrate the pitfalls of using IS, describe the proposed prior swapping methods for different types of false posterior inference results (e.g. exact or approximate density functions, or samples) and give theoretical guarantees for these methods. Finally, we show empirical results on heavy-tailed and sparsity priors in Bayesian generalized linear models, and relational priors over components in mixture and topic models.

## 2. Methodology

Suppose we have a dataset of  $n$  vectors  $x^n = \{x_1, \dots, x_n\}$ ,  $x_i \in \mathbb{R}^p$ , and we have chosen a family of models with the likelihood function  $L(\theta|x^n) = p(x^n|\theta)$ , parameterized by  $\theta \in \mathbb{R}^d$ . Suppose we have a prior distribution over the space of model parameters  $\theta$ , with probability density function (PDF)  $\pi(\theta)$ . The likelihood and prior define a joint model with PDF  $p(\theta, x^n) = \pi(\theta)L(\theta|x^n)$ . In Bayesian inference, we are interested in computing the posterior (conditional) distribution of this joint model, with PDF

$$p(\theta|x^n) = \frac{\pi(\theta)L(\theta|x^n)}{\int \pi(\theta)L(\theta|x^n) d\theta}. \quad (1)$$

Suppose we've chosen a different prior distribution  $\pi_f(\theta)$ , which we refer to as a *false prior* (while we refer to  $\pi(\theta)$  as the *target prior*). We can now define a new posterior

$$p_f(\theta|x^n) = \frac{\pi_f(\theta)L(\theta|x^n)}{\int \pi_f(\theta)L(\theta|x^n) d\theta} \quad (2)$$

which we refer to as a *false posterior*.

We are interested in the following task: given a false posterior inference result (i.e. samples from  $p_f(\theta|x^n)$ , or some exact or approximate PDF), choose an arbitrary target prior  $\pi(\theta)$  and efficiently sample from the associated target posterior  $p(\theta|x^n)$ —or, more generally, compute an expectation  $\mu_h = \mathbb{E}_p[h(\theta)]$  for some test function  $h(\theta)$  with respect to the target posterior.

### 2.1. Importance Sampling and Prior Sensitivity

We begin by describing an initial strategy, and existing work in a related task known as prior sensitivity analysis.

Suppose we have  $T$  false posterior samples  $\{\tilde{\theta}_t\}_{t=1}^T \sim p_f(\theta|x^n)$ . In importance sampling (IS), samples from an importance distribution are used to estimate the expectation of a test function with respect to a target distribution. A straightforward idea is to use the false posterior as an

importance distribution, and compute the IS estimate

$$\hat{\mu}_h^{\text{IS}} = \sum_{t=1}^T w(\tilde{\theta}_t)h(\tilde{\theta}_t) \quad (3)$$

where the weight function  $w(\theta) \propto \frac{p(\theta|x^n)}{p_f(\theta|x^n)} \propto \frac{\pi(\theta)}{\pi_f(\theta)}$ , and the  $T$  weights are normalized to sum to one.

IS-based methods have been developed for the task of *prior sensitivity analysis* (PSA). In PSA, the goal is to determine how the posterior varies over a sequence of priors (e.g. over a parameterized family of priors  $\pi(\theta; \gamma_i)$ ,  $i = 0, 1, \dots$ ). Existing work has proposed inferring a single posterior under prior  $\pi(\theta; \gamma_0)$ , and then using IS methods to infer further posteriors in the sequence (Besag et al., 1995; Hastings, 1970; Bornn et al., 2010).

This strategy is effective when subsequent priors are similar enough, but breaks down when two priors are sufficiently dissimilar, or are from ill-matched parametric families, which we illustrate in an example below.

Note that, in general for IS, as  $T \rightarrow \infty$ ,  $\hat{\mu}_h^{\text{IS}} \rightarrow \mu_h$  almost surely. However, IS estimates can still fail in practice if  $\hat{\mu}_h^{\text{IS}}$  has high or infinite variance. If so, the variance of the weights  $w(\tilde{\theta}_t)$  will be large (a problem often referred to as weight degeneracy), which can lead to inaccurate estimates. In our case, the variance of  $\hat{\mu}_h^{\text{IS}}$  is only finite if

$$\mathbb{E}_{p_f} \left[ h(\theta)^2 \frac{\pi(\theta)^2}{\pi_f(\theta)^2} \right] < \infty. \quad (4)$$

For a broad class of  $h$ , this is satisfied if there exists  $M \in \mathbb{R}$  such that  $\frac{\pi(\theta)}{\pi_f(\theta)} < M, \forall \theta$  (Geweke, 1989). Given some pre-inferred  $p_f(\theta|x^n)$  with false prior  $\pi_f(\theta)$ , the accuracy of IS thus depends on the target prior of interest. For example, if  $\pi(\theta)$  has heavier tails than  $\pi_f(\theta)$ , the variance of  $\hat{\mu}_h^{\text{IS}}$  will be infinite for many  $h$ . Intuitively, we expect the variance to be higher for  $\pi$  that are more dissimilar to  $\pi_f$ .

We show a concrete example of this in Fig. 1. Consider a normal model for data  $x^n \sim \mathcal{N}(\theta, 1)$ , with a standard normal false prior  $\pi_f(\theta) = \mathcal{N}(\theta|0, 1)$ . This yields a closed-form false posterior (due to the conjugate  $\pi_f$ ), which is also normal. Suppose we'd like to estimate the posterior expectation under a Laplace target prior, with mean 10 and variance 1, for test function  $h(\theta) = \theta$  (i.e. an estimate of the target posterior mean). We draw  $T$  false posterior samples  $\{\tilde{\theta}_t\}_{t=1}^T \sim p_f(\theta|x^n)$ , compute weights  $w(\tilde{\theta}_t)$  and IS estimate  $\hat{\mu}_h^{\text{IS}}$ , and compare it with the true expectation  $\mu_h$ .

We see in Fig. 1 that  $|\mu_h - \hat{\mu}_h^{\text{IS}}|$  slows significantly as  $T$  increases, and maintains a high error even as  $T$  is made very large. We can analyze this issue theoretically. Suppose we want  $|\mu_h - \hat{\mu}_h^{\text{IS}}| < \delta$ . Since we know  $p_f(\theta|x^n)$  is normal, we can compute a lower bound on the number of false posterior samples  $T$  that would be needed for

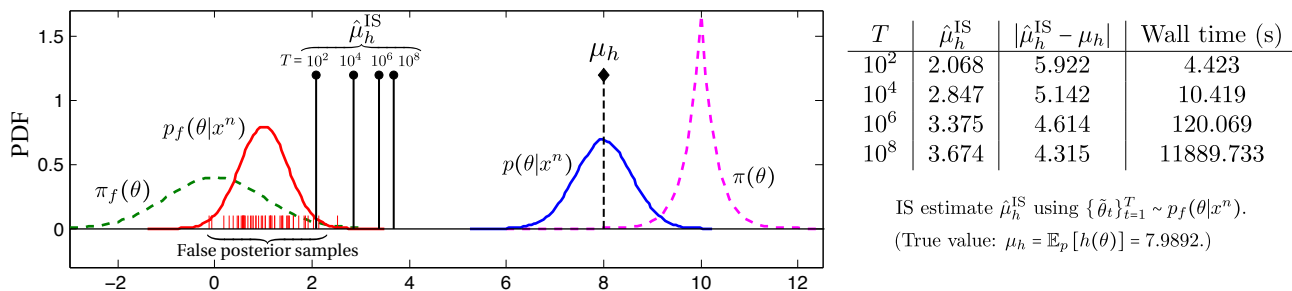


Figure 1. Importance sampling with false posterior samples. As the number of samples  $T$  grows, the difference between the IS estimate  $\hat{\mu}_h^{\text{IS}}$  and the true value  $\mu_h$  decreases increasingly slowly. The difference remains large even when  $T = 10^8$ . See text for analysis.

the expected estimate to be within  $\delta$  of  $\mu_h$ . Namely, if  $p_f(\theta|x^n) = \mathcal{N}(\theta|m, s^2)$ , in order for  $|\mu_h - \mathbb{E}_{p_f}[\hat{\mu}_h^{\text{IS}}]| < \delta$ , we’d need

$$T \geq \exp \left\{ \frac{1}{2s^2} (|\mu_h - m| - \delta)^2 \right\}.$$

In the example in Fig. 1, we have  $m = 1$ ,  $s^2 = 0.25$ , and  $\mu_h = 7.9892$ . Hence, for  $|\mu_h - \mathbb{E}_{p_f}[\hat{\mu}_h^{\text{IS}}]| < 1$ , we’d need  $T > 10^{31}$  samples (see appendix for full details of this analysis). Note that this bound actually has nothing to do with the parametric form of  $\pi(\theta)$ —it is based solely on the normal false posterior, and its distance to the target posterior mean  $\mu_h$ . However, even if this distance was small, the importance estimate would still have infinite variance due to the Laplace target prior. Further, note that the situation can significantly worsen in higher dimensions, or if the false posterior has a lower variance.

## 2.2. Prior Swapping

We’d like a method that will work well even when false and target priors  $\pi_f(\theta)$  and  $\pi(\theta)$  are significantly different, or are from different parametric families, with performance that does not worsen (in accuracy nor computational complexity) as the priors are made more dissimilar.

Redoing inference for each new target posterior can be very costly, especially when the data size  $n$  is large, because the per-iteration cost of most standard inference algorithms scales with  $n$ , and many iterations may be needed for accurate inference. This includes both MCMC and sequential monte carlo (SMC) algorithms (i.e. repeated-IS-methods that infer a sequence of distributions). In SMC, the per-iteration cost still scales with  $n$ , and the variance estimates can still be infinite if subsequent distributions are ill-matched.

Instead, we aim to leverage the inferred false posterior to more-efficiently compute any future target posterior. We begin by defining a *prior swap density*  $p_s(\theta)$ . Suppose for now that a false posterior inference algorithm has returned a density function  $\tilde{p}_f(\theta)$  (we will give more details on  $\tilde{p}_f$

later; assume for now that it is either equal to  $p_f(\theta|x^n)$  or approximates it). We then define the prior swap density as

$$p_s(\theta) \propto \frac{\tilde{p}_f(\theta)\pi(\theta)}{\pi_f(\theta)}. \quad (5)$$

Note that if  $\tilde{p}_f(\theta) = p_f(\theta|x^n)$ , then  $p_s(\theta) = p(\theta|x^n)$ . However, depending on how we represent  $\tilde{p}_f(\theta)$ ,  $p_s(\theta)$  can have a much simpler analytic representation than  $p(\theta|x^n)$ , which is typically defined via a likelihood function (i.e. a function of the data) and causes inference algorithms to have costs that scale with the data size  $n$ . Specifically, we will only use low-complexity  $\tilde{p}_f(\theta)$  that can be evaluated in constant time with respect to the data size  $n$ .

Our general strategy is to use  $p_s(\theta)$  as a surrogate for  $p(\theta|x^n)$  in standard MCMC or optimization procedures, to yield data-independent algorithms with constant cost per iteration. Intuitively, the likelihood information is captured by the false posterior—we make use of this instead of the likelihood function, which is costly to evaluate.

More concretely, at each iteration in standard inference algorithms, we must evaluate a data-dependent function associated with the posterior density. For example, we evaluate a function proportional to  $p(\theta|x^n)$  in Metropolis-Hastings (MH) (Metropolis et al., 1953), and  $\nabla_{\theta} \log p(\theta|x^n)$  in gradient-based MCMC methods (such as Langevin dynamics (LD) (Rosicky et al., 1978) and Hamiltonian Monte Carlo (HMC) (Neal, 2011)) and in optimization procedures that yield a MAP point estimate. In prior swapping, we instead evaluate  $p_s(\theta)$  in MH, or  $\nabla_{\theta} \log p_s(\theta)$  in LD, HMC, or gradient optimization to a MAP estimate (see appendix for algorithm pseudocode). Here, each iteration only requires evaluating a few simple analytic expressions, and thus has  $O(1)$  complexity with respect to data size.

We demonstrate prior swapping on our previous example (using a normal false prior and Laplace target prior) in Fig. 2, where we have a closed-form (normal PDF)  $\tilde{p}_f(\theta)$ . To do prior swapping, we run a Metropolis-Hastings algorithm on the target density  $p_s(\theta)$ . Note that drawing each

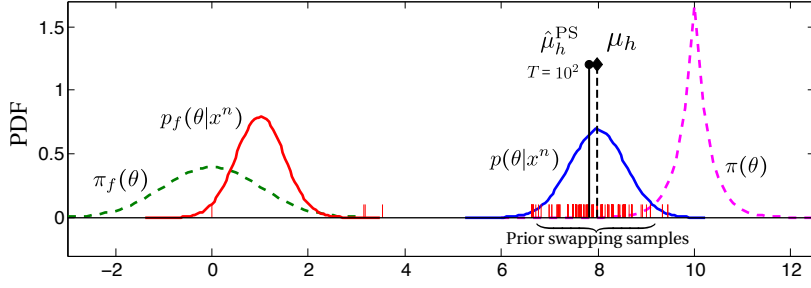


Figure 2. Using prior swapping to compute estimate  $\hat{\mu}_h^{\text{PS}}$  by drawing samples  $\{\theta_t\}_{t=1}^T \sim p_s(\theta)$ .

$T$	$\hat{\mu}_h^{\text{PS}}$	$ \hat{\mu}_h^{\text{PS}} - \mu_h $	Wall time (s)
$10^2$	7.823	0.166	0.039
$10^3$	8.021	0.032	0.381
$10^4$	7.979	0.010	3.808
$10^5$	7.996	0.007	38.027

Sample estimate  $\hat{\mu}_h^{\text{PS}}$  using  $\{\theta_t\}_{t=1}^T \sim p_s(\theta)$ .  
(True value:  $\mu_h = \mathbb{E}_p[h(\theta)] = 7.9892$ .)

sample in this Markov chain does not involve the data  $x^n$ , and can be done in constant time with respect to  $n$  (which we can see by viewing the wall time for different  $T$ ). In Fig. 2, we draw  $T$  samples  $\{\theta_t\}_{t=1}^T \sim p_s(\theta)$ , compute a sample estimate  $\hat{\mu}_h^{\text{PS}} = \frac{1}{T} \sum_{t=1}^T \theta_t$ , and compare it with the true value  $\mu_h$ . We see that  $\hat{\mu}_h^{\text{PS}}$  converges to  $\mu_h$  after a relatively small number of samples  $T$ .

### 2.3. Prior Swapping with False Posterior Samples

The previous method is only applicable if our false posterior inference result is a PDF  $\tilde{p}_f(\theta)$  (such as in closed-form inference or variational approximations). Here, we develop prior swapping methods for the setting where we only have access to samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f} \sim p_f(\theta|x^n)$ . We propose the following procedure:

1. Use  $\{\tilde{\theta}_t\}_{t=1}^{T_f}$  to form an estimate  $\tilde{p}_f(\theta) \approx p_f(\theta|x^n)$ .
2. Sample from  $p_s(\theta) \propto \frac{\pi(\theta)\tilde{p}_f(\theta)}{\pi_f(\theta)}$  with prior swapping, as before.

Note that, in general,  $p_s(\theta)$  only approximates  $p(\theta|x^n)$ . As a final step, after sampling from  $p_s(\theta)$ , we can:

3. Apply a correction to samples from  $p_s(\theta)$ .

We will describe two methods for applying a correction to  $p_s$  samples—one involving importance sampling, and one involving semiparametric density estimation. Additionally, we will discuss forms for  $\tilde{p}_f(\theta)$ , guarantees about these forms, and how to optimize the choice of  $\tilde{p}_f(\theta)$ . In particular, we will argue why (in contrast to the initial IS strategy) these methods do not fail when  $p(\theta|x^n)$  and  $p_f(\theta|x^n)$  are very dissimilar or have ill-matching parametric forms.

**Prior swap importance sampling.** Our first proposal for applying a correction to prior swap samples involves IS: after estimating some  $\tilde{p}_f(\theta)$ , and sampling  $\{\theta_t\}_{t=1}^T \sim p_s(\theta)$ , we can treat  $\{\theta_t\}_{t=1}^T$  as importance samples, and compute the IS estimate

$$\hat{\mu}_h^{\text{PSis}} = \sum_{t=1}^T w(\theta_t) h(\theta_t) \quad (6)$$

where the weight function is now

$$w(\theta) \propto \frac{p(\theta|x^n)}{p_s(\theta)} \propto \frac{p_f(\theta|x^n)}{\tilde{p}_f(\theta)} \quad (7)$$

and the weights are normalized so that  $\sum_{t=1}^T w(\theta_t) = 1$ .

The key difference between this and the previous IS strategy is the weight function. Recall that, previously, an accurate estimate depended on the similarity between  $\pi(\theta)$  and  $\pi_f(\theta)$ ; both the distance to and parametric form of  $\pi(\theta)$  could produce high or infinite variance estimates. This was an issue because we wanted the procedure to work well for any  $\pi(\theta)$ . Now, however, the performance depends on the similarity between  $\tilde{p}_f(\theta)$  and  $p_f(\theta|x^n)$ —and by using the false posterior samples, we can estimate a  $\tilde{p}_f(\theta)$  that well approximates  $p_f(\theta|x^n)$ . Additionally, we can prove that certain choices of  $\tilde{p}_f(\theta)$  guarantee a finite variance IS estimate. Note that the variance of  $\hat{\mu}_h^{\text{PSis}}$  is only finite if

$$\mathbb{E}_{p_f} \left[ h(\theta)^2 \frac{p_f(\theta|x^n)^2}{\tilde{p}_f(\theta)^2} \right] \propto \mathbb{E}_p \left[ h(\theta)^2 \frac{p_f(\theta|x^n)}{\tilde{p}_f(\theta)} \right] < \infty.$$

To bound this, it is sufficient to show that there exists  $M \in \mathbb{R}$  such that  $\frac{p_f(\theta|x^n)}{\tilde{p}_f(\theta)} < M$  for all  $\theta$  (assuming a test function  $h(\theta)$  with finite variance) (Geweke, 1989). To satisfy this condition, we will propose a certain parametric family  $\tilde{p}_f^\alpha(\theta)$ . Note that, to maintain a prior swapping procedure with  $O(1)$  cost, we want a  $\tilde{p}_f^\alpha(\theta)$  that can be evaluated in constant time. In general, a  $\tilde{p}_f^\alpha(\theta)$  with fewer terms will yield a faster procedure. With these in mind, we propose the following family of densities.

**Definition.** For a parameter  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_j \in \mathbb{R}^p$ ,  $k > 0$ , let density  $\tilde{p}_f^\alpha(\theta)$  satisfy

$$\tilde{p}_f^\alpha(\theta) \propto \pi_f(\theta) \prod_{j=1}^k p(\alpha_j|\theta)^{n/k} \quad (8)$$

where  $p(\alpha_j|\theta)$  denotes the model conditional PDF.

The number of terms in  $\tilde{p}_f^\alpha(\theta)$  (and cost to evaluate) is determined by the parameter  $k$ . Note that this family is



inspired by the true form of the false posterior  $p_f(\theta|x^n)$ . However,  $\tilde{p}_f^\alpha(\theta)$  has constant-time evaluation, and we can estimate its parameter  $\alpha$  using samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f} \sim p_f(\theta|x^n)$ . Furthermore, we have the following guarantees.

**Theorem 2.1.** *For any  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^p$  and  $k > 0$  let  $\tilde{p}_f^\alpha(\theta)$  be defined as in Eq. (8). Then, there exists  $M > 0$  such that  $\frac{p_f(\theta|x^n)}{\tilde{p}_f^\alpha(\theta)} < M$ , for all  $\theta \in \mathbb{R}^d$ .*

**Corollary 2.1.1.** *For  $\{\theta_t\}_{t=1}^T \sim p_s^\alpha(\theta) \propto \frac{\tilde{p}_f^\alpha(\theta)\pi(\theta)}{\pi_f(\theta)}$ ,  $w(\theta_t) = \frac{p_f(\theta_t|x^n)}{\tilde{p}_f^\alpha(\theta_t)} \left( \sum_{r=1}^T \frac{p_f(\theta_r|x^n)}{\tilde{p}_f^\alpha(\theta_r)} \right)^{-1}$ , and test function that satisfies  $\text{Var}_p[h(\theta)] < \infty$ , the variance of IS estimate  $\hat{\mu}_h^{\text{PSis}} = \sum_{t=1}^T h(\theta_t)w(\theta_t)$  is finite.*

Proofs for these theorems are given in the appendix.

Note that we do not know the normalization constant for  $\tilde{p}_f^\alpha(\theta)$ . This is not an issue for its use in prior swapping, since we only need access to a function proportional to  $p_s^\alpha(\theta) \propto \tilde{p}_f^\alpha(\theta)\pi(\theta)\pi_f(\theta)^{-1}$  in most MCMC algorithms. However, we still need to estimate  $\alpha$ , which is an issue because the unknown normalization constant is a function of  $\alpha$ . Fortunately, we can use the method of *score matching* (Hyvärinen, 2005) to estimate  $\alpha$  given a density such as  $\tilde{p}_f^\alpha(\theta)$  with unknown normalization constant.

Once we have found an optimal parameter  $\alpha^*$ , we draw samples from  $p_s^{\alpha^*}(\theta) \propto \tilde{p}_f^{\alpha^*}(\theta)\pi(\theta)\pi_f(\theta)^{-1}$ , compute weights for these samples (Eq. (7)), and compute the IS estimate  $\hat{\mu}_h^{\text{PSis}}$ . We give pseudocode for the full prior swap importance sampling procedure in Alg. 1.

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#### Algorithm 1: Prior Swap Importance Sampling

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**Input:** False posterior samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f} \sim p_f(\theta|x^n)$ .

**Output:** IS estimate  $\hat{\mu}_h^{\text{PSis}}$ .

- 1 Score matching: estimate  $\alpha^*$  using  $\{\tilde{\theta}_t\}_{t=1}^{T_f}$ .
  - 2 Prior swapping: sample  $\{\theta_t\}_{t=1}^T \sim p_s^{\alpha^*}(\theta) \propto \frac{\tilde{p}_f^{\alpha^*}(\theta)\pi(\theta)}{\pi_f(\theta)}$ .
  - 3 Importance sampling: compute  $\hat{\mu}_h^{\text{PSis}} = \sum_{t=1}^T h(\theta_t)w(\theta_t)$ .
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**Semiparametric prior swapping.** In the previous method, we chose a parametric form for  $\tilde{p}_f^\alpha(\theta)$ ; in general, even the optimal  $\alpha$  will yield an inexact approximation to  $p_f(\theta|x^n)$ . Here, we aim to incorporate methods that return an increasingly exact estimate  $\tilde{p}_f(\theta)$  when given more false posterior samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f}$ .

One idea is to use a nonparametric kernel density estimate  $\tilde{p}_f^{np}(\theta)$  and plug this into  $p_s^{np}(\theta) \propto \tilde{p}_f^{np}(\theta)\pi(\theta)\pi_f(\theta)^{-1}$ . However, nonparametric density estimates can yield inaccurate density tails and fare badly in high dimensions. To help mitigate these problems, we turn to a semiparametric estimate, which begins with a parametric estimate, and

adjusts it as samples are generated. In particular, we use a density estimate that can be viewed as the product of a parametric density estimate and a nonparametric correction function (Hjort & Glad, 1995). This density estimate is consistent as the number of samples  $T_f \rightarrow \infty$ . Instead of (or in addition to) correcting prior swap samples with importance sampling, we can correct them by updating the nonparametric correction function as we continue to generate false posterior samples.

Given  $T_f$  samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f} \sim p_f(\theta|x^n)$ , we write the semiparametric false posterior estimate as

$$\tilde{p}_f^{sp}(\theta) = \frac{1}{T_f} \sum_{t=1}^{T_f} \left[ \frac{1}{b^d} K \left( \frac{\|\theta - \tilde{\theta}_t\|}{b} \right) \frac{\tilde{p}_f^\alpha(\theta)}{\tilde{p}_f^\alpha(\tilde{\theta}_t)} \right], \quad (9)$$

where  $K$  denotes a probability density kernel, with bandwidth  $b$ , where  $b \rightarrow 0$  as  $T_f \rightarrow \infty$  (see (Wasserman, 2006) for details on probability density kernels and bandwidth selection). The semiparametric prior swap density is then

$$\begin{aligned} p_s^{sp}(\theta) &\propto \frac{\tilde{p}_f^{sp}(\theta)\pi(\theta)}{\pi_f(\theta)} = \frac{1}{T_f} \sum_{t=1}^{T_f} \frac{K \left( \frac{\|\theta - \tilde{\theta}_t\|}{b} \right) \tilde{p}_f^\alpha(\theta)\pi(\theta)}{\tilde{p}_f^\alpha(\tilde{\theta}_t)\pi_f(\theta)b^d} \\ &\propto [p_s^\alpha(\theta)] \left[ \frac{1}{T_f} \sum_{t=1}^{T_f} \frac{K \left( \frac{\|\theta - \tilde{\theta}_t\|}{b} \right)}{\tilde{p}_f^\alpha(\tilde{\theta}_t)} \right]. \end{aligned} \quad (10)$$

Hence, the prior swap density  $p_s^{sp}(\theta)$  is proportional to the product of two densities: the parametric prior swap density  $p_s^\alpha(\theta)$ , and a correction density. To estimate expectations with respect to  $p_s^{sp}(\theta)$ , we can follow Alg. 1 as before, but replace the weight function in the final IS estimate with

$$w(\theta) \propto \frac{p_s^{sp}(\theta)}{p_s^\alpha(\theta)} \propto \frac{1}{T_f} \sum_{t=1}^{T_f} \frac{K \left( \frac{\|\theta - \tilde{\theta}_t\|}{b} \right)}{\tilde{p}_f^\alpha(\tilde{\theta}_t)}. \quad (11)$$

One advantage of this strategy is that computing the weights doesn't require the data—it thus has constant cost with respect to data size  $n$  (though its cost does increase with the number of false posterior samples  $T_f$ ). Additionally, as in importance sampling, we can prove that this procedure yields an exact estimate of  $\mathbb{E}[h(\theta)]$ , asymptotically, as  $T_f \rightarrow \infty$  (and we can provide an explicit bound on the rate at which  $p_s^{sp}(\theta)$  converges to  $p(\theta|x^n)$ ). We do this by showing that  $p_s^{sp}(\theta)$  is consistent for  $p(\theta|x^n)$ .

**Theorem 2.2.** *Given false posterior samples  $\{\tilde{\theta}_t\}_{t=1}^{T_f} \sim p_f(\theta|x^n)$  and  $b \asymp T_f^{-1/(4+d)}$ , the estimator  $p_s^{sp}$  is consistent for  $p(\theta|x^n)$ , i.e. its mean-squared error satisfies*

$$\sup_{p(\theta|x^n)} \mathbb{E} \left[ \int (p_s^{sp}(\theta) - p(\theta|x^n))^2 d\theta \right] < \frac{c}{T_f^{4/(4+d)}}$$

for some  $c > 0$  and  $0 < b \leq 1$ .

The proof for this theorem is given in the appendix.

### 3. Empirical Results

We show empirical results on Bayesian generalized linear models (including linear and logistic regression) with sparsity and heavy tailed priors, and on latent factor models (including mixture models and topic models) with relational priors over factors (e.g. diversity-encouraging, agglomerate-encouraging, etc.). We aim to demonstrate empirically that prior swapping efficiently yields correct samples and, in some cases, allows us to apply certain inference algorithms to more-complex models than was previously possible. In the following experiments, we will refer to the following procedures:

- **Target posterior inference:** some standard inference algorithm (e.g. MCMC) run on  $p(\theta|x^n)$ .
- **False posterior inference:** some standard inference algorithm run on  $p_f(\theta|x^n)$ .
- **False posterior IS:** IS using samples from  $p_f(\theta|x^n)$ .
- **Prior swap exact:** prior swapping with closed-form  $\tilde{p}_f(\theta) = p_f(\theta|x^n)$ .
- **Prior swap parametric:** prior swapping with parametric  $\tilde{p}_f^\alpha(\theta)$  given by Eq. (8).
- **Prior swap IS:** correcting samples from  $\tilde{p}_f^\alpha(\theta)$  with IS.
- **Prior swap semiparametric:** correcting samples from  $\tilde{p}_f^\alpha(\theta)$  with the semiparametric estimate IS procedure.

To assess performance, we choose a test function  $h(\theta)$ , and compute the Euclidean distance between  $\mu_h = \mathbb{E}_p[h(\theta)]$  and some estimate  $\hat{\mu}_h$  returned by a procedure. We denote this performance metric by *posterior error* =  $\|\mu_h - \hat{\mu}_h\|_2$ . Since  $\mu_h$  is typically not available analytically, we run a single chain of MCMC on the target posterior for one million steps, and use these samples as ground truth to compute  $\mu_h$ . For timing plots, to assess error of a method at a given time point, we collect samples drawn before this time point, remove the first quarter as burn in, and add the time it takes to compute any of the corrections.

#### 3.1. Sparsity Inducing and Heavy Tailed Priors in Bayesian Generalized Linear Models

Sparsity-encouraging regularizers have gained a high level of popularity over the past decade due to their ability to produce models with greater interpretability and parsimony. For example, the  $L_1$  norm has been used to induce sparsity with great effect (Tibshirani, 1996), and has been shown to be equivalent to a mean-zero independent Laplace prior (Tibshirani, 1996; Seeger, 2008). In a Bayesian setting, inference given a sparsity prior can be difficult, and often requires a computationally intensive method (such as MH or

HMC) or posterior approximations (e.g. expectation propagation (Minka, 2001)) that make factorization or parametric assumptions (Seeger, 2008; Gerwinn et al., 2010). We propose a cheap yet accurate solution: first get an inference result with a more-tractable prior (such as a normal prior), and then use prior swapping to quickly convert the result to the posterior given a sparsity prior.

Our first set of experiments are on Bayesian linear regression models, which we can write as  $y_i = X_i\theta + \epsilon$ ,  $\epsilon \sim \mathcal{N}(0, \sigma^2)$ ,  $\theta \sim \pi$ ,  $i = 1, \dots, n$ . For  $\pi$ , we compute results on Laplace, Student’s t, and VerySparse (with PDF  $\text{VerySparse}(\sigma) = \prod_{i=1}^d \frac{1}{2\sigma} \exp\{-|\theta_i|^{0.4}/\sigma\}$  (Seeger, 2008)) priors. Here, a normal  $\pi_f$  is conjugate and allows for exact false posterior inference. Our second set of experiments are on Bayesian logistic regression models, which we write as  $y_i \sim \text{Bern}(p_i)$ ,  $p_i = \text{logistic}(X_i\theta)$ ,  $\theta \sim \pi$ ,  $i = 1, \dots, n$ . which we will pair with both heavy tailed priors and a hierarchical target prior  $\pi = \mathcal{N}(0, \alpha^{-1}I)$ ,  $\alpha \sim \text{Gamma}(\gamma, 1)$ . For these experiments, we also use a normal  $\pi_f$ . However, this false prior is no longer conjugate, and so we use MCMC to sample from  $p_f(\theta|x^n)$ .

For linear regression, we use the YearPredictionMSD data set\*, ( $n = 515345$ ,  $d = 90$ ), in which regression is used to predict the year associated with a song, and for logistic regression we use the MiniBooNE particle identification data set†, ( $n = 130065$ ,  $d = 50$ ), in which binary classification is used to distinguish particles.

In Fig. 3, we compare prior swapping and IS methods, in order to show that the prior swapping procedures yield accurate posterior estimates, and to compare their speeds of convergence. We plot posterior error vs. wall time for each method’s estimate of the posterior mean  $\mathbb{E}_p[h(\theta)] = \mathbb{E}_p[\theta]$  for two sparsity target priors (Laplace and VerySparse), for both linear and logistic regression. In linear regression (only), since the normal conjugate  $\pi_f$  allows us to compute a closed form  $p_f(\theta|x^n)$ , we can run the *prior swap exact* method, where  $\tilde{p}_f(\theta) = p_f(\theta|x^n)$ . However, we can also sample from  $p_f(\theta|x^n)$  to compute  $\tilde{p}_f^{\alpha*}(\theta)$ , and therefore compare methods such as *prior swap parametric* and the two correction methods. In logistic regression, we do not have a closed form  $p_f(\theta|x^n)$ ; here, we only compare the methods that make use of samples from  $p_f(\theta|x^n)$ . In Fig. 3, we see that the prior swapping methods (particularly prior swap IS) quickly converge to nearly zero posterior error. Additionally, in linear regression, we see that prior swap parametric, using  $\tilde{p}_f(\theta) = \tilde{p}_f^{\alpha*}(\theta)$ , yields similar posterior error as prior swap exact, which uses  $\tilde{p}_f(\theta) = p(\theta|x^n)$ .

\*<https://archive.ics.uci.edu/ml/datasets/YearPredictionMSD>

†<https://archive.ics.uci.edu/ml/datasets/MiniBooNE+particle+identification>

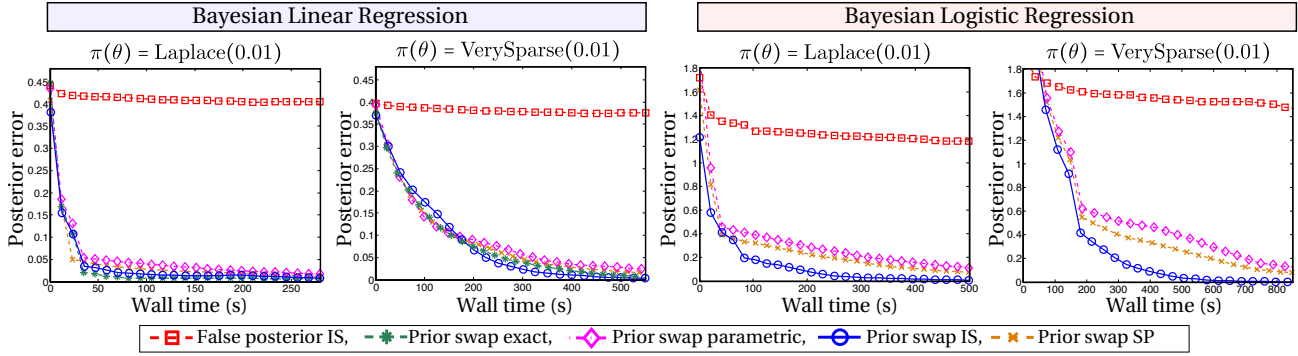


Figure 3. Comparison of prior swapping and IS methods for Bayesian linear and logistic regression under Laplace and VerySparse target priors. The prior swapping methods (particularly prior swap exact and prior swap IS) quickly converge to low posterior errors.

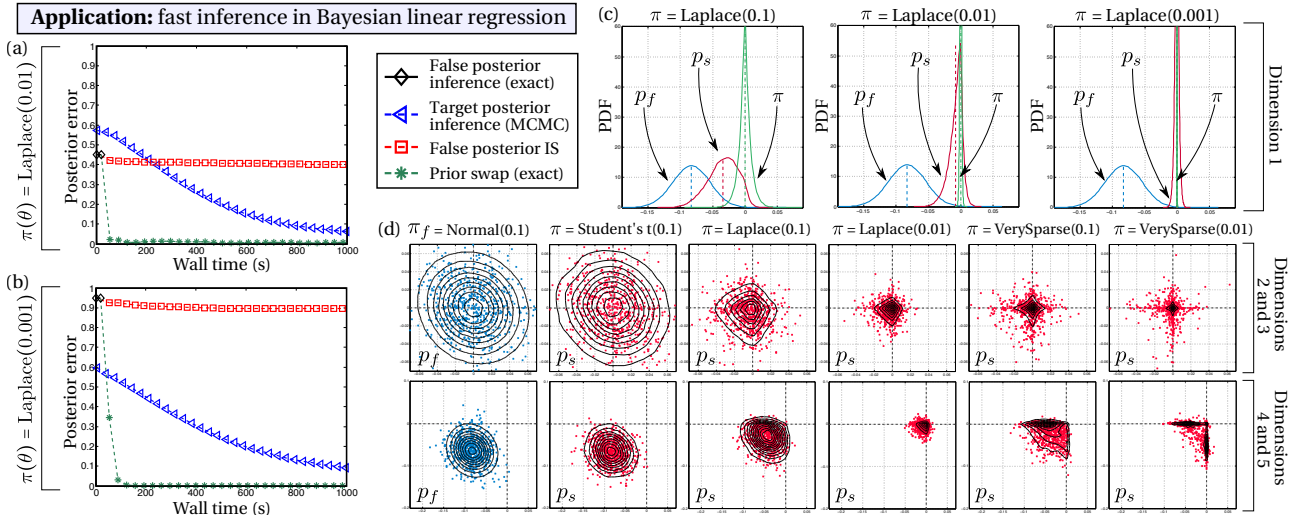


Figure 4. Prior swapping for fast inference in Bayesian linear models with sparsity and heavy-tailed priors: (a-b) Convergence plots showing that prior swapping performs accurate inference faster than the comparison methods and is robust to changing  $\pi$ . (c) Inferred 1-d density marginals when prior sparsity is increased. (d) Prior swapping results for a variety of different sparsity priors.

In Fig. 4, we show how prior swapping can be used for fast inference in Bayesian linear models with sparsity or heavy-tailed priors. We plot the time needed to first compute the false posterior (via exact inference) and then run prior swapping (via the MH procedure) on some target posterior, and compare this with the MH algorithm run directly on the target posterior. In (a) and (b) we show convergence plots and see that prior swapping performs faster inference (by a few orders of magnitude) than direct MH. In plot (b) we reduce the variance of the target prior; while this hurts the accuracy of false posterior IS, prior swapping still quickly converges to zero error. In (c) we show 1-d density marginals as we increase the prior sparsity, and in (d) we show prior swapping results for various sparsity priors.

In the appendix, we also include results on logistic regression with the hierarchical target prior, as well as results for synthetic data where we are able to compare timing and posterior error as we tune  $n$  and  $d$ .

### 3.2. Priors over Factors in Latent Variable Models

Many latent variable models in machine learning—such as mixture models, topic models, probabilistic matrix factorization, and others—involve a set of latent factors (e.g. components or topics). Often, we’d like to use priors that encourage interesting behaviors among the factors. For example, we might want dissimilar factors through a diversity-promoting prior (Kwok & Adams, 2012; Xie et al., 2016) or for the factors to show some sort of sparsity pattern (Mayrink et al., 2013; Knowles & Ghahramani, 2011). Inference in such models is often computationally expensive or designed on a case-by-case basis (Xie et al., 2016; Knowles & Ghahramani, 2011).

However, when conjugate priors are placed over the factor parameters, collapsed Gibbs sampling can be applied. In this method, the factor parameters are integrated out, leaving only a subset of variables; on these, the conditional

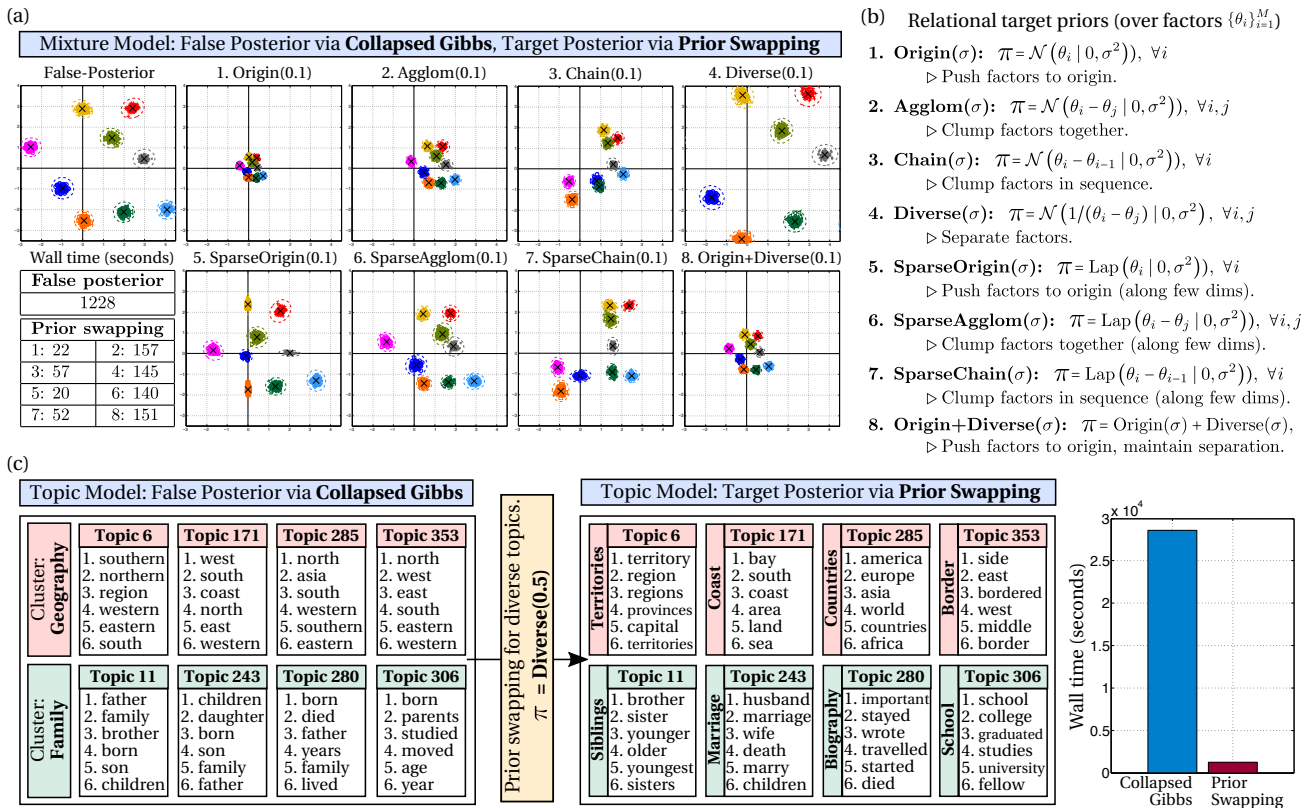


Figure 5. Latent factor models: (a) Prior swapping results for relational target priors (defined in (b)) over components in a mixture model. (c) Prior swapping with a diversity-promoting target prior on an LDA topic model (Simple English Wikipedia corpus) to separate redundant topic clusters; the top 6 words per topic are shown. In (a, c) we show wall times for the initial inference and prior swapping.

distributions can be computed analytically, which allows for Gibbs sampling over these variables. Afterwards, samples of the collapsed factor parameters can be computed. Hence, we propose the following strategy: first, assign a prior for the factor parameters that allows for collapsed Gibbs sampling; afterwards, reconstruct the factor samples and apply prior swapping for more complex relational priors over the factors. We can thus perform convenient inference in the collapsed model, yet apply more-sophisticated priors to variables in the uncollapsed model.

We first show results on a Gaussian mixture model (GMM), written  $x_i \sim \mathcal{N}(\mu_{z_i}, \Sigma_{z_i})$ ,  $z_i \sim \text{Dir}(\alpha)$ ,  $\{\mu_m\}_{m=1}^M \sim \pi$ ,  $i = 1, \dots, n$ . Using a normal  $\pi_f$  over  $\{\mu_m\}_{m=1}^M$  allows for collapsed Gibbs sampling. We also show results on a topic model (latent Dirichlet allocation (LDA) (Blei et al., 2003)) for text data (for the form of this model, see (Blei et al., 2003; Wang & Blei, 2011)). Here, using a Dirichlet  $\pi_f$  over topics allows for collapsed Gibbs sampling. For mixture models, we generate synthetic data from the above model ( $n=10,000$ ,  $d=2$ ,  $M=9$ ), and for topic models, we use the Simple English Wikipedia<sup>‡</sup> corpus ( $n=27,443$  documents, vocab=10,192 words), and set  $M=400$  topics.

<sup>‡</sup><https://simple.wikipedia.org/>

In Fig. 5, we show results for mixture and topic models. In (a) we show inferred posteriors over GMM components for a number of relational target priors, which we define in (b). In (c), we apply the diversity-promoting target prior to LDA, to separate redundant topics. Here, we show two topic clusters (“geography” and “family”) in  $p_f(\theta|x^n)$ , which are separated into distinct, yet thematically-similar, topics after prior swapping. In (a) and (c) we also show wall times of the inference methods.

## 4. Conclusion

Given some false posterior inference result, and an arbitrary target prior, we have studied methods to accurately compute the associated target posterior (or expectations with respect to it), and to do this efficiently by leveraging the pre-inferred result. We have argued and shown empirically that this strategy is effective even when the false and target posteriors are quite dissimilar. We believe that this strategy shows promise to allow a wider range of (and possibly less-costly) inference algorithms to be applied to certain models, and to allow updated or new prior information to be more-easily incorporated into models without re-incurring the full costs of standard inference algorithms.



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