A. Proof of Theorem 1

We generalize the analysis of Agrawal and Goyal (2013a). Since arm 1 is optimal, the regret can be written as

$$R(n) = \sum_{i=2}^{K} \Delta_i \mathbb{E}\left[T_{i,n}\right] .$$

In the rest of the proof, we bound $\mathbb{E}[T_{i,n}]$ for each suboptimal arm i. Fix arm $i > 1$. Let $E_{i,t} = \{\hat{\mu}_{i,t} \leq \tau_i\}$ and $\bar{E}_{i,t}$ be the complement of $E_{i,t}$. Then $\mathbb{E}[T_{i,n}]$ can be decomposed as

$$\mathbb{E}\left[T_{i,n}\right] = \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\{I_t = i\}\right] = \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\{I_t = i, E_{i,t} \text{ occurs}\}\right] + \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{I_t = i, \bar{E}_{i,t} \text{ occurs}\right\}\right] . \tag{10}$$

TERM b_i IN THE UPPER BOUND

We start with the second term in (10), which corresponds to b_i in our claim. This term can be tightly bounded based on the observation that event $\bar{E}_{t,i}$ is unlikely when $T_{i,t}$ is "large". Let $\mathcal{T} = \left\{t \in [n] : Q_{i,T_{i,t-1}}(\tau_i) > 1/n\right\}$. Then

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{I_t = i, \bar{E}_{i,t} \text{ occurs}\right\}\right] &\leq \mathbb{E}\left[\sum_{t \in \mathcal{T}} \mathbb{1}\{I_t = i\}\right] + \mathbb{E}\left[\sum_{t \notin \mathcal{T}} \mathbb{1}\left\{\bar{E}_{i,t}\right\}\right] \\
&\leq \mathbb{E}\left[\sum_{s=0}^{n-1} \mathbb{1}\{Q_{i,s}(\tau_i) > 1/n\}\right] + \mathbb{E}\left[\sum_{t \notin \mathcal{T}} \frac{1}{n}\right] \\
&\leq \sum_{s=0}^{n-1} \mathbb{P}\left(Q_{i,s}(\tau_i) > 1/n\right) + 1 .
\end{aligned}$$

TERM a_i IN THE UPPER BOUND

Now we focus on the first term in (10), which corresponds to a_i in our claim. Without loss of generality, we assume that Algorithm 1 is implemented as follows. When arm 1 is pulled for the s-th time, the algorithm generates an infinite i.i.d. sequence $(\hat{\mu}_\ell^{(s)})_\ell \sim p(\mathcal{H}_{1,s})$. Then, instead of sampling $\hat{\mu}_{1,t} \sim p(\mathcal{H}_{1,s})$ in round t when $T_{1,t-1} = s$, $\hat{\mu}_{1,t}$ is substituted with $\hat{\mu}_t^{(s)}$. Let $M = \{t \in [n] : \max_{j>1} \hat{\mu}_{j,t} \leq \tau_i\}$ be round indices where the values of all suboptimal arms are at most τ_i and

$$A_s = \left\{t \in M : \hat{\mu}_t^{(s)} \leq \tau_i,\, T_{1,t-1} = s\right\}$$

be its subset where the value of arm 1 is at most τ_i and the arm was pulled s times before. Then

$$\sum_{t=1}^{n} \mathbb{1}\{I_t = i,\, E_{i,t} \text{ occurs}\} \leq \sum_{t=1}^{n} \mathbb{1}\left\{\max_j \hat{\mu}_{j,t} \leq \tau_i\right\} = \sum_{s=0}^{n-1} \underbrace{\sum_{t=1}^{n} \mathbb{1}\left\{\max_j \hat{\mu}_{j,t} \leq \tau_i,\, T_{1,t-1} = s\right\}}_{|A_s|} .$$

In the next step, we bound $|A_s|$. Let

$$\Lambda_s = \min\left\{t \in M : \hat{\mu}_t^{(s)} > \tau_i,\, T_{1,t-1} \geq s\right\}$$

be the index of the first round in M where the value of arm 1 is larger than τ_i and the arm was pulled at least s times before. If such Λ_s does not exist, we set $\Lambda_s = n$. Let

$$B_s = \left\{t \in M \cap [\Lambda_s] : \hat{\mu}_t^{(s)} \leq \tau_i,\, T_{1,t-1} \geq s\right\}$$

be a subset of $M \cap [\Lambda_s]$ where the value of arm 1 is at most τ_i and the arm was pulled at least s times before.

We claim that $A_s \subseteq B_s$. By contradiction, suppose that there exists $t \in A_s$ such that $t \notin B_s$. Then it must be true that $\Lambda_s < t$, from the definitions of A_s and B_s. From the definition of Λ_s, we know that arm 1 was pulled in round Λ_s, after it was pulled at least s times before. Therefore, it cannot be true that $T_{1,t-1} = s$, and thus $t \notin A_s$. Therefore, $A_s \subseteq B_s$ and $|A_s| \leq |B_s|$. In the next step, we bound $|B_s|$ in expectation.

Let $\mathcal{F}_t = \sigma(\mathcal{H}_{1,T_{1,t}}, \ldots, \mathcal{H}_{K,T_{K,t}}, I_1, \ldots, I_t)$ be the σ-algebra generated by arm histories and pulled arms by the end of round t, for $t \in [n] \cup \{0\}$. Let $P_s = \min \{t \in [n] : T_{1,t-1} = s\}$ be the index of the first round where arm 1 was pulled s times before. If such P_s does not exist, we set $P_s = n + 1$. Note that P_s is a stopping time with respect to filtration $(\mathcal{F}_t)_t$. Hence, $\mathcal{G}_s = \mathcal{F}_{P_s - 1}$ is well-defined and thanks to $|A_s| \leq n$, we have

$$\mathbb{E}\left[|A_s|\right] = \mathbb{E}\left[\min\left\{\mathbb{E}\left[|A_s| \,|\, \mathcal{G}_s\right], n\right\}\right] \leq \mathbb{E}\left[\min\left\{\mathbb{E}\left[|B_s| \,|\, \mathcal{G}_s\right], n\right\}\right] .$$

We claim that $\mathbb{E}\left[|B_s| \,|\, \mathcal{G}_s\right] \leq 1/Q_{1,s}(\tau_i) - 1$. First, note that $|B_s|$ can be rewritten as

$$|B_s| = \sum_{t=P_s}^{\Lambda_s} \epsilon_t \rho_t ,$$

where $\epsilon_t = \mathbb{1}\{\max_{j>1} \hat{\mu}_{j,t} \leq \tau_i\}$ control which $\rho_t = \mathbb{1}\left\{\hat{\mu}_t^{(s)} \leq \tau_i\right\}$ contribute to the sum. Now recall Theorem 5.2 from Chapter III of Doob (1953).

Theorem 3. *Let $X_1, X_2, \ldots$ and $Z_1, Z_2, \ldots$ be two sequences of random variables and $(\mathcal{F}_t)_t$ be a filtration. Let $(X_t)_t$ be i.i.d., X_t be $\mathcal{F}_t$ measurable, $Z_t \in \{0,1\}$, and Z_t be $\mathcal{F}_{t-1}$ measurable. Let $N_t = \min\{t > N_{t-1} : Z_t = 1\}$ for $t \in [m]$, $N_0 = 0$, and assume that $N_m < \infty$ almost surely. Let $X'_t = X_{N_t}$ for $t \in [m]$. Then $(X'_t)_{t=1}^m$ is i.i.d. and its elements have the same distribution as X_1.*

By the above theorem and the definition of Λ_s, $|B_s|$ has the same distribution as the number of failed independent draws from $\mathrm{Ber}(Q_{1,s}(\tau_i))$ until the first success, capped at $n - P_s$. It is well known that the expected value of this quantity, without the cap, is bounded by $1/Q_{1,s}(\tau_i) - 1$.

Finally, we chain all inequalities and get

$$\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\{I_t = i, E_{i,t} \text{ occurs}\}\right] \leq \sum_{s=0}^{n-1} \mathbb{E}\left[\min\left\{1/Q_{1,s}(\tau_i) - 1, n\right\}\right] .$$

This concludes our proof.

B. Proof of Theorem 2

This proof has two parts.

Upper Bound on b_i in Theorem 1 (Section 5.1)

Fix suboptimal arm i. To simplify notation, we abbreviate $Q_{i,s}(\tau_i)$ as $Q_{i,s}$. Our first objective is to bound

$$b_i = \sum_{s=0}^{n-1} \mathbb{P}\left(Q_{i,s} > 1/n\right) + 1 .$$

Fix the number of pulls s. When the number of pulls is "small", $s \leq \dfrac{8\alpha}{\Delta_i^2} \log n$, we bound $\mathbb{P}\left(Q_{i,s} > 1/n\right)$ trivially by 1. When the number of pulls is "large", $s > \dfrac{8\alpha}{\Delta_i^2} \log n$, we divide the proof based on the event that $V_{i,s}$ is not much larger than its expectation. Define

$$E = \left\{V_{i,s} - (\mu_i + a)s \leq \frac{\Delta_i s}{4}\right\} .$$

On event E,

$$Q_{i,s} = \mathbb{P}\left(U_{i,s} - (\mu_i + a)s \geq \frac{\Delta_i s}{2} \,\middle|\, V_{i,s}\right) \leq \mathbb{P}\left(U_{i,s} - V_{i,s} \geq \frac{\Delta_i s}{4} \,\middle|\, V_{i,s}\right) \leq \exp\left[-\frac{\Delta_i^2 s}{8\alpha}\right] \leq n^{-1},$$

where the first inequality is from the definition of event E, the second inequality is by Hoeffding's inequality, and the third inequality is by our assumption on s. On the other hand, event $\bar{E}$ is unlikely because

$$\mathbb{P}\left(\bar{E}\right) \leq \exp\left[-\frac{\Delta_i^2 s}{8\alpha}\right] \leq n^{-1},$$

where the first inequality is by Hoeffding's inequality and the last inequality is by our assumption on s. Now we apply the last two inequalities to

$$\begin{aligned}\mathbb{P}\left(Q_{i,s} > 1/n\right) &= \mathbb{E}\left[\mathbb{P}\left(Q_{i,s} > 1/n \,|\, V_{i,s}\right) \mathbb{1}\{E\}\right] + \mathbb{E}\left[\mathbb{P}\left(Q_{i,s} > 1/n \,|\, V_{i,s}\right) \mathbb{1}\left\{\bar{E}\right\}\right] \\ &\leq 0 + \mathbb{E}\left[\mathbb{1}\left\{\bar{E}\right\}\right] \leq n^{-1}.\end{aligned}$$

Finally, we chain our upper bounds for all $s \in [n]$ and get the upper bound on b_i in (9).

UPPER BOUND ON a_i IN THEOREM 1 (SECTION 5.1)

Fix suboptimal arm i. Our second objective is to bound

$$a_i = \sum_{s=0}^{n-1} \mathbb{E}\left[\min\left\{\frac{1}{Q_{1,s}(\tau_i)} - 1, n\right\}\right].$$

We redefine τ_i as $\tau_i = (\mu_1 + a)/\alpha - \Delta_i/(2\alpha)$ and abbreviate $Q_{1,s}(\tau_i)$ as $Q_{1,s}$. Since i is fixed, this slight abuse of notation should not cause any confusion. For $s > 0$, we have

$$Q_{1,s} = \mathbb{P}\left(\frac{U_{1,s}}{\alpha s} \geq \frac{\mu_1 + a}{\alpha} - \frac{\Delta_i}{2\alpha} \,\middle|\, V_{1,s}\right).$$

Let $F_s = 1/Q_{1,s} - 1$. Fix the number of pulls s. When $s = 0$, $Q_{1,s} = 1$ and $\mathbb{E}\left[\min\left\{F_s, n\right\}\right] = 0$. When the number of pulls is "small", $0 < s \leq \frac{16\alpha}{\Delta_i^2}\log n$, we apply the upper bound from Theorem 4 in Appendix C and get

$$\mathbb{E}\left[\min\left\{F_s, n\right\}\right] \leq \mathbb{E}\left[1/Q_{1,s}\right] \leq \mathbb{E}\left[1/\mathbb{P}\left(U_{1,s} \geq (\mu_1 + a)s \,|\, V_{1,s}\right)\right] \leq c,$$

where c is defined in Theorem 2. The last inequality is by Theorem 4 in Appendix C.

When the number of pulls is "large", $s > \frac{16\alpha}{\Delta_i^2}\log n$, we divide the proof based on the event that $V_{1,s}$ is not much smaller than its expectation. Define

$$E = \left\{(\mu_1 + a)s - V_{1,s} \leq \frac{\Delta_i s}{4}\right\}.$$

On event E,

$$\begin{aligned}Q_{1,s} &= \mathbb{P}\left((\mu_1 + a)s - U_{1,s} \leq \frac{\Delta_i s}{2} \,\middle|\, V_{1,s}\right) = 1 - \mathbb{P}\left((\mu_1 + a)s - U_{1,s} > \frac{\Delta_i s}{2} \,\middle|\, V_{1,s}\right) \\ &\geq 1 - \mathbb{P}\left(V_{1,s} - U_{1,s} > \frac{\Delta_i s}{4} \,\middle|\, V_{1,s}\right) \geq 1 - \exp\left[-\frac{\Delta_i^2 s}{8\alpha}\right] \geq \frac{n^2 - 1}{n^2},\end{aligned}$$

where the first inequality is from the definition of event E, the second inequality is by Hoeffding's inequality, and the third inequality is by our assumption on s. The above lower bound yields

$$F_s = \frac{1}{Q_{1,s}} - 1 \leq \frac{n^2}{n^2 - 1} - 1 = \frac{1}{n^2 - 1} \leq n^{-1}$$

for $n \geq 2$. On the other hand, event $\bar{E}$ is unlikely because

$$\mathbb{P}\left(\bar{E}\right) \leq \exp\left[-\frac{\Delta_i^2 s}{8\alpha}\right] \leq n^{-2}\,,$$

where the first inequality is by Hoeffding's inequality and the last inequality is by our assumption on s. Now we apply the last two inequalities to

$$\begin{aligned}\mathbb{E}\left[\min\left\{F_s, n\right\}\right] &= \mathbb{E}\left[\mathbb{E}\left[\min\left\{F_s, n\right\} \mid V_{1,s}\right] \mathbb{1}\{E\}\right] + \mathbb{E}\left[\mathbb{E}\left[\min\left\{F_s, n\right\} \mid V_{1,s}\right] \mathbb{1}\left\{\bar{E}\right\}\right] \\ &\leq \mathbb{E}\left[n^{-1}\mathbb{1}\{E\}\right] + \mathbb{E}\left[n\mathbb{1}\left\{\bar{E}\right\}\right] \leq 2n^{-1}\,.\end{aligned}$$

Finally, we chain our upper bounds for all $s \in [n]$ and get the upper bound on a_i in (9). This concludes our proof.

C. Upper Bound on the Expected Inverse Probability of Being Optimistic

Theorem 4 provides an upper bound on the expected inverse probability of being optimistic,

$$\mathbb{E}\left[1/\mathbb{P}\left(U_{1,s} \geq (\mu_1 + a)s \mid V_{1,s}\right)\right]\,,$$

which is used in Section 5.2 and Appendix B. In the bound and its analysis, n is s, p is μ_1, x is $V_{1,s} - as$, and y is $U_{1,s}$.

Theorem 4. *Let* $m = (2a+1)n$ *and* $b = \dfrac{2a+1}{a(a+1)} < 2$. *Then*

$$W = \sum_{x=0}^{n} B(x; n, p) \left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{an + x}{m}\right)\right]^{-1} \leq \frac{2e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp\left[\frac{8b}{2-b}\right] \left(1 + \sqrt{\frac{2\pi}{4-2b}}\right)\,.$$

Proof. First, we apply the upper bound from Lemma 2 for

$$f(x) = \left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{an + x}{m}\right)\right]^{-1}\,.$$

Note that this function decreases in x, as required by Lemma 2, because the probability of observing at least $\lceil (a+p)n \rceil$ ones increases with x, for any fixed $\lceil (a+p)n \rceil$. The resulting upper bound is

$$W \leq \sum_{i=0}^{i_0-1} \exp[-2i^2] \left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{(a+p)n - (i+1)\sqrt{n}}{m}\right)\right]^{-1} + \exp[-2i_0^2] \left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{an}{m}\right)\right]^{-1}\,,$$

where i_0 is the smallest integer such that $(i_0+1)\sqrt{n} \geq pn$, as defined in Lemma 2.

Second, we bound both above reciprocals using Lemma 3. The first term is bounded for $x = pn - (i+1)\sqrt{n}$ as

$$\left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{(a+p)n - (i+1)\sqrt{n}}{m}\right)\right]^{-1} \leq \frac{e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp[b(i+2)^2]\,.$$

The second term is bounded for $x = 0$ as

$$\left[\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{an}{m}\right)\right]^{-1} \leq \frac{e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp\left[b\frac{(pn + \sqrt{n})^2}{n}\right] \leq \frac{e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp[b(i_0+2)^2]\,,$$

where the last inequality is from the definition of i_0. Then we chain the above three inequalities and get

$$W \leq \frac{e^2\sqrt{2a+1}}{\sqrt{2\pi}} \sum_{i=0}^{i_0} \exp[-2i^2 + b(i+2)^2]\,.$$

Now note that

$$2i^2 - b(i+2)^2 = (2-b)\left(i^2 - \frac{4bi}{2-b} + \frac{4b^2}{(2-b)^2}\right) - \frac{4b^2}{2-b} - 4b = (2-b)\left(i - \frac{2b}{2-b}\right)^2 - \frac{8b}{2-b}\,.$$

It follows that

$$\begin{aligned}
W &\leq \frac{e^2\sqrt{2a+1}}{\sqrt{2\pi}} \sum_{i=0}^{i_0} \exp\left[-(2-b)\left(i - \frac{2b}{2-b}\right)^2 + \frac{8b}{2-b}\right] \\
&\leq \frac{2e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp\left[\frac{8b}{2-b}\right] \sum_{i=0}^{\infty} \exp\left[-(2-b)i^2\right] \\
&\leq \frac{2e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp\left[\frac{8b}{2-b}\right] \left(1 + \int_{u=0}^{\infty} \exp\left[-\frac{u^2}{\frac{2}{4-2b}}\right] \mathrm{d}u\right) \\
&\leq \frac{2e^2\sqrt{2a+1}}{\sqrt{2\pi}} \exp\left[\frac{8b}{2-b}\right] \left(1 + \sqrt{\frac{2\pi}{4-2b}}\right).
\end{aligned}$$

This concludes our proof. ■

Lemma 2. *Let $f(x) \geq 0$ be a decreasing function of x and i_0 be the smallest integer such that $(i_0+1)\sqrt{n} \geq pn$. Then*

$$\sum_{x=0}^{n} B(x;n,p) f(x) \leq \sum_{i=0}^{i_0-1} \exp[-2i^2] f(pn - (i+1)\sqrt{n}) + \exp[-2i_0^2] f(0)\,.$$

Proof. Let

$$\mathcal{X}_i = \begin{cases} (\max\{pn - \sqrt{n},\, 0\},\, n]\,, & i = 0\,; \\ (\max\{pn - (i+1)\sqrt{n},\, 0\},\, pn - i\sqrt{n}]\,, & i > 0\,; \end{cases}$$

for $i \in [i_0] \cup \{0\}$. Then $\{\mathcal{X}_i\}_{i=0}^{i_0}$ is a partition of $[0,n]$. Based on this observation,

$$\begin{aligned}
\sum_{x=0}^{n} B(x;n,p) f(x) &= \sum_{i=0}^{i_0} \sum_{x=0}^{n} \mathbb{1}\{x \in \mathcal{X}_i\}\, B(x;n,p) f(x) \\
&\leq \sum_{i=0}^{i_0-1} f(pn - (i+1)\sqrt{n}) \sum_{x=0}^{n} \mathbb{1}\{x \in \mathcal{X}_i\}\, B(x;n,p) + f(0) \sum_{x=0}^{n} \mathbb{1}\{x \in \mathcal{X}_{i_0}\}\, B(x;n,p)\,,
\end{aligned}$$

where the inequality holds because $f(x)$ is a decreasing function of x. Now fix $i > 0$. Then from the definition of $\mathcal{X}_i$ and Hoeffding's inequality,

$$\sum_{x=0}^{n} \mathbb{1}\{x \in \mathcal{X}_i\}\, B(x;n,p) \leq \mathbb{P}\left(X \leq pn - i\sqrt{n} \,\middle|\, X \sim B(n,p)\right) \leq \exp[-2i^2]\,.$$

Trivially, $\sum_{x=0}^{n} \mathbb{1}\{x \in \mathcal{X}_0\}\, B(x;n,p) \leq 1 = \exp[-2 \cdot 0^2]$. Finally, we chain all inequalities and get our claim. ■

Lemma 3. *Let $x \in [0, pn]$, $m = (2a+1)n$, and $b = \dfrac{2a+1}{a(a+1)}$. Then for any integer $n > 0$,*

$$\sum_{y=\lceil (a+p)n \rceil}^{m} B\left(y; m, \frac{an+x}{m}\right) \geq \frac{\sqrt{2\pi}}{e^2\sqrt{2a+1}} \exp\left[-b\frac{(pn+\sqrt{n}-x)^2}{n}\right].$$

Proof. By Lemma 4,

$$B\left(y; m, \frac{an + x}{m}\right) \geq \frac{\sqrt{2\pi}}{e^2} \sqrt{\frac{m}{y(m - y)}} \exp\left[-\frac{(y - an - x)^2}{m \frac{an+x}{m} \frac{(a+1)n-x}{m}}\right].$$

Now note that

$$\frac{y(m - y)}{m} \leq \frac{1}{m} \frac{m^2}{4} = \frac{(2a + 1)n}{4}.$$

Moreover, since $x \in [0, pn]$,

$$m \frac{an + x}{m} \frac{(a + 1)n - x}{m} \geq m \frac{an}{m} \frac{(a + 1)n}{m} = \frac{a(a + 1)n}{2a + 1} = \frac{n}{b},$$

where b is defined in the claim of this lemma. Now we combine the above three inequalities and have

$$B\left(y; m, \frac{an + x}{m}\right) \geq \frac{2\sqrt{2\pi}}{e^2\sqrt{2a + 1}} \frac{1}{\sqrt{n}} \exp\left[-b\frac{(y - an - x)^2}{n}\right],$$

Finally, note the following two facts. First, the above lower bound decreases in y when $y \geq (a + p)n$ and $x \leq pn$. Second, by the pigeonhole principle, there exist at least $\lfloor\sqrt{n}\rfloor$ integers between $(a + p)n$ and $(a + p)n + \sqrt{n}$, starting with $\lceil(a + p)n\rceil$. These observations lead to a trivial lower bound

$$\begin{aligned}\sum_{y=\lceil(a+p)n\rceil}^{m} B\left(y; m, \frac{an + x}{m}\right) &\geq \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}} \frac{2\sqrt{2\pi}}{e^2\sqrt{2a + 1}} \exp\left[-b\frac{(pn + \sqrt{n} - x)^2}{n}\right] \\ &\geq \frac{\sqrt{2\pi}}{e^2\sqrt{2a + 1}} \exp\left[-b\frac{(pn + \sqrt{n} - x)^2}{n}\right].\end{aligned}$$

The last inequality is from $\lfloor\sqrt{n}\rfloor / \sqrt{n} \geq 1/2$, which holds for $n \geq 1$. This concludes our proof. ■

Lemma 4. *For any binomial probability,*

$$B(x; n, p) \geq \frac{\sqrt{2\pi}}{e^2} \sqrt{\frac{n}{x(n - x)}} \exp\left[-\frac{(x - pn)^2}{p(1 - p)n}\right].$$

Proof. By Stirling's approximation, for any integer $k \geq 0$,

$$\sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \leq k! \leq e k^{k+\frac{1}{2}} e^{-k}.$$

Therefore, any binomial probability can be bounded from below as

$$B(x; n, p) = \frac{n!}{x!(n - x)!} p^x q^{n-x} \geq \frac{\sqrt{2\pi}}{e^2} \sqrt{\frac{n}{x(n - x)}} \left(\frac{pn}{x}\right)^x \left(\frac{qn}{n - x}\right)^{n-x},$$

where $q = 1 - p$. Let

$$d(p_1, p_2) = p_1 \log \frac{p_1}{p_2} + (1 - p_1) \log \frac{1 - p_1}{1 - p_2}$$

be the KL divergence between Bernoulli random variables with means p_1 and p_2. Then

$$\begin{aligned}\left(\frac{pn}{x}\right)^x \left(\frac{qn}{n - x}\right)^{n-x} &= \exp\left[x \log\left(\frac{pn}{x}\right) + (n - x) \log\left(\frac{qn}{n - x}\right)\right] \\ &= \exp\left[-n\left(\frac{x}{n} \log\left(\frac{x}{pn}\right) + \frac{n - x}{n} \log\left(\frac{n - x}{qn}\right)\right)\right] \\ &= \exp\left[-nd\left(\frac{x}{n}, p\right)\right] \\ &\geq \exp\left[-\frac{(x - pn)^2}{p(1 - p)n}\right],\end{aligned}$$

where the inequality is from $d(p_1, p_2) \leq \frac{(p_1 - p_2)^2}{p_2(1 - p_2)}$. Finally, we chain all inequalities and get our claim. ■