A. Algorithm pseudocode

Algorithm 1 Optimization with randomized telescopes

Input: initial parameter θ, gradient routine $g(\theta, i)$ which returns $\bar{G}_i(\theta)$, compute costs $\bar{C}$, exponential decay α, tuning frequency K, horizon $\bar{H}$, reference learning rate $\bar{\eta}$
Initialize $B = 0$, next_tune= 0, $D_{i,j} = 0$
repeat
  if next_tune$<= B$ **then**
    $\bar{D}, q, W, S \leftarrow$ tune$(\theta, \bar{D}, g, \bar{C}, \alpha, \bar{H})$
    expectedCompute, expectedSquaredNorm = compute_and_variance$(\bar{D}, \bar{C}, S)$
    $\eta \leftarrow \bar{\eta}\frac{\text{expectedSquaredNorm}}{\bar{D}_{0,\bar{H}}}$
    $B+ = \sum_{i=1}^{\bar{H}} \bar{C}(\bar{H})$
    next_tune $+= \bar{C}(\bar{H})$
  end if
  $N \sim q$
  for $n = 1$ **to** N **do**
    $G_n \leftarrow g(\theta, S[n])$
  end for
  $\hat{G} \leftarrow \sum_{n=1}^{N} G_n W(n, N)$
  $\theta \leftarrow \theta - \eta\hat{G}$
  if compute reused **then**
    $B+ = \bar{C}(S[N])$
  else
    $B+ = \sum_{n=1}^{N} \bar{C}(S[n])$
  end if
until converged

Algorithm 2 tune

Input: current parameter θ, current squared distance estimates $\bar{D}_{i,j}$, gradient routine $g(\theta, i)$ which returns $\bar{G}_i(\theta)$, compute costs $\bar{C}$, exponential decay α, horizon $\bar{H}$
$\bar{G}_0(\theta) \leftarrow 0$
for $i = 1$ **to** $\bar{H}$ **do**
  $\bar{G}_i(\theta) \leftarrow g(\theta, i)$
end for
for $i = 0$ **to** $\bar{H}$ **do**
  for $j = 1$ **to** $\bar{H}$ **do**
    $D_{i,j} \leftarrow ||G_i - G_j||_2^2$
  end for
end for
$\bar{D} \leftarrow \alpha\bar{D} + (1 - \alpha)D$
$S \leftarrow$ greedy_subsequence_select$(\bar{D}, \bar{C})$
$q, W \leftarrow q$_and_$W(\bar{D}, \bar{C}, S)$
Return: updated estimates $\bar{D}_{i,j}$, sampling distribution q, weight function W, and subsequence S

Algorithm 3 greedy_subsequence_select

Input: Norm estimates $\bar{D}$, compute costs $\bar{C}$
Initialize $N = \text{len}(C)$
Initialize $S^+ = [N]$, $S^- = [1, ..., N]$, converged=FALSE, bestAddCost=cost($\bar{D}, S^+, \bar{C}$), bestRemoveCost=cost($\bar{D}, S^-, \bar{C}$)
while not converged **do**

  for $i \in [i$ for $i \in [1...N]$ if not $i \in S^+]$ **do**
    trial$S$ ←sort($S^+ + [i]$)
    trialCost←cost($\bar{D}, \bar{C}$,trial$S$)
    if trialCost < bestAddCost **then**
      S^+ ←trial$S$
      bestAddCost←trialCost
      converged ← False
      BREAK
    else
      converged ← True
    end if
  end for
end while
converged ← False
while not converged **do**
  for $i \in [i$ for $i \in S^-$ if$i \neq N$ **do**
    trial$S$ ← $[j$ for $j \in S^-$if$j! = i]$
    trialCost←sequence_cost($\bar{D}, C, trial$S)
    if trialCost $<$ bestRemoveCost **then**
      S^- ←trial$S$
      bestRemoveCost←trialCost
      converged ← False
      BREAK
    else
      converged ← True
    end if
  end for
end while
if bestRemoveCost$>$ bestAddCost **then**
  Return: S^-
else
  Return: S^+
end if

Algorithm 4 compute_and_variance

Input: Norm estimates $\bar{D}$, compute costs $\bar{C}$, sequence S
$q, W \leftarrow q$_and_$W(\bar{D}, \bar{C}, S)$
expectedCompute $\leftarrow \sum_{i\in[1...|S|]} q(S[i]])\bar{C}(S[i])$
if RT-SS **then**
  expectedSquaredNorm $\leftarrow \sum_{i\in[1...|S|]} q(S[i]])W(S[i], S[i])\bar{D}_{S[i-1],S[i]}$
else if RT-RR **then**
  expectedSquaredNorm $\leftarrow \sum_{i\in[1...|S|]} \sum_{j\in[1...i]} q(S[i]])W(S[j], S[i])\bar{D}_{S[j],S[i]}$
else
  Undefined: must specify RT-SS or RT-RR
end if
Return: expectedCompute, expectedSquaredNorm

Algorithm 5 sequence_cost

Input: Norm estimates $\bar{D}$, compute costs $\bar{C}$, sequence S
expectedCompute, expectedSquaredNorm = compute_and_variance$(\bar{D}, \bar{C}, S)$
Return: expectedCompute * expectedSquaredNorm

Algorithm 6 q_and_W

Input: $\bar{D}$, $\bar{C}$, and S
if RT-SS **then**
  $q(N) \leftarrow \sqrt{\frac{\bar{D}_{S[N],S[N-1]}}{\bar{C}(S[n])}}$
  $W(n, N) \leftarrow \frac{1}{q(N)}\mathbb{1}\{n = N\}$
else if RT-RR **then**
  $\tilde{Q}(N) \leftarrow \sqrt{\frac{\bar{D}_{S[N],S[N-1]}}{\bar{C}(S[n])-\bar{C}(S[n-1])}}$
  $\tilde{(q)}(N) \leftarrow$max$(0, \tilde{Q}(N) - \tilde{Q}(N-1))$
  $q(N) \leftarrow \frac{\tilde{q}(N)}{\sum_i \tilde{q}(i)}$
  $W(n, N) \leftarrow \frac{1}{1-\sum_i q(i)}\mathbb{1}\{n \leq N\}$
else
  Undefined: must specify RT-SS or RT-RR
end if
Return: q, W

B. Proofs

B.1. Proofs for section 2

B.1.1. PROPOSITION 2.1

Unbiasedness of RT estimators. The RT estimators in (2) are unbiased estimators of Y_H as long as

$$\mathbb{E}_{N\sim q}[W(n,N)\mathbb{1}\{N \geq n\}] = \sum_{N=n}^{H} W(n,N)q(N) = 1 \quad \forall n\,.$$

Proof. A randomized telescope estimator which satisfies the above linear constraint condition has expectation:

$$\begin{aligned}
\mathbb{E}[\hat{Y}_H] &= \sum_{N=1}^{H} q(N) \sum_{n=1}^{N} W(n,N)\Delta_n \\
&= \sum_{n=1}^{H}\sum_{N=1}^{H} \Delta_n W(n,N)q(N)\mathbb{1}\{n \leq N\} \\
&= \sum_{n=1}^{H} \Delta_n \sum_{N=n}^{H} W(n,N)q(N) = \sum_{n=1}^{H} \Delta_n = Y_H
\end{aligned}$$

□

B.2. Proofs for section 4

B.2.1. THEOREM 4.1

Bounded variance and compute with polynomial convergence of ψ**.** Assume ψ converges according to $\psi_n \leq \frac{c}{(n)^p}$ or faster, for constants $p > 0$ and $c > 0$. Choose the RT-SS estimator with $q(n) \propto 1/((n)^{p+1/2})$. The resulting estimator $\hat{G}$ achieves expected compute $C \leq (\mathcal{H}_H^{p-\frac{1}{2}})^2$, where $\mathcal{H}_H^i$ is the Hth generalized harmonic number of order i, and expected squared norm $\mathbb{E}[||\hat{G}||_2^2] \leq c_\psi^2 (\mathcal{H}_H^{p-\frac{1}{2}})^2 := \tilde{G}^2$. The limit $\lim_{H\to\infty} \mathcal{H}_H^{p-\frac{1}{2}}$ is finite iff $p > \frac{3}{2}$, in which case it is given by the Riemannian zeta function, $\lim_{H\to\infty} \mathcal{H}_H^{p-\frac{1}{2}} = \zeta(p - \frac{1}{2})$. Accordingly, the estimator achieves horizon-agnostic variance and expected compute bounds iff $p > \frac{3}{2}$.

Proof. Begin by noting the RT-SS estimator returns $\frac{\Delta_n}{q_n}$ with probability $q(n)$. Let $\bar{q}(n) = \frac{1}{n^{p+\frac{1}{2}}}$ and $\sum_{n=1}^{H} \bar{q}(n) = Z$, such that $q(n) = \frac{\bar{q}(n)}{Z}$. First, note $Z = \sum_{n=1}^{H} \frac{1}{n^{p+\frac{1}{2}}} = \mathrm{H}_H^{p+\frac{1}{2}}$. Now inspect the expected squared norm $\mathbb{E}||\hat{G}||_2^2$:

$$\begin{aligned}
\sum_{n=1}^{H} q(n)||\frac{\Delta_n}{q_n}||_2^2 &= \sum_{n=1}^{H} q(n)\frac{||\Delta_n||_2^2}{q_n^2} \\
&= Z\sum_{n=1}^{H} \bar{q}(n)\frac{||\Delta_n||_2^2}{\bar{q}_n^2} \\
&\leq Zc_\psi^2 \sum_{n=1}^{H} \bar{q}(n)\frac{n^{2p+1}}{n^{2p}} \\
&= Zc_\psi^2 \sum_{n=1}^{H} \frac{n^{2p+1}}{n^{3p+\frac{1}{2}}} \\
&= Zc_\psi^2 \sum_{n=1}^{H} \frac{1}{n^{p-\frac{1}{2}}} \\
&= Zc_\psi^2 \mathrm{H}_H^{p-\frac{1}{2}} \\
&= c_\psi^2 \mathrm{H}_H^{p-\frac{1}{2}} \mathrm{H}_H^{p+\frac{1}{2}} \\
&\leq c_\psi^2 (\mathrm{H}_H^{p-\frac{1}{2}})^2
\end{aligned}$$

Now inspect the expected compute, $\mathbb{E}_{n\sim q} n$:

$$\begin{aligned}
\mathbb{E}_{n\sim q} &= \sum_{n=1}^{N} q(n)n \\
&= Z\sum_{n=1}^{H} \frac{n}{n^{p+\frac{1}{2}}} \\
&= Z\sum_{n=1}^{H} \frac{1}{n^{p-\frac{1}{2}}} \\
&= Z\mathrm{H}_H^{p-\frac{1}{2}} \\
&= \mathrm{H}_H^{p-\frac{1}{2}} \mathrm{H}_H^{p+\frac{1}{2}} \\
&\leq (\mathrm{H}_H^{p-\frac{1}{2}})^2
\end{aligned}$$

□

B.2.2. THEOREM 4.2

Bounded variance and compute with geometric convergence of ψ. Assume ψ_n converges according to $\psi_n \leq cp^n$, or faster, for $0 < p < 1$. Choose RT-SS and with $q(n) \propto p^n$. The resulting estimator $\hat{G}$ achieves expected compute $C \leq (1-p)^{-2}$ and expected squared norm $||\hat{G}||_2^2 \leq \frac{c}{(1-p)^2} := \tilde{G}^2$. Thus, the estimator achieves horizon-agnostic variance and expected compute bounds for all $0 < p < 1$.

Proof. Let $q(n) = \frac{\bar{q}(n)}{Z}$, for $\bar{q}(n) = p^n$. Note $Z = \sum_{n=1}^{H} p^n = p\frac{1-p^H}{1-p} \leq \frac{1}{1-p}$. Now, note $\psi_n = c_\psi \bar{q}(n)$. It follows

$$\begin{aligned}
\mathbb{E}_{n\sim q}||\frac{\Delta_n}{q(n)}||_2^2 &= \sum_{n=1}^{H} q(n)\frac{||\Delta_n||_2^2}{q(n)^2} \\
&\leq \sum_{n=1}^{H} q(n)\frac{\psi_n^2}{q(n)^2} \\
&=\leq c_\psi^2 \sum_{n=1}^{H} q(n)\frac{\bar{q}(n)^2}{q(n)^2} \\
&= c_\psi^2 Z^2 \sum_{n=1}^{H} q(n) \\
&= c_\psi^2 Z^2
\end{aligned}$$

Now consider the expected compute. We have

$$\begin{aligned}
\mathbb{E}_{n\sim q} n &= \sum_{n=1}^{N} nq(n) \\
&= \sum_{n=1}^{N} \frac{np^n}{Z} \\
&= \frac{1}{Z}\sum_{n=1}^{N} np^n \\
&= p\frac{1}{Z}\frac{1 + Hp^{H+1} - (H+1)p^H}{(1-p)^2} \\
&= \frac{1 + Hp^{H+1} - (H+1)p^H}{(1-p)(1-p^H)} \\
&\leq \frac{1}{(1-p)(1-p^H)} \\
&\leq \frac{1}{(1-p)^2}
\end{aligned}$$

□

B.2.3. THEOREM 4.3

Asymptotic regret bounds for optimizing infinite-horizon programs. Assume the setting from 4.1 or 4.2, and the corresponding C and $\tilde{G}$ from those theorems. Let R_t be the instantaneous regret at the tth step of optimization, $R_t = \mathcal{L}(\theta_t) - \min_\theta \mathcal{L}(\theta)$. Let $t(B)$ be the greatest t such that a computational budget B is not exceeded. Use online gradient descent with step size $\eta_t = \frac{D}{\sqrt{t}\mathbb{E}[||\hat{G}||_2^2]}$. As $B \to \infty$, the asymptotic instantaneous regret is bounded by $R_{t(B)} \leq \mathcal{O}(\tilde{G}D\sqrt{\frac{C}{B}})$, independent of H.

Proof. First, we control $t(B)$ using the central limit theorem. Note $t \to \infty \iff B(t) \to \infty$. Consider B as a function $B(t)$ of t. We have $B(t) = \sum_{\tau=1}^{t} N_t$, where $N \sim q$. Thus, $\frac{B(t)}{t} \to \mathbb{E}_{N\sim q}N$ by the central limit theorem. This implies that in the limit, $t = \frac{B}{C}$.

To complete the proof, plug in $t(B)$ and η_t, as well as the upper bound on squared norm $\mathbb{E}||\hat{G}||_2^2 \leq \tilde{G}^2$ and upper bound on diameter D, into standard results for stochastic gradient descent with convex loss functions (e.g. section 3.4 in (Hazan et al., 2016)) □

B.3. Proofs for section 5

B.3.1. THEOREM 5.1

Optimality of RT-SS under adversarial correlation. Consider the family of estimators presented in Equation 2. Assume θ, ∇_θ, and G are univariate. For any fixed sampling distribution q, the single-sample RT estimator RT-SS minimizes the worst-case variance of $\hat{G}$ across an adversarial choice of covariances $\mathrm{Cov}(\Delta_i, \Delta_j) \leq \sqrt{\mathrm{Var}(\Delta_i)}\sqrt{\mathrm{Var}(\Delta_j)}$.

Proof. Recall $\hat{G} = \sum_{n=0}^{N} \Delta_n W(n, N)$. Let $\sigma_{i,j}^2 = \mathrm{Cov}(\Delta_i, \Delta_j)$ and $\sigma_i^2 = \mathrm{Var}(\Delta_i)$. The variance of $\hat{G}$ is:

$$\begin{aligned}\mathrm{Var}(\hat{G}) &= \sum_N q(N)\Big[\sum_{i=0}^{N}\sum_{j=0}^{N} W(i,N)W(j,N)\sigma_{i,j}^2\Big] \\ &\leq \sum_N q(N)\Big[\sum_{i=0}^{N}\sum_{j=0}^{N} W(i,N)W(j,N)\sigma_i\sigma_j\Big] \\ &= \sum_N q(N)\Big(\sum_{n=0}^{N} W(n,N)\sigma_n\Big)^2\end{aligned}$$

Note the above bound is tight as the adversary can choose $\mathrm{Cov}(\Delta_i, \Delta_j) = \sigma_i\sigma_j$. Introduce $\rho(n, N) = W(n, N)q(N)$, and note that the constraint from proposition 2.1 can equivalently be stated as $\sum_{N \geq n} \rho(n, N) = 1 \forall n$. We have the variance:

$$\mathrm{Var}(\hat{G}|N) \leq \sum_N \frac{1}{q(N)}\Big(\sum_{n=0}^{N} \rho(n,N)\sigma_n\Big)^2$$

Consider finding $\rho(n, N)$ which minimizes the variance for an arbitrary q. The constrained optimization has the Lagrangian:

$$J = \Big(\sum_N \frac{1}{q(N)}(\sum_{n=0}^{N} \rho(n,N)\sigma_n)^2\Big) + \sum_n \lambda_n(\sum_{N\geq n} \rho(n,N) - 1)$$

We can accordingly optimize by taking derivatives:

$$\begin{aligned}\frac{dJ}{d\rho(n,N)} &= 2Cq(N)(\sum_{i=0}^{N} w(i,N)\sigma_i)\sigma_n + \lambda_n \\ \frac{dJ}{d\rho(n,N)} = 0 &\implies \sigma_n q(N)\sum_{i=0}^{N} w(i,N)\sigma_i = k_n \\ &\implies \sigma_n \sum_{i=0}^{N} \rho(i,N)\sigma_i = k_n \forall N \geq n \\ &\implies \rho(n,N) = 0 \forall N > n\end{aligned}$$

□

B.3.2. THEOREM 5.2

Optimal q under adversarial correlation. Consider the family of estimators presented in Equation 2. Assume $\mathrm{Cov}(\Delta_i, \Delta_i)$ and $\mathrm{Cov}(\Delta_i, \Delta_j)$ are diagonal. The RT-SS estimator with $q_n \propto \sqrt{\frac{\mathbb{E}[||\Delta_n||_2^2}{C(n)}}$ maximizes the ROE across an adversarial choice of diagonal covariance matrices $\mathrm{Cov}(\Delta_i, \Delta_j)_{kk} \leq \sqrt{\mathrm{Cov}(\Delta_i, \Delta_i)_{kk}\mathrm{Cov}(\Delta_j, \Delta_j)_{kk}}$.

Proof. First, note that by the assumption of diagonal covariance between all terms, the expected squared norm decomposes over indices k:

$$\mathbb{E}||\hat{G}||_2^2 = \sum_k \mathbb{E}\hat{G}[k]^2$$

For all choices of q, the RT-SS estimator minimizes the worst-case variance and thus (due to unbiasedness) the expected squared value of each entry in $\hat{G}$. Because the squared norm decomposes, the RT-SS estimator minimizes the squared norm for all q.

It remains to optimize q. We know $\rho(n, N) = 0 \forall N > n$. Therefore to satisfy the constraint, we have $\rho(N, N) = 1$. It follows that:

$$\text{ROE}^{-}1 = \big(\sum_N q(N)C(N)\big)\big(\sum_N \frac{\mathbb{E}||\Delta_N||_2^2}{q(N)}\big)$$

We require $\sum_N q(N) = 1$. The constrained optimization has the Lagrangian:

$$J = \Big(\sum_N q(N)C(N)\Big)\Big(\sum_N \frac{\mathbb{E}||\Delta_N||_2^2}{q(N)}\Big) + \lambda(\sum_N q(N) - 1)$$

Let $C = \Big(\sum_N q(N)C(N)\Big)$ and $V = \Big(\sum_N \frac{\mathbb{E}||\Delta_N||_2^2}{q(N)}\Big)$. We optimize $q(N)$ by taking the derivative of the inverse ROE:

$$\frac{d\text{ROE}^{-1}}{dq(N)} = C(N)V - C\frac{\sigma_N^2}{q(N)^2}$$

$$\frac{d\text{ROE}^{-1}}{dq(N)} = 0 \implies q(N)^2 \propto \frac{\mathbb{E}||\Delta_N||_2^2 C}{C(N)V}$$

$$\implies q(N) \propto \sqrt{\frac{\mathbb{E}||\Delta_N||_2^2}{C(N)}}$$

□

B.3.3. THEOREM 5.3

Optimality of RT-RR under independence. Consider the family of estimators presented in Eq. 2. Assume the Δ_j are univariate. When the Δ_j are uncorrelated, for any importance sampling distribution q, the Russian roulette estimator achieves the minimum variance in this family and thus maximizes the optimization efficiency lower bound.

Proof. By independence, we have $\mathbb{E}\big(\sum_n W(n, N)\Delta_n\big)^2 = \sum_n W(n, N)^2 \mathbb{E}\Delta_n^2$. It follows that an RT estimator has variance:

$$\text{Var}(\hat{G}) = \sum_N q(N) \sum_{n \le N} W(n, N)^2 \mathbb{E}\Delta_n^2$$

$$= \sum_N \frac{1}{q(N)} \sum_{n \le N} \rho(n, N)^2 \mathbb{E}\Delta_n^2$$

Recall the constraint in proposition 2.1 requires $\sum_{N \ge n} \rho(n, N) = 1$ for all n. The Lagrangian of the constrained minimization of $\text{Var}(\hat{G})$ with respect to ρ is:

$$J = \text{Var}(\hat{G}) + \sum_n \lambda_n(\sum_{N \ge n} \rho_n - 1)$$

We optimize ρ by finding the minimum of the Lagrangian:

$$\frac{dJ}{d\rho(n, N)} = \frac{2}{q(N)}\rho(n, N)\mathbb{E}\Delta_n^2 + \lambda_n$$

$$\frac{dJ}{d\rho(n, N)} = 0 \implies \frac{\rho(n, N)}{q(N)} = -\frac{\lambda_n}{2\mathbb{E}\Delta_n^2}$$

$$\implies W(n, N) = -\frac{\lambda_n}{2\mathbb{E}\Delta_n^2}, \text{ which is independent of } N$$

$$\implies W(n, N) = \frac{1}{\sum_{N' \ge n} q(N')} \text{ to fulfill the constraint in proposition 2.1}$$

□

B.3.4. THEOREM 5.4

Optimal q under independence. Consider the family of estimators presented in Equation 2. Assume $\mathrm{Cov}(\Delta_i, \Delta_i)$ is diagonal and Δ_i and Δ_j are independent. The RT-RR estimator with $Q(i) \propto \sqrt{\frac{\mathbb{E}[||\Delta_i||_2^2}{C(i)-C(i-1)}}]$, where $Q(i) = \Pr(n \geq i) = \sum_{j=i}^{H} q(j)$, maximizes the ROE.

Proof. First note that by theorem 5.3, for any q and for each element in the vector $\hat{G}$, the RT-RR estimator minimizes the variance of that element. Now note that due to independence of Δ_i, Δ_j and diagonality of $\mathrm{Cov}(\Delta_i, \Delta_i)$:

$$\begin{aligned}\mathbb{E}||\sum_{n=1}^{N} W(n,N)\Delta_n||_2^2 &= \sum_{n=1}^{N} W(n,N)\mathbb{E}||\Delta_n||_2^2 \\ &= \sum_k \sum_{n=1}^{N} W(n,N)\mathbb{E}\Delta_n[k]^2 \qquad = \sum_k \mathbb{E}\hat{G}[k]^2\end{aligned}$$

As the RT-RR estimator minimizes $\mathbb{E}\hat{G}[k]^2$ for each coordinate k, it also minimizes $\mathbb{E}||\hat{G}||_2^2$. It remains to optimize Q. Consider the inverse ROE of the RT-RR estimator. By independence we have:

$$\mathrm{ROE}(\hat{G})^{-1} = \mathbb{E}||\hat{G}||_2^2 \mathbb{E}C = \Big(\sum_N q(N) \sum_{n \leq N} \frac{1}{Q(n)^2}\mathbb{E}||\Delta_n||_2^2\Big)\Big(\sum_N q(N)C(N)\Big)$$

Take the gradient of the inverse optimization efficiency lower bound w.r.t. $q(n)$:

$$\frac{d\mathrm{ROE}(\hat{G})^{-1}}{dq(N)} = C(N)\mathbb{E}||\hat{G}||_2^2 + \sum_{n \leq N} \frac{1}{Q(n)^2}\mathbb{E}||\Delta_n||_2^2 - \sum_i q(i) \sum_{j \leq min(i,N)} \frac{2}{Q(j)^3}\mathbb{E}||\Delta_j||_2^2$$

$$\begin{aligned}\sum_i q(i) \sum_{j \leq min(i,N)} \frac{2}{Q(j)^3}\mathbb{E}||\Delta_j||_2^2 &= \sum_{j \leq N} \frac{2}{Q(j)^2}\mathbb{E}||\Delta_j||_2^2 \frac{\sum_i q(i)\mathbb{1}\{i \geq j\}}{Q(j)} \\ &= \sum_{j \leq N} \frac{2}{Q(j)^2}\mathbb{E}||\Delta_j||_2^2 \quad \text{by definition of } Q(j)\end{aligned}$$

$$\implies \frac{d\mathrm{ROE}(\hat{G})^{-1}}{dq(N)} = C(N)\mathbb{E}||\hat{G}||_2^2 - \sum_{n \leq N} \frac{1}{Q(n)^2}\mathbb{E}||\Delta_n||_2^2$$

Now optimize the objective w.r.t. Q by finding the critical point:

$$\begin{aligned}\frac{d\mathrm{ROE}(\hat{G})^{-1}}{dq(N)} = 0 &\implies C(N)\mathbb{E}||\hat{G}||_2^2 = \sum_{n \leq N} \frac{1}{Q(n)^2}\mathbb{E}||\Delta_n||_2^2 \\ &\implies \mathbb{E}||\hat{G}||_2^2\Big(C(N) - C(N-1)\Big) = \frac{1}{2}\frac{\mathbb{E}||\Delta_N||_2^2}{Q(N)^2} \\ &\implies Q(N)^2 \propto \frac{\mathbb{E}||\Delta_n||_2^2}{C(N) - C(N-1)}\end{aligned}$$

□