6 Appendix

Proof of modified Proposition 3.1. In this version, we assume $(\mathbf{w}, \mathbf{x}, y)$ is a trajectory of (1) rather than being a trajectory of (8).

All we need to show is that for any pair of $(\mathbf{x}, y)$, there exist another pair $(\tilde{\mathbf{x}}, R_y)$, such that they give the same update. In particular, we set $\tilde{\mathbf{x}} = a\mathbf{x}$ and show that there always exists an $a \in [-1, 1]$ such that

$$(y - \mathbf{w}^\mathsf{T}\mathbf{x})\mathbf{x} = (R_y - \mathbf{w}^\mathsf{T}a\mathbf{x})a\mathbf{x}.$$

This simplifies to

$$g(a) := (\mathbf{w}^\mathsf{T}\mathbf{x})a^2 - R_y a + (y - \mathbf{w}^\mathsf{T}\mathbf{x}) = 0. \tag{15}$$

The discriminant of the quadratic (15) is

$$\begin{aligned} R_y^2 - 4\mathbf{w}^\mathsf{T}\mathbf{x}(y - \mathbf{w}^\mathsf{T}\mathbf{x}) &\geq R_y^2 - 4|\mathbf{w}^\mathsf{T}\mathbf{x}|\left(R_y + |\mathbf{w}^\mathsf{T}\mathbf{x}|\right) \\ &= \left(R_y - 2|\mathbf{w}^\mathsf{T}\mathbf{x}|\right)^2 \geq 0 \end{aligned}$$

So there always exists a solution $a \in \mathbb{R}$. Moreover, $g(-1) = R_y + y \geq 0$ and $g(1) = -R_y + y \leq 0$, so there must be a real root in $[-1, 1]$. ■

Proof of Theorem 3.2. We showed in Section 3 that Regime V trajectories are 2D. We also argued that solutions that reach $\mathbf{w}_\star$ via Regime III–IV are not unique and need not be 2D. We will now show that it's always possible to construct a 2D solution.

We begin by characterizing the set of $\mathbf{w}_\star$ reachable via Regime III–IV. Recall from Section 3 that the transition between III and IV occurs when $\|\mathbf{w}\| = R := \frac{R_y}{2R_x}$. If t_0 is the time at which this transition occurs, then for $0 \leq t \leq t_0$, the solution is $\mathbf{x} = \frac{R_x}{\|\mathbf{w}\|}\mathbf{w}$, which leads to a straight-line trajectory from $\mathbf{w}_0$ to $\mathbf{w}(t_0)$.

Now consider the part of the trajectory in Regime IV, where $t_0 \leq t \leq t_f$. As derived in Section 3, Regime IV trajectories satisfy $\dot{\mathbf{w}} = \mathbf{w}^\mathsf{T}\mathbf{x} = \frac{R_y}{2}$. These lead to $\frac{\mathrm{d}\|\mathbf{w}\|^2}{\mathrm{d}t} = \frac{R_y^2}{2}$, which means that $\|\mathbf{w}\|$ grows at the same rate regardless of $\mathbf{x}$. If our trajectory reaches $\mathbf{w}(t_f) = \mathbf{w}_\star$, then we can deduce via integration that

$$\|\mathbf{w}_\star\|^2 - \|\mathbf{w}(t_0)\|^2 = \tfrac{R_y^2}{2}(t_f - t_0), \tag{16}$$

Suppose $(\mathbf{w}(t), \mathbf{x}(t))$ for $t_0 \leq t \leq t_f$ is a trajectory that reaches $\mathbf{w}_\star$. Refer to Figure 8. The reachable set at time t_f is a spherical sector whose boundary requires a trajectory that maximizes curvature. We will now derive this fact.

Let $\theta_{\max}$ be the largest possible angle between $\mathbf{w}(t_0)$ and any reachable $\mathbf{w}(t_f) = \mathbf{w}_\star$, where we have fixed t_f. Define $\theta(t)$ to be the angle between $\mathbf{w}(t)$ and $\mathbf{w}(t_f)$.

$$\theta(t_0) = \int_{t_0}^{t_f} \dot{\theta}\,\mathrm{d}t \leq \int_{t_0}^{t_f} |\dot{\theta}|\,\mathrm{d}t$$

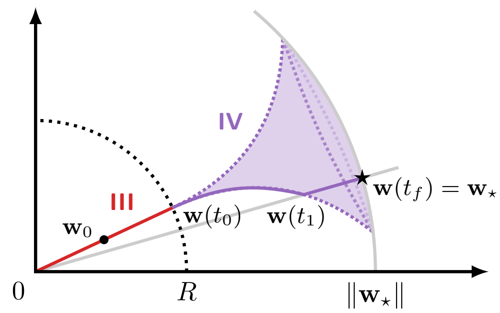

Figure 8: If a reachable $\mathbf{w}_\star$ is contained in the concave funnel shape, which is the reachable set in Regime IV, it can be reached by some trajectory $(\mathbf{w}(t), \mathbf{x}(t))$ lying entirely in the 2D subspace defined by $\text{span}\{\mathbf{w}_0, \mathbf{w}_\star\}$: follow the max-curvature solution until t_1 and then transition to a radial solution until t_f.

An alternative expression for this rate of change is the projection of $\dot{\mathbf{w}}$ onto the orthogonal complement of $\mathbf{w}$:

$$|\dot{\theta}| = \frac{\left\|\dot{\mathbf{w}} - \left(\dot{\mathbf{w}}^\mathsf{T}\frac{\mathbf{w}}{\|\mathbf{w}\|}\right)\frac{\mathbf{w}}{\|\mathbf{w}\|}\right\|}{\|\mathbf{w}\|} = \frac{R_y\left\|\mathbf{x} - \frac{R_y}{2\|\mathbf{w}\|^2}\mathbf{w}\right\|}{2\|\mathbf{w}\|}$$

Where we used the fact that $\dot{\mathbf{w}} = \mathbf{w}^\mathsf{T}\mathbf{x} = \frac{R_y}{2}$ in Regime IV. Now,

$$\begin{aligned} \theta_{\max} &= \max_{\substack{\mathbf{x}:\, \mathbf{w}^\mathsf{T}\mathbf{x} = R_y/2 \\ \|\mathbf{x}\| \leq R_x}} \theta(t_0) \\ &\leq \max_{\substack{\mathbf{x}:\, \mathbf{w}^\mathsf{T}\mathbf{x} = R_y/2 \\ \|\mathbf{x}\| \leq R_x}} \int_{t_0}^{t_f} \frac{R_y\left\|\mathbf{x} - \frac{R_y}{2\|\mathbf{w}\|^2}\mathbf{w}\right\|}{2\|\mathbf{w}\|}\,\mathrm{d}t \\ &\leq \int_{t_0}^{t_f} \frac{\sqrt{R_x^2 - \left(\frac{R_y}{2\|\mathbf{w}\|}\right)^2}}{\|\mathbf{w}\|}\,\mathrm{d}t \end{aligned} \tag{17}$$

In the final step, we maximized over $\mathbf{x}$. Notice that the integrand (17) is an upper bound that only depends on t_0 and $\|\mathbf{w}_\star\|$ but not on $\mathbf{x}$. One can also verify that this upper bound is achieved by the choice

$$\mathbf{x} = \frac{R_y}{2\|\mathbf{w}\|}\hat{\mathbf{w}} + \sqrt{R_x^2 - \left(\frac{R_y}{2\|\mathbf{w}\|}\right)^2}\,\frac{\mathbf{w}_\star - (\hat{\mathbf{w}}^\mathsf{T}\mathbf{w}_\star)\hat{\mathbf{w}}}{\|\mathbf{w}_\star - (\hat{\mathbf{w}}^\mathsf{T}\mathbf{w}_\star)\hat{\mathbf{w}}\|}.$$

where $\hat{\mathbf{w}} := \mathbf{w}/\|\mathbf{w}\|$ and $\mathbf{w}_\star$ is any vector that satisfies (16) with angle $\theta_{\max}$ with $\mathbf{w}(t_0)$. Any $\mathbf{w}_\star$ with this norm but angle $\theta_f < \theta_{\max}$ can also be reached by using the max-curvature control until time t_1, where t_1 is chosen such that $\theta_f = \int_{t_0}^{t_1} \frac{\sqrt{R_x^2 - \left(\frac{R_y}{2\|\mathbf{w}\|}\right)^2}}{\|\mathbf{w}\|}\,\mathrm{d}t$, and then using $\mathbf{x} = \frac{R_y}{2\|\mathbf{w}\|^2}\mathbf{w}$ for $t_1 \leq t \leq t_f$. This piecewise path is illustrated in Figure 8.

Our constructed optimal trajectory lies in the 2D span of $\mathbf{w}_\star$ and $\mathbf{w}_0$. This shows that all reachable $\mathbf{w}_\star$ can be reached via a 2D trajectory. ■