A. Multiclass Perceptron

MULTICLASS PERCEPTRON is an algorithm for ONLINE MULTICLASS CLASSIFICATION. Both the protocol for the problem and the algorithm are stated below. The algorithm assumes that the feature vectors come from an inner product space $(V, \langle \cdot, \cdot \rangle)$.

Two results are folklore. The first result is Theorem 10 which states that if examples are linearly separable with margin γ and examples have norm at most R then the algorithm makes at most $\lfloor 2(R/\gamma)^2 \rfloor$ mistakes. The second result is Theorem 11 which states that under the same assumptions as in Theorem 11 *any* deterministic algorithm for ONLINE MULTICLASS CLASSIFICATION must make at least $\lfloor (R/\gamma)^2 \rfloor$ mistakes in the worst case.

Protocol 2 ONLINE MULTICLASS CLASSIFICATION

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Require: Number of classes $K$, number of rounds $T$.
Require: Inner product space $(V, \langle \cdot, \cdot \rangle)$.
for $t = 1, 2, \dots, T$ do
    Adversary chooses example $(x_t, y_t) \in V \times \{1, 2, \dots, K\}$, where $x_t$ is revealed to the learner.
    Predict class label $\widehat{y}_t \in \{1, 2, \dots, K\}$.
    Observe feedback $y_t$.
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Algorithm 3 MULTICLASS PERCEPTRON

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Require: Number of classes $K$, number of rounds $T$.
Require: Inner product space $(V, \langle \cdot, \cdot \rangle)$.
Initialize $w_1^{(1)} = w_2^{(1)} = \cdots = w_K^{(1)} = 0$
for $t = 1, 2, \dots, T$ do
    Observe feature vector $x_t \in V$
    Predict $\widehat{y}_t = \operatorname{argmax}_{i \in \{1,2,\dots,K\}} \left\langle w_t^{(i)}, x_t \right\rangle$
    Observe $y_t \in \{1, 2, \dots, K\}$
    if $\widehat{y}_t \neq y_t$ then
        Set $w_i^{(t+1)} = w_i^{(t)}$
        for all $i \in \{1, 2, \dots, K\} \setminus \{y_t, \widehat{y}_t\}$
        Update $w_{y_t}^{(t+1)} = w_{y_t}^{(t)} + x_t$
        Update $w_{\widehat{y}_t}^{(t+1)} = w_{\widehat{y}_t}^{(t)} - x_t$
    else
        Set $w_i^{(t+1)} = w_i^{(t)}$ for all $i \in \{1, 2, \dots, K\}$
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Theorem 10 (Mistake upper bound (Crammer & Singer, 2003)). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let K be a positive integer, let γ be a positive real number and let R be a non-negative real number. If $(x_1, y_1), (x_2, y_2), \dots, (x_T, y_T)$ is a sequence of labeled examples in $V \times \{1, 2, \dots, K\}$ that are weakly linearly separable with margin γ and $\|x_1\|, \|x_2\|, \dots, \|x_T\| \le R$ then* MULTICLASS PERCEPTRON *algorithm makes at most $\lfloor 2(R/\gamma)^2 \rfloor$ mistakes.*

Proof. Let $M = \sum_{t=1}^{T} \mathbb{1}[\widehat{y}_t \neq y_t]$ be the number of mistakes the algorithm makes. Since the K-tuple $(w_1^{(t)}, w_2^{(t)}, \dots, w_K^{(t)})$ changes only if a mistake is made, we can upper bound $\sum_{i=1}^{K} \left\| w_i^{(t)} \right\|^2$ in terms of number of mistakes. If a mistake happens in round t then

$$\begin{aligned}
\sum_{i=1}^K \left\| w_i^{(t+1)} \right\|^2 &= \left(\sum_{i \in \{1,2,\dots,K\} \setminus \{y_t, \widehat{y}_t\}} \left\| w_i^{(t)} \right\|^2 \right) + \left\| w_{y_t}^{(t)} + x_t \right\|^2 + \left\| w_{\widehat{y}_t}^{(t)} - x_t \right\|^2 \\
&= \left(\sum_{i \in \{1,2,\dots,K\} \setminus \{y_t, \widehat{y}_t\}} \left\| w_i^{(t)} \right\|^2 \right) + \left\| w_{y_t}^{(t)} \right\|^2 + \left\| w_{\widehat{y}_t}^{(t)} \right\|^2 + 2\|x_t\|^2 + 2 \left\langle w_{y_t}^{(t)} - w_{\widehat{y}_t}^{(t)}, x_t \right\rangle \\
&= \left(\sum_{i=1}^K \left\| w_i^{(t)} \right\|^2 \right) + 2\|x_t\|^2 + 2 \left\langle w_{y_t}^{(t)} - w_{\widehat{y}_t}^{(t)}, x_t \right\rangle \\
&\le \left(\sum_{i=1}^K \left\| w_i^{(t)} \right\|^2 \right) + 2\|x_t\|^2 \\
&\le \left(\sum_{i=1}^K \left\| w_i^{(t)} \right\|^2 \right) + 2R^2 \; .
\end{aligned}$$

So each time a mistake happens, $\sum_{i=1}^K \left\| w_i^{(t)} \right\|^2$ increases by at most $2R^2$. Thus,

$$\sum_{i=1}^K \left\| w_i^{(T+1)} \right\|^2 \le 2R^2 M \; . \tag{14}$$

Let $w_1^*, w_2^*, \dots, w_K^* \in V$ be vectors satisfying (1) and (2). We lower bound $\sum_{i=1}^K \left\langle w_i^*, w_i^{(t)} \right\rangle$. This quantity changes only when a mistakes happens. If mistake happens in round t, we have

$$\begin{aligned}
\sum_{i=1}^K \left\langle w_i^*, w_i^{(t+1)} \right\rangle &= \left(\sum_{i \in \{1,2,\dots,K\} \setminus \{y_t, \widehat{y}_t\}} \left\langle w_i^*, w_i^{(t)} \right\rangle \right) \\
&\quad + \left\langle w_{y_t}^*, w_{y_t}^{(t)} + x_t \right\rangle + \left\langle w_{\widehat{y}_t}^*, w_{\widehat{y}_t}^{(t)} - x_t \right\rangle \\
&= \left(\sum_{i=1}^K \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + \left\langle w_{y_t}^* - w_{\widehat{y}_t}^*, x_t \right\rangle \\
&\ge \left(\sum_{i=1}^K \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + \gamma \; .
\end{aligned}$$

Thus, after M mistakes,

$$\sum_{i=1}^K \left\langle w_i^*, w_i^{(T+1)} \right\rangle \ge \gamma M \; .$$

We upper bound the left hand side by using Cauchy-Schwartz inequality twice and the condition (1) on $w_1^*, w_2^*, \dots, w_K^*$. We have

$$\begin{aligned}
\sum_{i=1}^K \left\langle w_i^*, w_i^{(T+1)} \right\rangle &\le \sum_{i=1}^K \|w_i^*\| \cdot \left\| w_i^{(T+1)} \right\| \\
&\le \sqrt{\sum_{i=1}^K \|w_i^*\|^2} \sqrt{\sum_{i=1}^K \left\| w_i^{(T+1)} \right\|^2} \\
&\le \sqrt{\sum_{i=1}^K \left\| w_i^{(T+1)} \right\|^2} \; .
\end{aligned}$$

Combining the above inequality with Equations (14) and (A), we get

$$(\gamma M)^2 \le \sum_{i=1}^{K} \left\| w_i^{(T+1)} \right\|^2 \le 2R^2 M \ .$$

We conclude that $M \le 2(R/\gamma)^2$. Since M is an integer, $M \le \lfloor 2(R/\gamma)^2 \rfloor$. □

Theorem 11 (Mistake lower bound)**.** *Let K be a positive integer, let γ be a positive real number and let R be a non-negative real number. For any (possibly randomized) algorithm $\mathcal{A}$ for the* Online Multiclass Classification *problem there exists an inner product space $(V, \langle \cdot, \cdot \rangle)$, a non-negative integer T and a sequence of labeled examples $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T)$ examples in $V \times \{1, 2, \ldots, K\}$ that are weakly linearly separable with margin γ, the norms satisfy $\|x_1\|, \|x_2\|, \ldots, \|x_T\| \le R$ and the algorithm makes at least $\frac{1}{2} \lfloor (R/\gamma)^2 \rfloor$ mistakes.*

Proof. Let $T = \lfloor (R/\gamma)^2 \rfloor$, $V = \mathbb{R}^T$, and for all t in $\{1, \ldots, T\}$, define instance $x_t = Re_t$ where e_t is t-th element of the standard orthonormal basis of $\mathbb{R}^T$. Let labels $y_1, \ldots, y_T$ be chosen i.i.d uniformly at random from $\{1, 2, \ldots, K\}$ and independently of any randomness used by the algorithm $\mathcal{A}$.

We first show that the set of examples (x_1, y_1), …, (x_T, y_T) we have constructed is weakly linearly separable with margin γ. To prove that, we demonstrate vectors $w_1, w_2, \ldots, w_K$ satisfying conditions (1) and (2). We define

$$w_i = \frac{\gamma}{R} \sum_{\substack{t : 1 \le t \le T \\ y_t = i}} e_t \qquad \text{for } i = 1, 2, \ldots, K.$$

Let $a_i = |\{t \ : \ 1 \le t \le T, \ y_t = i\}|$ be the number of occurrences of label i. It is easy to see that

$$\|w_i\|^2 = \frac{\gamma^2}{R^2} \sum_{\substack{t : 1 \le t \le T \\ y_t = i}} \|e_t\|^2 = \frac{a_i \gamma^2}{R^2} \qquad \text{for } i = 1, 2, \ldots, K.$$

Since $\sum_{i=1}^K a_i = T$, $\sum_{i=1}^K \|w_i\|^2 = T \cdot \frac{\gamma^2}{R^2} \le 1$, i.e. the condition (1) holds. To verify condition (2) consider any labeled example (x_t, y_t). Then, for any i in $\{1, \ldots, K\}$, by the definition of w_i, we have

$$\begin{aligned} \langle w_i, x_t \rangle &= \frac{\gamma}{R} \sum_{\substack{s : 1 \le s \le T \\ y_s = i}} \langle e_s, Re_t \rangle \\ &= \gamma \cdot \sum_{\substack{s : 1 \le s \le T \\ y_s = i}} \mathbb{1}\left[s = t \right] \\ &= \gamma \cdot \mathbb{1}\left[y_t = i \right] \ . \end{aligned}$$

Therefore, if $i = y_t$, $\langle w_i, x_t \rangle = \gamma$; otherwise $i \neq y_t$, in which case $\langle w_i, x_t \rangle = 0$. Hence, condition (2) holds.

We now give a lower bound on the number of mistakes $\mathcal{A}$ makes. As y_t is chosen uniformly from $\{1, 2, \ldots, K\}$, independently from $\mathcal{A}$'s randomization and the first $t-1$ examples,

$$\mathbf{E}[\mathbb{1}\left[\widehat{y}_t \neq y_t\right]] \ge 1 - \frac{1}{K} \ge \frac{1}{2} \ .$$

Summing over all t in $\{1, \ldots, T\}$, we conclude that

$$\mathbf{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\widehat{y}_t \neq y_t\right] \right] \ge \frac{T}{2} = \frac{1}{2} \lfloor (R/\gamma)^2 \rfloor,$$

which completes the proof. □

B. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let $M = \sum_{t=1}^T z_t$ be the number of mistakes Algorithm 1 makes. Let $A = \sum_{t=1}^T \mathbb{1}\left[S_t \neq \emptyset\right] z_t$ be the number of mistakes in the rounds when $S_t \neq \emptyset$, i.e. the number of rounds line 18 is executed. In addition, let $B = \sum_{t=1}^T \mathbb{1}\left[S_t = \emptyset\right] z_t$ be the number of mistakes in the rounds when $S_t = \emptyset$. It can be easily seen that $M = A + B$.

Let $C = \sum_{t=1}^T \mathbb{1}\left[S_t = \emptyset\right](1 - z_t)$ be the number of rounds line 12 gets executed. Let $U = \sum_{t=1}^T (\mathbb{1}\left[S_t \neq \emptyset\right] z_t + \mathbb{1}\left[S_t = \emptyset\right](1 - z_t))$ be the number of rounds line 12 or 18 gets executed. In other words, U is the number of times the K-tuple of vectors $(w_1^{(t)}, w_2^{(t)}, \ldots, w_K^{(t)})$ gets updated. It can be easily seen that $U = A + C$.

The key observation is that $\mathbf{E}[B] = (K-1)\,\mathbf{E}[C]$. To see this, note that if $S_t = \emptyset$, there is $1/K$ probability that the algorithm guesses the correct label ($z_t = 0$) and with probability $(K-1)/K$ algorithm's guess is incorrect ($z_t = 1$). Therefore,

$$\mathbf{E}[z_t | S_t = \emptyset] = \frac{K-1}{K},$$

$$\mathbf{E}[B] = \frac{K-1}{K}\,\mathbf{E}\left[\sum_{t=1}^T \mathbb{1}\left[S_t = \emptyset\right]\right],$$

$$\mathbf{E}[C] = \frac{1}{K}\,\mathbf{E}\left[\sum_{t=1}^T \mathbb{1}\left[S_t = \emptyset\right]\right].$$

Putting all the information together, we get that

$$\begin{aligned}\mathbf{E}[M] &= \mathbf{E}[A] + \mathbf{E}[B] \\ &= \mathbf{E}[A] + (K-1)\,\mathbf{E}[C] \\ &\leq (K-1)\,\mathbf{E}[A + C] \\ &= (K-1)\,\mathbf{E}[U]\,. \end{aligned} \tag{15}$$

To finish the proof, we need to upper bound the number of updates U. We claim that $U \leq \lfloor 4(R/\gamma)^2 \rfloor$ with probability 1. The proof of this upper bound is similar to the proof of the mistake bound for MULTICLASS PERCEPTRON algorithm. Let $w_1^*, w_2^*, \ldots, w_K^* \in V$ be vectors that satisfy (3), (4) and (5). The K-tuple $(w_1^{(t)}, w_2^{(t)}, \ldots, w_K^{(t)})$ changes only if there is an update in round t. We investigate how $\sum_{i=1}^K \left\|w_i^{(t)}\right\|^2$ and $\sum_{i=1}^K \left\langle w_i^*, w_i^{(t)}\right\rangle$ change. If there is an update in round t, by lines 12 and 18, we always have $w_{\widehat{y}_t}^{(t+1)} = w_{\widehat{y}_t}^{(t)} + (-1)^{z_t} x_t$, and for all $i \neq \widehat{y}_t$, $w_i^{(t+1)} = w_i^{(t)}$. Therefore,

$$\begin{aligned}\sum_{i=1}^K \left\|w_i^{(t+1)}\right\|^2 &= \left(\sum_{i \in \{1,2,\ldots,K\} \setminus \{\widehat{y}_t\}} \left\|w_i^{(t)}\right\|^2\right) + \left\|w_{\widehat{y}_t}^{(t+1)}\right\|^2 \\ &= \left(\sum_{i \in \{1,2,\ldots,K\} \setminus \{\widehat{y}_t\}} \left\|w_i^{(t)}\right\|^2\right) + \left\|w_{\widehat{y}_t}^{(t)} + (-1)^{z_t} x_t\right\|^2 \\ &= \left(\sum_{i=1}^K \left\|w_i^{(t)}\right\|^2\right) + \|x_t\|^2 + \underbrace{(-1)^{z_t} 2 \left\langle w_{\widehat{y}_t}^{(t)}, x_t \right\rangle}_{\leq 0} \\ &\leq \left(\sum_{i=1}^K \left\|w_i^{(t)}\right\|^2\right) + \|x_t\|^2 \\ &\leq \left(\sum_{i=1}^K \left\|w_i^{(t)}\right\|^2\right) + R^2\,. \end{aligned}$$

The inequality that $(-1)^{z_t} 2 \left\langle w^{(t)}_{\widehat{y}_t}, x_t \right\rangle \le 0$ is from a case analysis: if line 12 is executed, then $z_t = 0$ and $\left\langle w^{(t)}_{\widehat{y}_t}, x_t \right\rangle < 0$; otherwise line 18 is executed, in which case $z_t = 1$ and $\left\langle w^{(t)}_{\widehat{y}_t}, x_t \right\rangle \ge 0$.

Hence, after U updates,

$$\sum_{i=1}^{K} \left\| w_i^{(T+1)} \right\|^2 \le R^2 U \ . \tag{16}$$

Similarly, if there is an update in round t, we have

$$\begin{aligned}
\sum_{i=1}^{K} \left\langle w_i^*, w_i^{(t)} \right\rangle &= \left(\sum_{i \in \{1,2,\dots,K\} \setminus \{\widehat{y}_t\}} \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + \left\langle w^*_{\widehat{y}_t}, w^{(t+1)}_{\widehat{y}_t} \right\rangle \\
&= \left(\sum_{i \in \{1,2,\dots,K\} \setminus \{\widehat{y}_t\}} \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + \left\langle w^*_{\widehat{y}_t}, w^{(t)}_{\widehat{y}_t} + (-1)^{z_t} x_t \right\rangle \\
&= \left(\sum_{i=1}^{K} \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + (-1)^{z_t} \left\langle w^*_{\widehat{y}_t}, x_t \right\rangle \\
&\ge \left(\sum_{i=1}^{K} \left\langle w_i^*, w_i^{(t)} \right\rangle \right) + \frac{\gamma}{2},
\end{aligned}$$

where the last inequality follows from a case analysis on z_t and Definition 1: if $z_t = 0$, then $\widehat{y}_t = y_t$, by Equation (4), we have that $\left\langle w^*_{\widehat{y}_t}, x_t \right\rangle \ge \frac{\gamma}{2}$; if $z_t = 1$, then $\widehat{y}_t \neq y_t$, by Equation (5), we have that $\left\langle w^*_{\widehat{y}_t}, x_t \right\rangle \le -\frac{\gamma}{2}$.

Thus, after U updates,

$$\sum_{i=1}^{K} \left\langle w_i^*, w_i^{(T+1)} \right\rangle \ge \frac{\gamma U}{2} \ . \tag{17}$$

Applying Cauchy-Schwartz's inequality twice, and using assumption (3), we get that

$$\begin{aligned}
\sum_{i=1}^{K} \left\langle w_i^*, w_i^{(T+1)} \right\rangle &\le \sum_{i=1}^{K} \| w_i^* \| \cdot \left\| w_i^{(T+1)} \right\| \\
&\le \sqrt{\sum_{i=1}^{K} \| w_i^* \|^2} \sqrt{\sum_{i=1}^{K} \left\| w_i^{(T+1)} \right\|^2} \\
&\le \sqrt{\sum_{i=1}^{K} \left\| w_i^{(T+1)} \right\|^2} \ .
\end{aligned}$$

Combining the above inequality with Equations (16) and (17), we get

$$\left(\frac{\gamma U}{2} \right)^2 \le \sum_{i=1}^{K} \left\| w_i^{(T+1)} \right\|^2 \le R^2 U \ .$$

We conclude that $U \le 4(R/\gamma)^2$. Since U is an integer, $U \le \lfloor 4(R/\gamma)^2 \rfloor$.

Applying Equation (15), we get

$$\mathbf{E}[M] \le (K-1)\, \mathbf{E}[U] \le (K-1) \lfloor 4(R/\gamma)^2 \rfloor \ . \quad \square$$

Proof of Theorem 3. Let $M = \left\lfloor \frac{1}{4}(R/\gamma)^2 \right\rfloor$. Let $V = \mathbb{R}^{M+1}$ equipped with the standard inner product. Let $e_1, e_2, \dots, e_{M+1}$ be the standard orthonormal basis of V. We define vectors $v_1, v_2, \dots, v_M \in V$ where $v_j = \frac{R}{\sqrt{2}}(e_j +$

$e_{M+1})$ for $j = 1, 2, \dots, M$. Let $\ell_1, \ell_2, \dots, \ell_M$ be chosen i.i.d. uniformly at random from $\{1, 2, \dots, K\}$ and independently of any randomness used the by algorithm $\mathcal{A}$. Let $T = M(K-1)$. We define examples $(x_1, y_1), (x_2, y_2), \dots, (x_T, y_T)$ as follows. For any $j = 1, 2, \dots, M$ and any $h = 1, 2, \dots, K-1$,

$$(x_{(j-1)(K-1)+h}, y_{(j-1)(K-1)+h}) = (v_j, \ell_j)$$

The norm of each example is exactly R. The examples are strongly linearly separable with margin γ. To see that, consider $w_1^*, w_2^*, \dots, w_K^* \in V$ defined by

$$w_i^* = \sqrt{2}\frac{\gamma}{R}\left(\sum_{j\,:\,\ell_j = i} e_j\right) - \frac{\sqrt{2}}{2}\frac{\gamma}{R} e_{M+1}$$

for $i = 1, 2, \dots, K$.

For $i \in \{1, 2, \dots, K\}$ and $j \in \{1, 2, \dots, M\}$, consider the inner product of w_i^* and v_j. If $i = \ell_j$, $\left\langle w_i^*, v_j \right\rangle = \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}$; otherwise $i \neq \ell_j$, in which case $\left\langle w_i^*, v_j \right\rangle = 0 - \frac{\gamma}{2} = -\frac{\gamma}{2}$. This means that $w_1^*, w_2^*, \dots, w_K^*$ satisfy conditions (4) and (5). Condition (3) is satisfied since

$$\begin{aligned}\sum_{i=1}^{K} \|w_i^*\|^2 &= 2\frac{\gamma^2}{R^2}\sum_{j=1}^{M}\|e_j\|^2 + \frac{\gamma^2}{2R^2}K\|e_{M+1}\|^2 \\ &= 2\frac{\gamma^2}{R^2}M + \frac{\gamma^2}{2R^2}K \le \frac{1}{2} + \frac{1}{2} = 1\,.\end{aligned}$$

It remains to lower bound the expected number of mistakes of $\mathcal{A}$. For any $j \in \{1, 2, \dots, M\}$, consider the expected number of mistakes the algorithm makes in rounds $(K-1)(j-1)+1, (K-1)(j-1)+2, \dots, (K-1)j$.

Define a filtration of σ-algebras $\{\mathcal{B}_j\}_{j=0}^{M}$, where $\mathcal{B}_j = \sigma((x_1, y_1, \hat{y}_1), \dots, (x_{(K-1)j}, y_{(K-1)j}, \hat{y}_{(K-1)j}))$ for every j in $\{1, 2, \dots, M\}$. By Claim 2 of Daniely & Helbertal (2013), as ℓ_j is chosen uniformly from $\{1, \dots, K\}$ and independent of $\mathcal{B}_{j-1}$ and $\mathcal{A}$'s randomness,

$$\mathbf{E}\left[\sum_{t=(K-1)(j-1)+1}^{(K-1)j} z_t \,\middle|\, \mathcal{B}_{j-1}\right] \ge \frac{K-1}{2}\,.$$

This implies that

$$\mathbf{E}\left[\sum_{t=(K-1)(j-1)+1}^{(K-1)j} z_t\right] \ge \frac{K-1}{2}\,.$$

Summing over all j in $\{1, 2, \dots, M\}$,

$$\mathbf{E}\left[\sum_{t=1}^{(K-1)M} z_t\right] \ge \frac{K-1}{2}\cdot M = \frac{K-1}{2}\left\lfloor \frac{1}{4}(R/\gamma)^2 \right\rfloor\,.$$

Thus there exists a particular sequence of examples for which the algorithm makes at least $\frac{K-1}{2}\left\lfloor \frac{1}{4}(R/\gamma)^2 \right\rfloor$ mistakes in expectation over its internal randomization. □

C. Proof of Lemma 9

Proof. Note that the polynomial p can be written as $p(x) = \sum_{\alpha_1, \alpha_2, \dots, \alpha_d} c'_{\alpha_1, \alpha_2, \dots, \alpha_d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$. We define $c \in \ell_2$ using the multi-index notation as

$$c_{\alpha_1, \alpha_2, \dots, \alpha_d} = \frac{c'_{\alpha_1, \alpha_2, \dots, \alpha_d} 2^{(\alpha_1 + \alpha_2 + \dots + \alpha_d)/2}}{\sqrt{\binom{\alpha_1 + \alpha_2 + \dots + \alpha_d}{\alpha_1, \alpha_2, \dots, \alpha_d}}}$$

for all tuples $(\alpha_1, \alpha_2, \dots, \alpha_d)$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_d \le \deg(p)$. Otherwise, we define $c_{\alpha_1, \alpha_2, \dots, \alpha_d} = 0$. By the definition of ϕ, $\left\langle c, \phi(x) \right\rangle_{\ell_2} = p(x)$.

Whether $\alpha_1 + \ldots + \alpha_d \leq \deg(p)$, we always have:

$$|c_{\alpha_1,\alpha_2,\ldots,\alpha_d}| \leq 2^{(\alpha_1+\alpha_2+\cdots+\alpha_d)/2}|c'_{\alpha_1,\alpha_2,\ldots,\alpha_d}| \leq 2^{\deg(p)/2}|c'_{\alpha_1,\alpha_2,\ldots,\alpha_d}| \ .$$

Therefore,

$$\|c\|_{\ell_2} \leq 2^{\deg(p)/2}\sqrt{\sum_{\alpha_1,\alpha_2,\ldots,\alpha_d}(c'_{\alpha_1,\alpha_2,\ldots,\alpha_d})^2} = 2^{\deg(p)/2}\|p\| \ . \qquad \square$$

D. Proofs of Theorems 7 and 8

In this section, we follow the construction of Klivans & Servedio (2008) (which in turn uses the constructions of Beigel et al. (1995)) to establish two polynomials of low norm, such that it takes large positive values in

$$\bigcap_{i=1}^{m}\left\{x \in \mathbb{R}^d \ : \|x\| \leq 1, \ \langle v_i, x\rangle \geq \gamma\right\}$$

and takes large negative values in

$$\bigcup_{i=1}^{m}\left\{x \in \mathbb{R}^d \ : \|x\| \leq 1, \ \langle v_i, x\rangle \leq -\gamma\right\}.$$

We improve the norm bound analysis of Klivans & Servedio (2008) in two aspects:

1. Our upper bounds on the norm of the polynomials do not have any dependency on the dimensionality d.
2. We remove the requirement that the fractional part of input x must be above some threshold in Theorem 8.

A lot of the proof details are similar to those of Klivans & Servedio (2008); nevertheless, we provide a self-contained full proof here.

For the proofs of the theorems we need several auxiliary results.

Lemma 12 (Simple inequality)**.** *For any real numbers $b_1, b_2, \ldots, b_n$,*

$$\left(\sum_{i=1}^{n} b_i\right)^2 \leq n\sum_{i=1}^{n} b_i^2 \ .$$

Proof. The lemma follows from Cauchy-Schwartz inequality applied to vectors $(b_1, b_2, \ldots, b_n)$ and $(1, 1, \ldots, 1)$. □

Lemma 13 (Bound on binomial coefficients)**.** *For any integers n, k such that $n \geq k \geq 0$,*

$$\binom{n}{k} \leq (n-k+1)^k \ .$$

Proof. If $k = 0$, the inequality trivially holds. For the rest of the proof we can assume $k \geq 1$. We write the binomial coefficient as

$$\begin{aligned}\binom{n}{k} &= \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \\ &= \frac{n}{k}\cdot\frac{n-1}{k-1}\cdots\frac{n-k+1}{1} \ .\end{aligned}$$

We claim that

$$\frac{n}{k} \leq \frac{n-1}{k-1} \leq \cdots \leq \frac{n-k+1}{1}$$

from which the lemma follows by upper bounding all the fractions by $n-k+1$. It remains to prove that for any $j = 0, 1, \ldots, k-1$,

$$\frac{n-j+1}{k-j+1} \leq \frac{n-j}{k-j} \; .$$

Multiplying by the (positive) denominators, we get an equivalent inequality

$$(n-j+1)(k-j) \leq (n-j)(k-j+1) \; .$$

We multiply out the terms and get

$$nk - kj + k - nj + j^2 - j \leq nk - nj + n - kj + j^2 - j \; .$$

We cancel common terms and get an equivalent inequality $k \geq n$, which holds by the assumption. □

Lemma 14 (Properties of the norm of polynomials)**.**

1. *Let $p_1, p_2, \ldots, p_n$ be multivariate polynomials and let $p(x) = \prod_{j=1}^n p_j(x)$ be their product. Then, $\|p\|^2 \leq n^{\sum_{j=1}^n \deg(p_j)} \prod_{j=1}^n \|p_j\|^2$.*

2. *Let q be a multivariate polynomial of degree at most s and let $p(x) = (q(x))^n$. Then, $\|p\|^2 \leq n^{ns} \|q\|^{2n}$.*

3. *Let be $p_1, p_2, \ldots, p_n$ be multivariate polynomials. Then, $\left\|\sum_{j=1}^n p_j\right\| \leq \sum_{j=1}^n \|p_j\|$. Consequently, $\left\|\sum_{j=1}^n p_j\right\|^2 \leq n \sum_{j=1}^n \|p_j\|^2$.*

Proof. Using multi-index notation we can write any multivariate polynomial p as

$$p(x) = \sum_A c_A x^A$$

where $A = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index (i.e. a d-tuple of non-negative integers), $x^A = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}$ is a monomial and $c_A = c_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ is the corresponding real coefficient. The sum is over a finite subset of d-tuples of non-negative integers. Using this notation, the norm of a polynomial p can be written as

$$\|p\| = \sqrt{\sum_A (c_A)^2} \; .$$

For a multi-index $A = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ we define its 1-norm as $\|A\|_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

To prove the part 1, we express p_j as

$$p_j(x) = \sum_{A_j} c_{A_j}^{(j)} x^{A_j} \; .$$

Since $p(x) = \prod_{i=1}^n p_j(x)$, the coefficients of its expansion $p(x) = \sum_A c_A x^A$ are

$$c_A = \sum_{\substack{(A_1, A_2, \ldots, A_n) \\ A_1 + A_2 + \cdots + A_n = A}} c_{A_1}^{(1)} c_{A_2}^{(2)} \cdots c_{A_n}^{(n)} \; .$$

Therefore,

$$\begin{aligned}
\|p\|^2 &= \sum_A (c_A)^2 \\
&= \sum_A \left(\sum_{\substack{(A_1,A_2,\dots,A_n) \\ A_1+A_2+\cdots+A_n=A}} c^{(1)}_{A_1} c^{(2)}_{A_2} \cdots c^{(n)}_{A_n} \right)^2 \\
&= \sum_A \left(\sum_{\substack{(A_1,A_2,\dots,A_n) \\ A_1+A_2+\cdots+A_n=A}} \prod_{j=1}^n c^{(j)}_{A_j} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
\prod_{i=1}^n \|p_i\|^2 &= \prod_{i=1}^n \left(\sum_{A_i} (c^{(i)}_{A_i})^2 \right) \\
&= \sum_{(A_1,A_2,\dots,A_n)} \prod_{j=1}^n (c^{(j)}_{A_j})^2 \\
&= \sum_{(A_1,A_2,\dots,A_n)} \left(\prod_{j=1}^n c^{(j)}_{A_j} \right)^2 \\
&= \sum_A \sum_{\substack{(A_1,A_2,\dots,A_n) \\ A_1+A_2+\cdots+A_n=A}} \left(\prod_{j=1}^n c^{(j)}_{A_j} \right)^2
\end{aligned}$$

where in both cases the outer sum is over multi-indices A such that $\|A\|_1 \le \deg(p)$. Lemma 12 implies that for any multi-index A,

$$\left(\sum_{\substack{(A_1,A_2,\dots,A_n) \\ A_1+A_2+\cdots+A_n=A}} \prod_{j=1}^n c^{(j)}_{A_j} \right)^2 \le M_A \sum_{\substack{(A_1,A_2,\dots,A_n) \\ A_1+A_2+\cdots+A_n=A}} \left(\prod_{j=1}^n c^{(j)}_{A_j} \right)^2 .$$

where M_A is the number of n-tuples $(A_1, A_2, \dots, A_n)$ such that $A_1 + A_2 + \cdots + A_n = A$.

To finish the proof, it is sufficient to prove that $M_A \le n^{\deg(p)}$ for any A such that $\|A\|_1 \le \deg(p)$. To prove this inequality, consider a multi-index $A = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and consider its i-th coordinate α_i. In order for $A_1 + A_2 + \cdots + A_n = A$ to hold, the i-th coordinates of $A_1, A_2, \dots, A_n$ need to sum to α_i. There are exactly $\binom{\alpha_i+n-1}{\alpha_i}$ possibilities for the choice of i-th coordinates of $A_1, A_2, \dots, A_n$. The total number of choices is thus

$$M_A = \prod_{i=1}^d \binom{\alpha_i + n - 1}{\alpha_i} .$$

Using Lemma 13, we upper bound it as

$$M_A \le \prod_{i=1}^d n^{\alpha_i} = n^{\|A\|_1} \le n^{\deg(p)} .$$

Part 2 follows from the part 1 by setting $p_1 = p_2 = \dots p_n = q$.

The first inequality of part 3 follows from triangle inequality in Euclidean spaces, by viewing the polynomials $p = \sum_A c_A x^A$ as multidimensional vectors (c_A), and $\|p\| = \|(c_A)\|$.

For the second inequality, by Lemma 12, we have

$$\left\|\sum_{j=1}^{n} p_j\right\|^2 = \left(\left\|\sum_{j=1}^{n} p_j\right\|\right)^2 \leq \left(\sum_{j=1}^{n} \|p_j\|\right)^2 \leq n \sum_{j=1}^{n} \|p_j\|^2 \ .$$

□

D.1. Proof of Theorem 7

To construct the polynomial p we use Chebyshev polynomials of the first kind. Chebyshev polynomials of the fist kind form an infinite sequence of polynomials $T_0(z), T_1(z), T_2(z), \ldots$ of single real variable z. They are defined by the recurrence

$$\begin{aligned} T_0(z) &= 1 \ , \\ T_1(z) &= z \ , \\ T_{n+1}(z) &= 2zT_n(z) - T_{n-1}(z), \quad \text{for } n \geq 1. \end{aligned}$$

Chebyshev polynomials have a lot of interesting properties. We will need properties listed in Proposition 15 below. Interested reader can learn more about Chebyshev polynomials from the book by Mason & Handscomb (2002).

Proposition 15 (Properties of Chebyshev polynomials)**.** *Chebyshev polynomials satisfy*

1. $\deg(T_n) = n$ *for all* $n \geq 0$.
2. *If* $n \geq 1$*, the leading coefficient of* $T_n(z)$ *is* 2^{n-1}.
3. $T_n(\cos(\theta)) = \cos(n\theta)$ *for all* $\theta \in \mathbb{R}$ *and all* $n \geq 0$.
4. $T_n(\cosh(\theta)) = \cosh(n\theta)$ *for all* $\theta \in \mathbb{R}$ *and all* $n \geq 0$.
5. $|T_n(z)| \leq 1$ *for all* $z \in [-1, 1]$ *and all* $n \geq 0$.
6. $T_n(z) \geq 1 + n^2(z-1)$ *for all* $z \geq 1$ *and all* $n \geq 0$.
7. $\|T_n\| \leq (1+\sqrt{2})^n$ *for all* $n \geq 0$

Proof of Proposition 15. The first two properties can be easily proven by induction on n using the recurrence.

We prove the third property by induction on n. Indeed, by definition

$$T_0(\cos(\theta)) = 1 = \cos(0\theta) \quad \text{and} \quad T_1(\cos(\theta)) = \cos(\theta) \ .$$

For $n \geq 1$, we have

$$\begin{aligned} T_{n+1}(\cos(\theta)) &= 2\cos(\theta)T_n(\cos(\theta)) - T_{n-1}(\cos(\theta)) \\ &= 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta)) \ , \end{aligned}$$

where the last step follow by induction hypothesis. It remains to show that the last expression equals $\cos((n+1)\theta)$. This can be derived from the trigonometric formula

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \ .$$

By substituting $\alpha = n\theta$ and $\beta = \theta$, we get two equations

$$\begin{aligned} \cos((n+1)\theta) &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \ , \\ \cos((n-1)\theta) &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta) \ . \end{aligned}$$

Summing them yields

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)\cos(\theta)$$

which finishes the proof.

The fourth property has the similar proof as the third property. It suffices to replace $\cos$ and $\sin$ with $\cosh$ and $\sinh$ respectively.

The fifth property follows from the third property. Indeed, for any $z \in [-1, 1]$ there exists $\theta \in \mathbb{R}$ such that $\cos\theta = z$. Thus, $|T_n(z)| = |T_n(\cos(\theta))| = |\cos(n\theta)| \le 1$.

The sixth property is equivalent to

$$T_n(\cosh(\theta)) \ge 1 + n^2(\cosh(\theta) - 1) \qquad \text{for all } \theta \ge 0,$$

since $\cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2}$ is an even continuous function that maps $\mathbb{R}$ onto $[1, +\infty)$, is strictly decreasing on $(-\infty, 0]$, and is strictly increasing on $[0, \infty)$. Using the fourth property the last inequality is equivalent to

$$\cosh(n\theta) \ge 1 + n^2(\cosh(\theta) - 1) \qquad \text{for all } \theta \ge 0.$$

For $\theta = 0$, both sides are equal to 1. Thus, it is sufficient to prove that the derivative of the left hand side is greater or equal to the derivative of the right hand side. Recalling that $[\cosh(\theta)]' = \sinh(\theta)$, this means that we need to show that

$$\sinh(n\theta) \ge n\sinh(\theta) \qquad \text{for all } \theta \ge 0.$$

To prove this inequality we use the summation formula

$$\sinh(\alpha + \beta) = \sinh(\alpha)\cosh(\beta) + \sinh(\beta)\cosh(\beta)\ .$$

If α, β are non-negative then $\sinh(\alpha), \sinh(\beta)$ are non-negative and $\cosh(\alpha), \cosh(\beta) \ge 1$. Hence,

$$\sinh(\alpha + \beta) \ge \sinh(\alpha) + \sinh(\beta) \qquad \text{for any } \alpha, \beta \ge 0.$$

This implies that (using induction on n) that $\sinh(n\theta) \ge n\sinh(\theta)$ for all $\theta \ge 0$.

We verify the seventh property by induction on n. For $n = 0$ and $n = 1$ the inequality trivially holds, since $\|T_0\| = \|T_1\| = 1$. For $n \ge 1$, since $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$,

$$\begin{aligned}
\|T_{n+1}\| &\le 2\|T_n\| + \|T_{n-1}\| \\
&\le 2(1+\sqrt{2})^n + (1+\sqrt{2})^{n-1} \\
&= (1+\sqrt{2})^{n-1}(2(1+\sqrt{2}) + 1) \\
&= (1+\sqrt{2})^{n-1}(3 + 2\sqrt{2}) \\
&= (1+\sqrt{2})^{n-1}(1+\sqrt{2})^2 \\
&= (1+\sqrt{2})^{n+1}\ .
\end{aligned}$$

□

We are now ready to prove Theorem 7. Let $r = \lceil \log_2(2m) \rceil$ and $s = \left\lceil \sqrt{\frac{1}{\gamma}} \right\rceil$. We define the polynomial $p : \mathbb{R}^d \to \mathbb{R}$ as

$$p(x) = m + \frac{1}{2} - \sum_{i=1}^{m} \left(T_s(1 - \langle v_i, x \rangle)\right)^r\ .$$

It remains to show that p has properties 1–5.

To verify the first property notice that if $x \in \mathbb{R}^d$ satisfies $\|x\| \le 1$ and $\langle v_i, x \rangle \ge \gamma$ then since $\|v_i\| \le 1$ we have $\langle v_i, x \rangle \in [0, 1]$. Thus, $T_s(1 - \langle v_i, x \rangle)$ and $\left(T_s(1 - \langle v_i, x \rangle)\right)^r$ lie in the interval $[-1, 1]$. Therefore,

$$p(x) \ge m + \frac{1}{2} - m \ge \frac{1}{2}\ .$$

To verify the second property consider any $x \in \bigcup_{i=1}^m \left\{x \in \mathbb{R}^d \ : \|x\| \le 1, \ \langle v_i, x \rangle \le -\gamma\right\}$. Clearly, $\|x\| \le 1$ and there exists at least one $i \in \{1, 2, \dots, m\}$ such that $\langle v_i, x \rangle \le -\gamma$. Therefore, $1 - \langle v_i, x \rangle \ge 1 + \gamma$ and Proposition 15 (part 6) imply that

$$T_s(1 - \langle v_i, x \rangle) \ge 1 + s^2 \gamma \ge 2$$

and thus

$$\big(T_s(1 - \langle v_i, x \rangle)\big)^r \ge 2^r \ge 2m \ .$$

On the other hand for any $j \in \{1, 2, \dots, m\}$, we have $\left\langle v_j, x \right\rangle \in [-1, 1]$ and thus $1 - \left\langle v_j, x \right\rangle$ lies in the interval $[0, 2]$. According to Proposition 15 (parts 5 and 6), $T_s(1 - \left\langle v_j, x \right\rangle) \ge -1$. Therefore,

$$\begin{aligned} p(x) &= m + \frac{1}{2} - \big(T_s(1 - \langle v_i, x \rangle)\big)^r - \sum_{\substack{j \,:\, 1 \le j \le m \\ j \neq i}} \Big(T_s(1 - \left\langle v_j, x \right\rangle)\Big)^r \\ &\le m + \frac{1}{2} - 2m + (m - 1) \le -\frac{1}{2} \ . \end{aligned}$$

The third property follows from the observation that the degree of p is the same as the degree of any one of the terms $\big(T_s(1 - \langle v_i, x \rangle)\big)^r$ which is $r \cdot s$.

To prove the fourth property, we need to upper bound the norm of p. Let $f_i(x) = 1 - \langle v_i, x \rangle$, let $g_i(x) = T_s(1 - \langle v_i, x \rangle)$ and let $h_i(x) = (T_s(1 - \langle v_i, x \rangle))^r$. We have

$$\|f_i\|^2 = 1 + \|v_i\|^2 \le 1 + 1 = 2 \ .$$

Let $T_s(z) = \sum_{j=0}^s c_j z^j$ be the expansion of s-th Chebyshev polynomial. Then,

$$\begin{aligned} \|g_i\|^2 &= \left\| \sum_{j=0}^s c_j (f_i)^j \right\|^2 \\ &\le (s+1) \sum_{j=0}^s \left\| c_j (f_i)^j \right\|^2 \qquad \text{(by part 3 of Lemma 14)} \\ &= (s+1) \sum_{j=0}^s (c_j)^2 \left\| (f_i)^j \right\|^2 \\ &\le (s+1) \sum_{j=0}^s (c_j)^2 j^j \|f_i\|^{2j} \qquad \text{(by part 2 of Lemma 14)} \\ &\le (s+1) \sum_{j=0}^s (c_j)^2 j^j 2^{2j} \\ &\le (s+1) s^s 2^{2s} \sum_{j=0}^s (c_j)^2 \\ &= (s+1) s^s 2^{2s} \|T_s\|^2 \\ &= (s+1) s^s 2^{2s} (1 + \sqrt{2})^{2s} \qquad \text{(by part 7 of Proposition 15)} \\ &= (s+1) \left(4(1 + \sqrt{2})^2 s\right)^s \\ &\le \left(8(1 + \sqrt{2})^2 s\right)^s \\ &\le (47s)^s \ . \end{aligned}$$

where we used that $s+1 \le 2^s$ for any non-negative integer s. Finally,

$$\begin{aligned}
\|p\| &\le m + \frac{1}{2} + \sum_{i=1}^m \left\| (g_i)^r \right\| \\
&= m + \frac{1}{2} + \sum_{i=1}^m \sqrt{\left\| (g_i)^r \right\|^2} \\
&\le m + \frac{1}{2} + \sum_{i=1}^m \sqrt{r^{rs} \|g_i\|^{2r}} \\
&\le m + \frac{1}{2} + m r^{rs/2} \left(47s\right)^{rs/2} \\
&= m + \frac{1}{2} + m \left(47rs\right)^{rs/2} \ .
\end{aligned}$$

We can further upper bound the last expression by using that $m \le \frac{1}{2} 2^r$. Since $r, s \ge 1$,

$$\begin{aligned}
\|p\| &\le m + \frac{1}{2} + m \left(47rs\right)^{rs/2} \\
&\le \frac{1}{2} 2^r + \frac{1}{2} + \frac{1}{2} 2^r \left(47rs\right)^{rs/2} \\
&\le 2^r + \frac{1}{2} 2^r \left(47rs\right)^{rs/2} \\
&= 2^r \left(1 + \frac{1}{2} \left(47rs\right)^{rs/2}\right) \\
&= 2^r \left(47rs\right)^{rs/2} \\
&\le 4^{rs/2} \left(47rs\right)^{rs/2} \\
&\le \left(188rs\right)^{rs/2} \ .
\end{aligned}$$

Substituting for r and s finishes the proof.

D.2. Proof of Theorem 8

We define several univariate polynomials

$$\begin{aligned}
P_n(z) &= (z-1) \prod_{i=1}^n (z - 2^i)^2, \quad \text{for } n \ge 0, \\
A_{n,k}(z) &= (P_n(z))^k - (P_n(-z))^k, \quad \text{for } n, k \ge 0, \\
B_{n,k}(z) &= -(P_n(z))^k - (P_n(-z))^k, \quad \text{for } n, k \ge 0.
\end{aligned}$$

We define the polynomial $q : \mathbb{R}^d \to \mathbb{R}$ as

$$q(x) = \left[\sum_{i=1}^m A_{s,r}\left(\frac{\langle v_i, x\rangle}{\gamma}\right) \prod_{\substack{j \,:\, 1 \le j \le m \\ j \ne i}} B_{s,r}\left(\frac{\langle v_j, x\rangle}{\gamma}\right) \right] - \left(m - \frac{1}{2}\right) \prod_{j=1}^m B_{s,r}\left(\frac{\langle v_j, x\rangle}{\gamma}\right) \ .$$

Finally, we define $p(x) = 2^{-s(s+1)rm+1} q(x)$. We are going to show that this polynomial p satisfies the required properties.

For convenience we define univariate rational function

$$S_{n,k}(z) = \frac{A_{n,k}(z)}{B_{n,k}(z)}, \quad \text{for } n, k \ge 0,$$

and a multivariate rational function

$$Q(x) = \left(\sum_{i=1}^{m} S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma} \right) \right) - \left(m - \frac{1}{2} \right) .$$

It is easy to verify that

$$q(x) = Q(x) \prod_{j=1}^{m} B_{s,r}\left(\frac{\langle v_j, x \rangle}{\gamma} \right) .$$

Lemma 16 (Properties of P_n)**.**

1. *If $z \in [0,1]$ then $P_n(-z) \le P_n(z) \le 0$.*
2. *If $z \in [1, 2^n]$ then $0 \le 4P_n(z) \le -P_n(-z)$.*
3. *If $z \ge 0$ then $-P_n(-z) \ge 2^{n(n+1)}$.*

Proof. To prove the first part, note that $P_n(z)$ and $P_n(-z)$ are non-positive for $z \in [0,1]$. We can write $\frac{P_n(z)}{P_n(-z)}$ as a product of $n+1$ non-negative fractions

$$\frac{P_n(z)}{P_n(-z)} = \frac{1-z}{1+z} \prod_{i=1}^{n} \frac{(z+2^i)^2}{(z-2^i)^2} .$$

The first part follows from the observation that each fraction is upper bounded by 1.

To prove the second part, notice that $P_n(z)$ is non-negative and $P_n(-z)$ is non-positive for any $z \in [1, 2^n]$. Now, fix $z \in [1, 2^n]$ and let $j \in \{1, 2, \ldots, n\}$ be such that $2^{j-1} \le z \le 2^j$. This implies that $(z+2^j)^2 \ge (2^j)^2 \ge 4(z-2^j)^2$. We can write $\frac{P_n(z)}{-P_n(-z)}$ as a product of $n+1$ non-negative fractions

$$\frac{P_n(z)}{-P_n(-z)} = \frac{z-1}{z+1} \cdot \frac{(z-2^j)^2}{(z+2^j)^2} \prod_{\substack{i \,:\, 1 \le i \le n \\ i \ne j}} \frac{(z-2^i)^2}{(z+2^i)^2} .$$

The second part follows from the observation that the second fraction is upper bounded by $1/4$ and all other fractions are upper bounded by 1.

The third part follows from

$$-P_n(-z) = (1+z) \prod_{i=1}^{n} (z+2^i)^2 \ge \prod_{i=1}^{n} 2^{2i} = 2^{n(n+1)} .$$

□

Lemma 17 (Properties of $S_{n,r}$ and $B_{n,r}$)**.** *Let n, m be non-negative integers. Let $r = 2\left\lceil \frac{1}{4}\log_2(4m+1) \right\rceil + 1$. Then,*

1. *If $z \in [1, 2^n]$ then $S_{n,r}(z) \in [1, 1 + \frac{1}{2m}]$.*
2. *If $z \in [-2^n, -1]$ then $S_{n,r}(z) \in [-1 - \frac{1}{2m}, -1]$.*
3. *If $z \in [-1, 1]$ then $|S_{n,r}(z)| \le 1$.*
4. *If $z \in [-2^n, 2^n]$ then $B_{n,r}(z) \ge \left(1 - \frac{1}{4m+1}\right) 2^{n(n+1)r}$.*

Proof. Note that $B_{n,r}(z)$ is an even function and $A_{n,r}(z)$ is an odd function. Therefore, $S_{n,r}(z)$ is odd. Also notice that r is an odd integer.

1. Observe that $S_{n,r}(z)$ can be written as
$$S_{n,r}(z) = \frac{1 + \left(-\dfrac{P_n(z)}{P_n(-z)}\right)^r}{1 - \left(-\dfrac{P_n(z)}{P_n(-z)}\right)^r} = \frac{1+c}{1-c}$$
where $c = \left(-\frac{P_n(z)}{P_n(-z)}\right)^r$. Since $z \in [1, 2^n]$, by part 2 of Lemma 16, $c \in [0, \frac{1}{4^r}]$. Since $r \geq \frac{1}{2}\log_2(4m+1)$, this means that $c \in [0, \frac{1}{4m+1}]$. Thus, $S_{n,r}(z) = \frac{1+c}{1-c} \in [1, 1 + \frac{1}{2m}]$.

2. Since $S_{n,r}(z)$ is odd, the statement follows from part 1.

3. Recall that $S_{n,r}(z)$ can be written as
$$S_{n,r}(z) = \frac{1+c}{1-c}$$
where $c = \left(-\frac{P_n(z)}{P_n(-z)}\right)^r$. If $z \in [0, 1]$, by part 1 of Lemma 16 and the fact that r is odd, $c \in [-1, 0]$, and thus, $S_{n,r}(z) = \frac{1+c}{1-c} \in [0, 1]$. Since $S_{n,r}(z)$ is odd, for $z \in [-1, 0]$, $S_{n,r}(z) \in [-1, 0]$.

4. Since $B_{n,r}(z)$ is even, we can without loss generality assume that $z \geq 0$. We consider two cases.
Case $z \in [0, 1]$. Since r is odd and $P_n(z)$ is non-positive,
$$\begin{aligned} B_{n,r}(z) &= -(P_n(z))^r + \big(-P_n(-z)\big)^r \\ &\geq \big(-P_n(-z)\big)^r \geq 2^{n(n+1)r} \\ &\geq 2^{n(n+1)r}\left(1 - \frac{1}{4m+1}\right) . \end{aligned}$$
where the second last inequality follows from part 3 of Lemma 16.
Case $z \in [1, 2^n]$. Since r is odd,
$$\begin{aligned} B_{n,r}(z) &= \big(-P_n(-z)\big)^r \left(1 - \left(-\frac{P_n(z)}{P_n(-z)}\right)^r\right) \\ &= \big(-P_n(-z)\big)^r (1 - c) \end{aligned}$$
where $c = \left(-\frac{P_n(z)}{P_n(-z)}\right)^r$. Since $z \in [1, 2^n]$, by part 2 of Lemma 16, $c \in [0, \frac{1}{4^r}]$. By the definition of r that means that $c \in [0, \frac{1}{4m+1}]$. Thus,
$$\begin{aligned} B_{n,r}(z) &\geq \big(-P_n(-z)\big)^r \left(1 - \frac{1}{4m+1}\right) \\ &\geq 2^{n(n+1)r}\left(1 - \frac{1}{4m+1}\right) . \end{aligned}$$
where the last inequality follows from part 3 of Lemma 16.

□

Lemma 18 (Properties of $Q(x)$)**.** *The rational function $Q(x)$ satisfies*

1. $Q(x) \geq \frac{1}{2}$ *for all* $x \in \bigcap_{i=1}^{m} \left\{ x \in \mathbb{R}^d \,:\, \|x\| \leq 1,\ \langle v_i, x\rangle \geq \gamma \right\}$,

2. $Q(x) \leq -\frac{1}{2}$ *for all* $x \in \bigcup_{i=1}^{m} \left\{ x \in \mathbb{R}^d \,:\, \|x\| \leq 1,\ \langle v_i, x\rangle \leq -\gamma \right\}$.

Proof. To prove part 1, consider any $x \in \bigcap_{i=1}^m \left\{x \in \mathbb{R}^d \ : \ \|x\| \le 1, \ \langle v_i, x \rangle \ge \gamma \right\}$. Then, $\frac{\langle v_i, x \rangle}{\gamma} \in [1, \frac{1}{\gamma}]$. By part 1 of Lemma 17, $S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \in [1, 1 + \frac{1}{2m}]$ and in particular $S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \ge 1$. Thus,

$$\begin{aligned} Q(x) &= \left(\sum_{i=1}^m S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \right) - (m - 1/2) \\ &\ge m - (m - 1/2) \\ &= 1/2 \ . \end{aligned}$$

To prove part 2, consider any $x \in \bigcup_{i=1}^m \left\{x \in \mathbb{R}^d \ : \|x\| \le 1, \ \langle v_i, x \rangle \le -\gamma \right\}$. Observe that $\frac{\langle v_i, x \rangle}{\gamma} \in [-\frac{1}{\gamma}, \frac{1}{\gamma}]$. Consider $S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right)$ for any $i \in \{1, 2, \dots, m\}$. Parts 1,2, and 3 of Lemma 17 and the fact $1/\gamma \le 2^s$ imply that $S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \le 1 + \frac{1}{2m}$ for all $i \in \{1, 2, \dots, m\}$. By the choice of x, there exists $j \in \{1, 2, \dots, m\}$ such that $\langle v_j, x \rangle \le -\gamma$. Part 2 of Lemma 17 implies that $S_{s,r}\left(\frac{\langle v_j, x \rangle}{\gamma}\right) \in [-1 - \frac{1}{2m}, -1]$. Thus,

$$\begin{aligned} Q(x) &= \left(\sum_{i=1}^m S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \right) - \left(m - \frac{1}{2}\right) \\ &= S_{s,r}\left(\frac{\langle v_j, x \rangle}{\gamma}\right) + \left(\sum_{\substack{i \, : \, 1 \le i \le m \\ i \neq j}} S_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right) \right) - \left(m - \frac{1}{2}\right) \\ &\le -1 + (m-1)\left(1 + \frac{1}{2m}\right) - \left(m - \frac{1}{2}\right) \\ &\le -1/2 \ . \end{aligned}$$

□

To prove parts 1 and 2 of Theorem 8 first note that part 4 of Lemma 17 implies that for any x such that $\|x\| \le 1$, $B_{s,r}\left(\frac{\langle v_i, x \rangle}{\gamma}\right)$ is positive. Thus $p(x)$ and $Q(x)$ have the same sign on the unit ball. Consider any x in either $\bigcap_{i=1}^m \left\{x \in \mathbb{R}^d \ : \ \|x\| \le 1, \ \langle v_i, x \rangle \ge \gamma \right\}$ or in $\bigcup_{i=1}^m \left\{x \in \mathbb{R}^d \ : \ \|x\| \le 1, \ \langle v_i, x \rangle \le -\gamma \right\}$. Lemma 18 states that $|Q(x)| \ge 1/2$ and the sign depends on which of the two sets x lies in. Since signs of $Q(x)$ and $p(x)$ are the same, it remains to show that $|p(x)| \ge \frac{1}{4} \cdot 2^{s(s+1)rm}$. Indeed,

$$\begin{aligned} |p(x)| &= 2^{-s(s+1)rm+1} \cdot |Q(x)| \prod_{j=1}^m B_{s,r}\left(\frac{\langle v_j, x \rangle}{\gamma}\right) \\ &\ge 2^{-s(s+1)rm+1} \cdot |Q(x)| \left(2^{s(s+1)r} \left(1 - \frac{1}{4m+1}\right)\right)^m \\ &\ge |Q(x)| \ge \frac{1}{2} \quad \text{(Lemma 18)} \ . \end{aligned}$$

where we used that $\left(1 - \frac{1}{4m+1}\right)^m \ge e^{-\frac{1}{4}} \ge 1/2$.

To prove part 3 of Theorem 8 note that $\deg(P_s) = 2s+1$. Thus, $\deg(A_{s,r})$ and $\deg(B_{s,r})$ are at most $(2s+1)r$. Therefore, $\deg(p) \le (2s+1)rm$.

It remains to prove part 4 of Theorem 8. For any $i \in \{0, 1, 2, \dots, s\}$ and any $v \in \mathbb{R}^d$ such that $\|v\| \le 1$ define multivariate polynomials

$$\begin{aligned} f_{i,v}(x) &= \frac{\langle v, x\rangle}{\gamma} - 2^i \ , \\ q_v(x) &= P_s\left(\frac{\langle v, x\rangle}{\gamma}\right) \ , \\ a_v(x) &= A_{s,r}\left(\frac{\langle v, x\rangle}{\gamma}\right) \ , \\ b_v(x) &= B_{s,r}\left(\frac{\langle v, x\rangle}{\gamma}\right) \ . \end{aligned}$$

Note that

$$q(x) = \left[\sum_{i=1}^m a_{v_i}(x) \prod_{\substack{j \,:\, 1 \le j \le m \\ j \neq i}} b_{v_j}(x)\right] - \left(m - \frac{1}{2}\right) \prod_{j=1}^n b_{v_j}(x) \ .$$

We bound the norms of these polynomials. We have

$$\left\|f_{i,v}\right\|^2 = \|v\|^2 / \gamma^2 + 2^{2i} \le 2 \cdot 2^{2s} \ .$$

where we used that $1/\gamma \le 2^s$ and $\|v\| \le 1$. Since $q_v(x) = f_{i,v}(\frac{\langle v,x\rangle}{\gamma}) \prod_{i=1}^s \left(f_{i,v}(\frac{\langle v,x\rangle}{\gamma})\right)^2$, using part 1 of Lemma 14 we upper bound the norm of q_v as

$$\begin{aligned} \|q_v\|^2 &\le (2s+1)^{2s+1} \left\|f_{0,v}\right\|^2 \prod_{i=1}^s \left\|f_{i,v}\right\|^4 \\ &\le (2s+1)^{2s+1} (2 \cdot 2^{2s})^{2s+1} \ . \end{aligned}$$

Using parts 3 and 2 of Lemma 14 we upper bound the norm of a_v as

$$\begin{aligned} \|a_v\|^2 &\le 2\left\|(q_v)^r\right\|^2 + 2\left\|(q_{-v})^r\right\|^2 \\ &\le 2r^{r(2s+1)} (\|q_v\|^2)^r + 2r^{r(2s+1)} (\|q_{-v}\|^2)^r \\ &\le 4r^{r(2s+1)} \left((2s+1)^{2s+1} (2 \cdot 2^s)^{2s+1}\right)^r \\ &= 4\left(2^{2s} r(4s+2)\right)^{(2s+1)r} \ . \end{aligned}$$

The same upper bound holds for $\|b_v\|^2$. Therefore,

$$\begin{aligned} \|q\| &\le \left[\sum_{i=1}^m \left\| a_{v_i} \prod_{\substack{j \,:\, 1 \le j \le m \\ j \neq i}} b_{v_j} \right\|\right] + \left(m - \frac{1}{2}\right) \left\|\prod_{j=1}^m b_{v_j}\right\| \\ &\le \left[\sum_{i=1}^m m^{(s+1/2)rm} \|a_{v_i}\| \prod_{\substack{j \,:\, 1 \le j \le m \\ j \neq i}} \|b_{v_j}\|\right] \\ &\qquad + \left(m - \frac{1}{2}\right) m^{(s+1/2)rm} \prod_{j=1}^m \|b_{v_j}\| \\ &\le (2m - 1/2) m^{(s+1/2)rm} \left(4\left(2^{2s} r(4s+2)\right)^{(2s+1)r}\right)^{m/2} \\ &= (2m - 1/2) 2^m \cdot \left(2^{2s} rm(4s+2)\right)^{(s+1/2)rm} \ . \end{aligned}$$

Finally, $\|p\| = 2^{-s(s+1)rm+1}\|q\| \leq (4m-1)2^m \cdot \left(2^s rm(4s+2)\right)^{(s+1/2)rm}$. The theorem follows.

E. Proof of Theorem 5

Proof of Theorem 5. Since the examples (x_1, y_1), (x_2, y_2), $\ldots$, (x_T, y_T) are weakly linearly separable with margin γ,, there are vectors $w_1, w_2, \ldots, w_K$ satisfying (1) and (2).

Fix any $i \in \{1, 2, \ldots, K\}$. Consider the $K-1$ vectors $(w_i - w_j)/2$ for $j \in \{1, 2, \ldots, K\} \setminus \{i\}$. Note that the vectors have norm at most 1. We consider two cases regarding the relationship between γ_1 and γ_2.

Case 1: $\gamma_1 \geq \gamma_2$**.** In this case, Theorem 7 implies that there exist a multivariate polynomial $p_i : \mathbb{R}^d \to \mathbb{R}$,

$$\deg(p_i) = \lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil ,$$

such that all examples x in R_i^+ (resp. R_i^-) satisfy $p_i(x) \geq 1/2$ (resp. $p_i(x) \leq -1/2$). Therefore, for all $t = 1, 2, \ldots, T$, if $y_t = i$ then $p_i(x_t) \geq 1/2$, and if $y_t \neq i$ then $p_i(x_t) \leq -1/2$, and

$$\|p_i\| \leq \left(188 \lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil \right)^{\frac{1}{2}\lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil} .$$

By Lemma 9, there exists $c_i \in \ell_2$ such that $\langle c_i, \phi(x) \rangle = p_i(x)$, and

$$\|c_i\|_{\ell_2} \leq \left(376 \lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil \right)^{\frac{1}{2}\lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil} .$$

Define vectors $u_i \in \ell_2$ as

$$u_i = \frac{1}{\sqrt{K}} \cdot \frac{c_i}{\left(376 \lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil \right)^{\frac{1}{2}\lceil \log_2(2K-2) \rceil \cdot \left\lceil \sqrt{\frac{2}{\gamma}} \right\rceil}} .$$

Then, $\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_K\|^2 \leq 1$. Furthermore, for all $t = 1, 2, \ldots, T$, $\langle u_{y_t}, \phi(x_t) \rangle \geq \gamma_1$ and for all $j \in \{1, 2, \ldots, K\} \setminus \{y_t\}$, $\langle u_j, \phi(x_t) \rangle \leq -\gamma_1$. In other words, $(\phi(x_1), y_1), (\phi(x_2), y_2), \ldots, (\phi(x_T), y_T)$ are strongly linearly separable with margin $\gamma_1 = \max\{\gamma_1, \gamma_2\}$.

Case 2: $\gamma_1 < \gamma_2$**.** In this case, Theorem 8 implies that there exist a multivariate polynomial $q_i : \mathbb{R}^d \to \mathbb{R}$,

$$\deg(q_i) = (2s+1)r(K-1) ,$$

such that all examples x in R_i^+ (resp. R_i^-) satisfy $q_i(x) \geq 1/2$ (resp. $q_i(x) \leq -1/2$), and

$$\|q_i\| \leq (4K-5)2^{K-1} \cdot \left(2^s r(K-1)(4s+2)\right)^{(s+1/2)r(K-1)} .$$

Recall that here,

$$r = 2\left\lceil \frac{1}{4}\log_2(4K-3) \right\rceil + 1 \quad \text{and} \quad s = \lceil \log_2(1/\gamma) \rceil .$$

Therefore, for all $t = 1, 2, \ldots, T$, if $y_t = i$ then $q_i(x_t) \geq 1/2$, and if $y_t \neq i$ then $q_i(x_t) \leq -1/2$.

By Lemma 9, there exists $c_i' \in \ell_2$ such that $\langle c_i', \phi(x) \rangle = p_i(x)$, and

$$\|c_i'\|_{\ell_2} \leq (4K-5)2^{K-1} \cdot \left(2^{s+1} r(K-1)(4s+2)\right)^{(s+1/2)r(K-1)} .$$

Define vectors $u'_i \in \ell_2$ as

$$u'_i = \frac{c'_i \cdot \left(2^{s+1} r(K-1)(4s+2)\right)^{-(s+1/2)r(K-1)}}{\sqrt{K}(4K-5)2^{K-1}} \, .$$

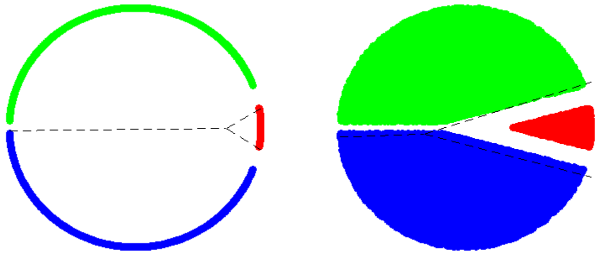
Then, $\left\|u'_1\right\|^2 + \left\|u'_2\right\|^2 + \cdots + \left\|u'_K\right\|^2 \leq 1$. Furthermore, for all $t = 1, 2, \ldots, T$, $\left\langle u'_{y_t}, \phi(x_t) \right\rangle \geq \gamma_2$ and for all $j \in \{1, 2, \ldots, K\} \setminus \{y_t\}$, $\left\langle u'_j, \phi(x_t) \right\rangle \leq -\gamma_2$. In other words, $(\phi(x_1), y_1), (\phi(x_2), y_2), \ldots, (\phi(x_T), y_T)$ are strongly linearly separable with margin $\gamma_2 = \max\{\gamma_1, \gamma_2\}$.

In summary, the examples are strongly linearly separable with margin $\gamma' = \max\{\gamma_1, \gamma_2\}$. Finally, observe that for any $t = 1, 2, \ldots, T$,

$$k(x_t, x_t) = \frac{1}{1 - \frac{1}{2}\|x_t\|^2} \leq 2 \, . \qquad \square$$

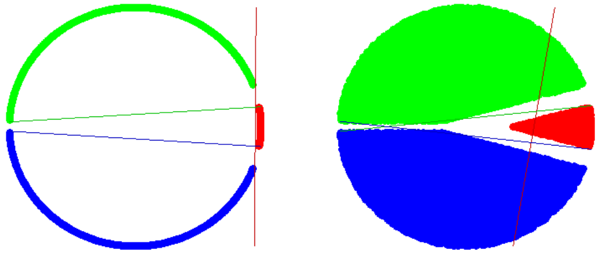
F. Supplementary Materials for Section 6

Figures 6, 7, and 8 show the final decision boundaries learned by each algorithm on the two datasets (Figures 4 and 5), after $T = 5 \times 10^6$ rounds. We used the version of Banditron with exploration rate of 0.02, which explores the most.



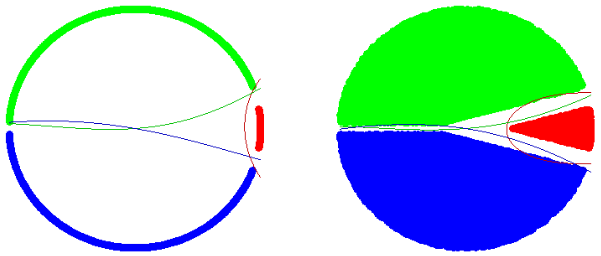
(a) Strongly separable case (b) Weakly separable case

Figure 6. BANDITRON's final decision boundaries



(a) Strongly separable case (b) Weakly separable case

Figure 7. Algorithm 1's final decision boundaries



(a) Strongly separable case (b) Weakly separable case

Figure 8. Algorithm 2 (with rational kernel)'s final decision boundaries

G. Nearest neighbor algorithm

```
Algorithm 4 NEAREST-NEIGHBOR ALGORITHM
Require: Number of classes $K$, number of rounds $T$.
Require: Inner product space $(V, \langle\cdot,\cdot\rangle)$.
Initialize $S \leftarrow \emptyset$
for $t = 1, 2, \ldots, T$: do
    if $\min_{(x,y)\in S} \|x_t - x\| \le \gamma$ then
        Find nearest neighbor
        $(\widetilde{x}, \widetilde{y}) = \text{argmin}_{(x,y)\in S} \|x_t - x\|$
        Predict $\widehat{y}_t = \widetilde{y}$
    else
        Predict $\widehat{y}_t \sim$ Uniform$(\{1, 2, \ldots, K\})$
        Receive feedback $z_t = \mathbb{1}[\widehat{y}_t \neq y_t]$
        if $z_t = 0$ then
            $S \leftarrow S \cup \{(x_t, \widehat{y}_t)\}$
```

In this section we analyze NEAREST-NEIGHBOR ALGORITHM shown as Algorithm 4. The algorithm is based on the obvious idea that, under the weak linear separability assumption, two examples that are close to each other must have the same label. The lemma below formalizes this intuition.

Lemma 19 (Non-separation lemma)**.** *Let $(V, \langle\cdot,\cdot\rangle)$ be a vector space, K be a positive integer and let γ be a positive real number. Suppose $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T) \in V \times \{1, 2, \ldots, K\}$ are labeled examples that are weakly linearly separable with margin γ. For i, j in $\{1, 2, \ldots, T\}$, if $\left\|x_i - x_j\right\|_2 \le \gamma$ then $y_i = y_j$.*

Proof. Suppose for the sake on contradiction that $y_i \neq y_j$. By Definition 1, there exists vectors $w_1, \ldots, w_K$ such that conditions (1) and (2) are satisfied.

Specifically,

$$\left\langle w_{y_i} - w_{y_j}, x_i \right\rangle \ge \gamma \; ,$$
$$\left\langle w_{y_j} - w_{y_i}, x_j \right\rangle \ge \gamma \; .$$

This implies that

$$\left\langle w_{y_i} - w_{y_j}, x_i - x_j \right\rangle \ge 2\gamma \; .$$

On the other hand,

$$\left\langle w_{y_i} - w_{y_j}, x_i - x_j \right\rangle \le \left\| w_{y_i} - w_{y_j} \right\| \cdot \left\| x_i - x_j \right\| \le \sqrt{2}\gamma$$

where the first inequality is from Cauchy-Schwartz inequality, the second inequality is from that $\left\| w_{y_i} - w_{y_j} \right\| \le \sqrt{2(\left\|w_{y_i}\right\|^2 + \left\|w_{y_j}\right\|^2)} \le \sqrt{2}$ and our assumption on x_i and x_j. Therefore, we reach a contradiction. □

We also need to define several notions. A subset $S \subseteq \mathbb{R}^d$ is called a γ-packing if for any $x, x' \in S$ such that $x \neq x'$ we have $\left\|x - x'\right\| > \gamma$. The following lemma is standard. Also recall that $\mathrm{B}(x, R) = \{x' \in \mathbb{R}^d \ : \ \left\|x' - x\right\| \leq R\}$ denotes the closed ball of radius R centered a point x. For set $S \subseteq \mathbb{R}^d$, denote by $\mathrm{Vol}(S)$ the volume of S.

Lemma 20 (Size of γ-packing)**.** *Let γ and R be positive real numbers. If $S \subseteq \mathrm{B}(\mathbf{0}, R) \subseteq \mathbb{R}^d$ is a γ-packing then*

$$|S| \leq \left(\frac{2R}{\gamma} + 1\right)^d \ .$$

Proof. If S is a γ-packing then $\{\mathrm{B}(x, \gamma/2) \ : \ x \in S\}$ is a collection of disjoint balls of radius γ that fit into $\mathrm{B}(\mathbf{0}, R + \gamma/2)$. Thus,

$$|S| \cdot \mathrm{Vol}(\mathrm{B}(\mathbf{0}, \gamma/2)) \leq \mathrm{Vol}(\mathrm{B}(\mathbf{0}, R + \gamma/2))$$

Hence,

$$|S| \leq \frac{\mathrm{Vol}(\mathrm{B}(\mathbf{0}, R + \gamma/2))}{\mathrm{Vol}(\mathrm{B}(\mathbf{0}, \gamma/2))} = \left(\frac{R + \gamma/2}{\gamma/2}\right)^d = \left(\frac{2R}{\gamma} + 1\right)^d \ .$$

□

Theorem 21 (Mistake upper bound for NEAREST-NEIGHBOR ALGORITHM)**.** *Let K and d be positive integers and let γ, R be a positive real numbers. Suppose $(x_1, y_1), \ldots, (x_T, y_T) \in \mathbb{R}^d \times \{1, 2, \ldots, K\}$ are labeled examples that are weakly linearly separable with margin γ and satisfy $\|x_1\|, \|x_2\|, \ldots, \|x_T\| \leq R$. Then, the expected number of mistakes made by Algorithm 4 is at most*

$$(K - 1)\left(\frac{2R}{\gamma} + 1\right)^d \ .$$

Proof. Let M be the number of mistakes made by the algorithm. Let b_t be the indicator that line 7 is executed at time step t, i.e. we fall into the "else" case. Note that if $b_t = 0$, then by Lemma 19, the prediction $\widehat{y}_t$ must equal y_t, i.e. $z_t = 0$. Therefore, $M = \sum_{t=1}^T z_t = \sum_{t=1}^T b_t z_t$. Let $U = \sum_{t=1}^T b_t (1 - z_t)$. Clearly, $|S| = U$. Since $S \subseteq \mathrm{B}(\mathbf{0}, R)$ is a γ-packing, $U = |S| \leq (\frac{2R}{\gamma} + 1)^d$.

Note that when $b_t = 1$, $\widehat{y}_t$ is chosen uniformly at random, we have

$$\mathbf{E}[z_t \mid b_t = 1] = \frac{K-1}{K} \ .$$

Therefore,

$$\mathbf{E}[M] = \mathbf{E}\left[\sum_{t=1}^T b_t z_t\right] = \frac{K-1}{K}\,\mathbf{E}\left[\sum_{t=1}^T b_t\right] \ .$$

On the other hand,

$$\mathbf{E}[U] = \mathbf{E}\left[\sum_{t=1}^T b_t (1 - z_t)\right] = \frac{1}{K}\,\mathbf{E}\left[\sum_{t=1}^T b_t\right] \ .$$

Therefore,

$$\mathbf{E}[M] = (K - 1)\,\mathbf{E}[U] \leq (K - 1)\left(\frac{2R}{\gamma} + 1\right)^d \ .$$

□

H. NP-hardness of the weak labeling problem

Any algorithm for the bandit setting collects information in the form of so called *strongly labeled* and *weakly labeled* examples. Strongly-labeled examples are those for which we know the class label. Weakly labeled example is an example for which we know that class label can be anything except for a particular one class.

A natural strategy for each round is to find vectors $w_1, w_2, \ldots, w_K$ that linearly separate the examples seen in the previous rounds and use the vectors to predict the label in the next round. More precisely, we want to find both the vectors $w_1, w_2, \ldots, w_K$ and label for each example consistent with its weak and/or strong labels such that $w_1, w_2, \ldots, w_K$ linearly separate the labeled examples. We show this problem is NP-hard even for $K = 3$.

Clearly, the problem is at least as hard as the decision version of the problem where the goal is to determine if such vectors and labeling exist. We show that this problem is NP-complete.

We use symbols $[K] = \{1, 2, \ldots, K\}$ for strong labels and $[\overline{K}] = \{\overline{1}, \overline{2}, \ldots, \overline{K}\}$ for weak labels. Formally, the weak labeling problem can be described as below:

Weak Labeling

Given: Feature-label pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T)$ in $\{0,1\}^d \times \{1, 2, \ldots, K, \overline{1}, \overline{2}, \ldots, \overline{K}\}$.
Question: Do there exist $w_1, w_2, \ldots, w_K \in \mathbb{R}^d$ such that for all $t = 1, 2, \ldots, T$,

$$y_t \in [K] \Longrightarrow \forall i \in [K] \setminus \{y_t\} \quad \left\langle w_{y_t}, x_t \right\rangle > \langle w_i, x_t \rangle \ ,$$

and

$$y_t \in [\overline{K}] \Longrightarrow \exists i \in [K] \quad \langle w_i, x_t \rangle > \left\langle w_{\overline{y_t}}, x_t \right\rangle \ ?$$

The hardness proof is based on a reduction from the set splitting problem, which is proven to be NP-complete by Lovász (Garey & Johnson, 1979), to our weak labeling problem. The reduction is adapted from (Blum & Rivest, 1993).

Set Splitting

Given: A finite set S and a collection C of subsets c_i of S.
Question: Do there exist disjoint sets S_1 and S_2 such that $S_1 \cup S_2 = S$ and $\forall i, c_i \not\subseteq S_1$ and $c_i \not\subseteq S_2$?

Below we show the reduction. Suppose we are given an instance of the set splitting problem

$$S = \{1, 2, \ldots, N\} \ , C = \{c_1, c_2, \ldots, c_M\} \ .$$

We create the weak labeling instance as follows. Let $d = N + 1$ and $K = 3$. Define $\mathbf{0}$ as the zero vector $(0, \ldots, 0) \in \mathbb{R}^N$ and $\mathbf{e}_i$ as the i-th standard vector $(0, \ldots, 1, \ldots, 0) \in \mathbb{R}^N$. Then we include all the following feature-label pairs:

- Type 1: $(x, y) = ((\mathbf{0}, 1), 3)$,
- Type 2: $(x, y) = ((\mathbf{e}_i, 1), \overline{3})$ for all $i \in \{1, 2, \ldots, N\}$,
- Type 3: $(x, y) = \left(\left(\sum_{i \in c_j} \mathbf{e}_i, 1\right), 3\right)$ for all $j \in \{1, 2, \ldots, M\}$.

For example, if we have $S = \{1, 2, 3\}$, $C = \{c_1, c_2\}$, $c_1 = \{1, 2\}$, $c_2 = \{2, 3\}$, then we create the weak labeling sample set as:

$$\{((0, 0, 0, 1), 3), ((1, 0, 0, 1), \overline{3}), ((0, 1, 0, 1), \overline{3}), ((0, 0, 1, 1), \overline{3}), ((1, 1, 0, 1), 3), ((0, 1, 1, 1), 3)\} \ .$$

The following lemma shows that answering this weak labeling problem is equivalent to answering the original set splitting problem.

Lemma 22. *Any instance of the set splitting problem is a YES instance if and only if the corresponding instance of the weak labeling problem (as described above) is a YES instance.*

Proof. ($\Longrightarrow$) Let S_1, S_2 be the solution of the set splitting problem. Define

$$w_1 = \left(a_1, a_2, \cdots, a_N, -\frac{1}{2}\right),$$

where for all $i \in \{1, 2, \ldots, N\}$, $a_i = 1$ if $i \in S_1$ and $a_i = -N$ if $i \notin S_1$. Similarly, define

$$w_2 = \left(b_1, b_2, \cdots, b_N, -\frac{1}{2}\right),$$

where for all $i \in \{1, 2, \ldots, N\}$, $b_i = 1$ if $i \in S_2$ and $b_i = -N$ if $i \notin S_2$. Finally, define

$$w_3 = (0, 0, \cdots, 0),$$

the zero vector. To see this is a solution for the weak labeling problem, we verify separately for Type 1-3 samples defined above. For Type 1 sample, we have

$$\langle w_3, x \rangle = 0 > -\frac{1}{2} = \langle w_1, x \rangle = \langle w_2, x \rangle .$$

For a Type 2 sample that corresponds to index i, we have either $i \in S_1$ or $i \in S_2$ because $S_1 \cup S_2 = \{1, 2, \ldots, N\}$ is guaranteed. Thus, either $a_i = 1$ or $b_i = 1$. If $a_i = 1$ is the case, then

$$\langle w_1, x \rangle = a_i - \frac{1}{2} = \frac{1}{2} > 0 = \langle w_3, x \rangle ;$$

similarly if $b_i = 1$, we have $\langle w_2, x \rangle > \langle w_3, x \rangle$.
For a Type 3 sample that corresponds to index j, Since $c_j \not\subset S_1$, there exists some $i' \in c_j$ and $i' \notin S_1$. Thus we have $x_{i'} = 1$, $a_{i'} = -N$, and therefore

$$\begin{aligned}\langle w_1, x \rangle &= a_{i'} x_{i'} + \sum_{i \in \{1,2,\ldots,N\} \setminus \{i'\}} a_i x_i - \frac{1}{2} \\ &\le -N + (N-1) - \frac{1}{2} < 0 = \langle w_3, x \rangle \ .\end{aligned}$$

Because $c_j \not\subset S_2$ also holds, we also have $\langle w_2, x \rangle < \langle w_3, x \rangle$. This direction is therefore proved.

($\Longleftarrow$) Given the solution w_1, w_2, w_3 of the weak labeling problem, we define

$$\begin{aligned} S_1 &= \Big\{ i \in \{1, 2, \ldots, n\} \ : \ \big\langle w_1 - w_3, (\mathbf{e}_i, 1) \big\rangle > 0 \Big\}, \\ S_2 &= \Big\{ i \in \{1, 2, \ldots, n\} \ : \ \big\langle w_2 - w_3, (\mathbf{e}_i, 1) \big\rangle > 0 \text{ and } i \notin S_1 \Big\}. \end{aligned}$$

It is not hard to see $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = \{1, 2, \ldots, N\}$. The former is because S_2 only includes elements that are not in S_1. For the latter, note that $(\mathbf{e}_i, 1)$ is the feature vector for Type 2 samples. Because Type 2 samples all have label $\overline{3}$, for any $i \in \{1, 2, \ldots, N\}$, one of the following must hold: $\big\langle w_1 - w_3, (\mathbf{e}_i, 1) \big\rangle > 0$ or $\big\langle w_2 - w_3, (\mathbf{e}_i, 1) \big\rangle > 0$. This implies $i \in S_1$ or $i \in S_2$.

Now we show $\forall j$, $c_j \not\subset S_1$ and $c_j \not\subset S_2$ by contradiction. Assume there exists some j such that $c_j \subset S_1$. By our definition of S_1, we have $\big\langle w_1 - w_3, (\mathbf{e}_i, 1) \big\rangle > 0$ for all $i \in c_j$. Therefore,

$$\sum_{i \in c_j} \big\langle w_1 - w_3, (\mathbf{e}_i, 1) \big\rangle = \left\langle w_1 - w_3, \left(\sum_{i \in c_j} \mathbf{e}_i, |c_j| \right) \right\rangle > 0.$$

Because Type 1 sample has label 3, we also have

$$\left\langle w_1 - w_3, (\mathbf{0}, 1) \right\rangle < 0.$$

Combining the above two inequalities, we get

$$\left\langle w_1 - w_3, \left(\sum_{i \in c_j} \mathbf{e}_i, 1 \right) \right\rangle = \left\langle w_1 - w_3, \left(\sum_{i \in c_j} \mathbf{e}_i, |c_j| \right) \right\rangle - (|c_j| - 1) \left\langle w_1 - w_3, (\mathbf{0}, 1) \right\rangle > 0 \ .$$

Note that $\left(\sum_{i \in c_j} \mathbf{e}_i, 1 \right)$ is a feature vector for Type 3 samples. Thus the above inequality contradicts that Type 3 samples have label 3. Therefore, $c_j \not\subset S_1$. If we assume there exists some $c_j \subset S_2$, same arguments apply and also lead to contradiction. □

I. Mistake lower bound for ignorant algorithms

In this section, we prove a mistake lower bound for a family of algorithms called *ignorant algorithms*. Ignorant algorithms ignore the examples on which they make mistakes. This assumption seems strong, but as we will explain below, it is actually natural, and several recently proposed bandit linear classification algorithms that achieve $\sqrt{T}$ regret bounds belong to this family, e.g., SOBA (Beygelzimer et al., 2017), OBAMA (Foster et al., 2018). Also, NEAREST-NEIGHBOR ALGORITHM (Algorithm 4) presented in Appendix G is an ignorant algorithm.

Under the assumption that the examples lie in in the unit ball of $\mathbb{R}^d$ and are weakly linearly separable with margin γ, we show that any ignorant algorithm must make at least $\Omega\left(\left(\frac{1}{160\gamma} \right)^{(d-2)/4} \right)$ mistakes in the worst case. In other words, an algorithm that achieves a better mistake bound cannot ignore examples on which it makes a mistake and it must make a meaningful update on such examples.

To formally define ignorant algorithms, we define the conditional distribution from which an algorithm draws its predictions. Formally, given an algorithm $\mathcal{A}$ and an adversarial strategy, we define

$$p_t(y|x) = \Pr[y_t = y \mid (x_1, y_1), (x_2, y_2) \dots, (x_{t-1}, y_{t-1}), x_t = x] \ .$$

In other words, in any round t, conditioned on the past $t-1$ rounds, the algorithm $\mathcal{A}$ chooses y_t from probability distribution $p_t(\cdot|x_t)$. Formally, p_t is a function $p : \{1, 2, \dots, K\} \times \mathbb{R}^d \to [0, 1]$ such that $\sum_{y=1}^K p_t(y|x) = 1$ for any $x \in \mathbb{R}^d$.

Definition 23 (Ignorant algorithm)**.** *An algorithm $\mathcal{A}$ for* ONLINE MULTICLASS LINEAR CLASSIFICATION WITH BANDIT FEEDBACK *is called* ignorant *if for every $t = 1, 2, \dots, T$, p_t is determined solely by the sequence (x_{a_1}, y_{a_1}),(x_{a_2}, y_{a_2}), $\dots$, (x_{a_n}, y_{a_n}) of labeled examples from the rounds $1 \le a_1 < a_2 < \dots < a_n < t$ in which the algorithm makes a correct prediction.*

An equivalent definition of an ignorant algorithm is that the memory state of the algorithm does not change after it makes a mistake. Equivalently, the memory state of an ignorant algorithm is completely determined by the sequence of labeled examples on which it made correct prediction.

To explain the definition, consider an ignorant algorithm $\mathcal{A}$. Suppose that on a sequence of examples (x_1, y_1), (x_2, y_2), $\dots$, (x_{t-1}, y_{t-1}) generated by some adversary the algorithm $\mathcal{A}$ makes correct predictions in rounds $a_1, a_2, \dots, a_n$ where $1 \le a_1 < a_2 < \dots < a_n < t$ and errors on rounds $\{1, 2, \dots, t-1\} \setminus \{a_1, a_2, \dots, a_n\}$. Suppose that on another sequence of examples $(x'_1, y'_1), (x'_2, y'_2), \dots, (x'_{s-1}, y'_{s-1})$ generated by another adversary the algorithm $\mathcal{A}$ makes correct predictions in rounds $b_1, b_2, \dots, b_n$ where $1 \le b_1 < b_2 < \dots < b_n < s$ and errors on rounds $\{1, 2, \dots, s-1\} \setminus \{b_1, b_2, \dots, b_n\}$. Futhermore, suppose

$$\begin{aligned} (x_{a_1}, y_{a_1}) &= (x'_{b_1}, y'_{b_1}) \ , \\ (x_{a_2}, y_{a_2}) &= (x'_{b_2}, y'_{b_2}) \ , \\ &\vdots \\ (x_{a_n}, y_{a_n}) &= (x'_{b_2}, y'_{b_n}) \ . \end{aligned}$$

Then, as $\mathcal{A}$ is ignorant,

$$\Pr[y_t = y \mid (x_1, y_1), (x_2, y_2) \ldots, (x_{t-1}, y_{t-1}), x_t = x] = \Pr[y'_t = y \mid (x'_1, y'_1), (x'_2, y'_2) \ldots, (x'_{t-1}, y'_{t-1}), x'_t = x].$$

Note that the sequences (x_1, y_1), (x_2, y_2), $\ldots$, (x_{t-1}, y_{t-1}) and (x'_1, y'_1), (x'_2, y'_2), $\ldots$, (x'_{s-1}, y'_{s-1}) might have different lengths and and $\mathcal{A}$ might error in different sets of rounds. As a special case, if an ignorant algorithm makes a mistake in round t then $p_{t+1} = p_t$.

Our main result is the following lower bound on the expected number of mistakes for ignorant algorithms.

Theorem 24 (Mistake lower bound for ignorant algorithms)**.** *Let $\gamma \in (0, 1)$ and let d be a positive integer. Suppose $\mathcal{A}$ is an ignorant algorithm for* Online Multiclass Linear Classification with Bandit Feedback. *There exists T and an adversary that sequentially chooses labeled examples $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T) \in \mathbb{R}^d \times \{1, 2\}$ such that the examples are strongly linearly separable with magin γ and $\|x_1\|, \|x_2\|, \ldots, \|x_T\| \le 1$, and the expected number of mistakes made by $\mathcal{A}$ is at least*

$$\frac{1}{10}\left(\frac{1}{160\gamma}\right)^{\frac{d-2}{4}} .$$

Before proving the theorem, we need the following lemma.

Lemma 25. *Let $\gamma \in (0, \frac{1}{160})$, let d be a positive integer and let $N = (\frac{1}{2\sqrt{40\gamma}})^{d-2}$. There exist vectors $u_1, u_2, \ldots, u_N$, $v_1, v_2, \ldots, v_N$ in $\mathbb{R}^d$ such that for all $i, j \in \{1, 2, \ldots, N\}$,*

$$\begin{aligned} \|u_i\| &\le 1 \ , \\ \|v_j\| &\le 1 \ , \\ \langle u_i, v_j \rangle &\ge \gamma, \quad \textit{if } i = j, \\ \langle u_i, v_j \rangle &\le -\gamma, \quad \textit{if } i \neq j. \end{aligned}$$

Proof. By Lemma 6 of Long (1995), there exists vectors $z_1, z_2, \ldots, z_N \in \mathbb{R}^{d-1}$ such that $\|z_1\| = \|z_2\| = \cdots = \|z_N\| = 1$ and the angle between the vectors is $\measuredangle(z_i, z_j) \ge \sqrt{40\gamma}$ for $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$. Since $\cos\theta \le 1 - \theta^2/5$ for any $\theta \in [-\pi, \pi]$, this implies that

$$\begin{aligned} \langle z_i, z_j \rangle &= 1, \quad \text{if } i = j, \\ \langle z_i, z_j \rangle &\le 1 - 8\gamma, \quad \text{if } i \neq j. \end{aligned}$$

Define $v_i = (\frac{1}{2}z_i, \frac{1}{2})$, and $u_i = (\frac{1}{2}z_i, -\frac{1}{2}(1 - 4\gamma))$ for all $i \in \{1, 2, \ldots, N\}$. It can be easily checked that for all i, $\|v_i\| \le 1$ and $\|u_i\| \le 1$. Additionally,

$$\langle u_i, v_j \rangle = \frac{1}{4}\langle z_i, z_j \rangle - \frac{1 - 4\gamma}{4} \ .$$

Thus,

$$\begin{aligned} \langle u_i, v_j \rangle &\ge \gamma, \quad \text{if } i = j, \\ \langle u_i, v_j \rangle &\le -\gamma, \quad \text{if } i \neq j. \end{aligned}$$

□

Proof of Theorem 24. We consider the strategy for the adversary described in Algorithm 5.

Let τ be the time step t in which the adversary sets PHASE $\leftarrow 2$. If the adversary never sets PHASE $\leftarrow 2$, we define $\tau = T + 1$. Then,

$$\mathbf{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] \ge \mathbf{E}\left[\sum_{t=1}^{\tau-1} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] + \mathbf{E}\left[\sum_{t=\tau}^{T} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] .$$

We upper bound each of last two terms separately.

Algorithm 5 ADVERSARY'S STRATEGY

```
Define $T = N$ and $v_1, v_2, \ldots, v_N$ as in Lemma 25.
Define $q_0 = \frac{1}{\sqrt{T}}$.
Initialize PHASE $= 1$.
for $t = 1, 2, \ldots, T$ do
    if PHASE $= 1$ then
        if $p_t(1|v_t) < 1 - q_0$ then
            $(x_t, y_t) \leftarrow (v_t, 1)$
        else
            $(x_t, y_t) \leftarrow (v_t, 2)$
            PHASE $\leftarrow 2$
    else
        $(x_t, y_t) \leftarrow (x_{t-1}, y_{t-1})$
```

In rounds $1, 2, \ldots, \tau - 1$, the algorithm predicts the incorrect class 2 with probability at least q_0. Thus,

$$\mathbf{E}\left[\sum_{t=1}^{\tau-1} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] = q_0\, \mathbf{E}[(\tau - 1)]\ . \tag{18}$$

In rounds $\tau, \tau + 1, \ldots, T$, all the examples are the same and are equal to $(v_\tau, 2)$. Let s be the first time step t such that $t \geq \tau$ and the algorithm makes a correct prediction. If the algorithm makes mistakes in all rounds $\tau, \tau + 1, \ldots, T$, we define $s = T + 1$. By definition the algorithm makes mistakes in rounds $\tau, \tau + 1, \ldots, s - 1$. Therefore,

$$\mathbf{E}\left[\sum_{t=\tau}^{T} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] \geq \mathbf{E}[s - \tau]. \tag{19}$$

Since the algorithm is ignorant, conditioned on τ and $q \triangleq p_\tau(2|v_\tau)$, $s - \tau$ follows a truncated geometric distribution with parameter q (i.e., $s - \tau$ is 0 with probability q, 1 with probability $(1-q)q$, 2 with probability $(1-q)^2 q, \ldots$). Its conditional expectation can be calculated as follows:

$$\begin{aligned}
\mathbf{E}[s - \tau \mid \tau, q] &= \sum_{i=1}^{T+1-\tau} i \times \Pr[s - \tau = i |\, \tau, q] \\
&= \sum_{j=1}^{T+1-\tau} \Pr[s - \tau \geq j |\, \tau, q] \\
&= \sum_{j=1}^{T+1-\tau} (1 - q)^j \geq \sum_{j=1}^{T+1-\tau} (1 - q_0)^j \\
&= \frac{1 - q_0}{q_0}\left(1 - (1 - q_0)^{T-\tau+1}\right).
\end{aligned}$$

Therefore, by the tower property of conditional expectation,

$$\mathbf{E}[s - \tau \mid \tau] = \mathbf{E}\left[\mathbf{E}\left[s - \tau \mid \tau, q\right] \,\middle|\, \tau\right] \geq \frac{1 - q_0}{q_0}\left(1 - (1 - q_0)^{T-\tau+1}\right).$$

Combining this fact with Equations (18) and (19), we have that

$$\begin{aligned}
\mathbf{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\widehat{y}_t \neq y_t\right]\right] &\geq q_0\, \mathbf{E}[\tau - 1] + \mathbf{E}\left[\frac{1 - q_0}{q_0}\left(1 - (1 - q_0)^{T-\tau+1}\right)\right] \\
&= \mathbf{E}\left[q_0(\tau - 1) + \frac{1 - q_0}{q_0}\left(1 - (1 - q_0)^{T-\tau+1}\right)\right]\ .
\end{aligned}$$

We lower bound the last expression by considering two cases for τ. If $\tau \geq \frac{1}{2}T + 1$, then the last expression is lower bounded by $\frac{1}{2}q_0 T = \frac{1}{2}\sqrt{T}$. If $\tau < \frac{1}{2}T + 1$, it is lower bounded by

$$\begin{aligned}
&\frac{1-q_0}{q_0}\left(1 - (1-q_0)^{\frac{1}{2}T}\right) \\
&= \frac{1-q_0}{q_0}\left(1 - (1-q_0)^{\frac{1}{2q_0^2}}\right) \\
&\geq \frac{1-\frac{1}{\sqrt{2}}}{q_0}\left(1 - \frac{1}{\sqrt{e}}\right) \\
&\geq \frac{1}{10}\sqrt{T} \; .
\end{aligned}$$

Observe that in phase 1, the labels are equal to 1 and in phase 2 the labels are equal to 2. Note that $(x_\tau, y_\tau) = (x_{\tau+1}, y_{\tau+1}) = \cdots = (x_T, y_T) = (v_\tau, 2)$. Consider the vectors $u_1, u_2, \ldots, u_N$ as defined in Lemma 25. We claim that $w_1 = -u_\tau/2$ and $w_2 = u_\tau/2$ satisfy the conditions of strong linear separability.

Clearly $\|w_1\|^2 + \|w_2\|^2 \leq (\|w_1\| + \|w_2\|)^2 \leq (\frac{1}{2} + \frac{1}{2})^2 \leq 1$. By Lemma 25, we have $\left\langle w_2/2, x_t \right\rangle = \left\langle u_\tau/2, v_\tau \right\rangle \geq \gamma/2, \forall t \geq \tau$ and $\left\langle w_2/2, x_t \right\rangle = \left\langle u_\tau/2, v_t \right\rangle \leq -\gamma/2$ for all $t < \tau$. Similarly, $\left\langle w_1/2, x_t \right\rangle \leq -\gamma/2$ for all $t \geq \tau$ and $\left\langle w_1/2, x_t \right\rangle \geq \gamma/2$ for all $t < \tau$. Thus, the examples are strongly linearly separable with margin γ. □