Spectral Clustering of Signed Graphs via Matrix Power Means

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Abstract

Signed graphs encode positive (attractive) and negative (repulsive) relations between nodes. We extend spectral clustering to signed graphs via the one-parameter family of *Signed Power Mean Laplacians*, defined as the matrix power mean of normalized standard and signless Laplacians of positive and negative edges. We provide a thorough analysis of the proposed approach in the setting of a general Stochastic Block Model that includes models such as the Labeled Stochastic Block Model and the Censored Block Model. We show that in expectation the signed power mean Laplacian captures the ground truth clusters under reasonable settings where state-of-the-art approaches fail. Moreover, we prove that the eigenvalues and eigenvector of the signed power mean Laplacian concentrate around their expectation under reasonable conditions in the general Stochastic Block Model. Extensive experiments on random graphs and real world datasets confirm the theoretically predicted behaviour of the signed power mean Laplacian and show that it compares favourably with state-of-the-art methods.

1. Introduction

The analysis of graphs has received a significant amount of attention due to their capability to encode interactions that naturally arise in social networks. Yet, the vast majority of graph methods has been focused on the case where interactions are of the same type, leaving aside the case where different kinds of interactions are available (Leskovec et al., 2010b). Graphs and networks with both positive and negative edge weights arise naturally in a number of social, biological and economic contexts. Social dynamics and relationships are intrinsically positive and negative: users of online social networks such as Slashdot and Epinions, for example, can express positive interactions, like friendship and trust, and negative ones, like enmity and distrust. Other important application settings are the analysis of gene expressions in biology (Fujita et al., 2012) or the analysis of financial and economic time sequences (Ziegler et al., 2010; Pavlidis et al., 2006), where similarity and variable dependence measures commonly used may attain both positive and negative values (e.g. the Pearson correlation coefficient).

Although the majority of the literature has focused on graphs that encode only positive interactions, the analysis of signed graphs can be traced back to social balance theory (Cartwright & Harary, 1956; Harary, 1953; Davis, 1967), where the concept of a k-balance signed graph is introduced. The analysis of signed networks has been then pushed forward through the study of a variety of tasks in signed graphs, as for example edge prediction (Kumar et al., 2016; Leskovec et al., 2010a; Falher et al., 2017), node classification (Bosch et al., 2018; Tang et al., 2016a), node embeddings (Chiang et al., 2011; Derr et al., 2018; Kim et al., 2018; Wang et al., 2017; Yuan et al., 2017), node ranking (Chung et al., 2013; Shahriari & Jalili, 2014), and clustering (Chiang et al., 2012; Kunegis et al., 2010; Mercado et al., 2016; Sedoc et al., 2017; Doreian & Mrvar, 2009; Knyazev, 2018; Kirkley et al., 2018; Cucuringu et al., 2019; Cucuringu et al., 2018). See (Tang et al., 2016b; Gallier, 2016) for recent surveys on the topic.

In this paper we present a novel extension of spectral clustering for signed graphs. Spectral clustering (Luxburg, 2007) is a well established technique for non-signed graphs, which partitions the set of the nodes based on a k-dimensional node embedding obtained using the first eigenvectors of the graph Laplacian. **Our contributions are as follows:** We introduce the family of *Signed Power Mean (SPM) Laplacians*: a one-parameter family of graph matrices for signed graphs that blends the information from positive and negative interactions through the matrix power mean, a general class of matrix means that contains the arithmetic, geometric, and harmonic mean as special cases. This is inspired by recent extensions of spectral clustering which merge the information encoded by positive and negative interactions through different types of arithmetic (Chiang et al., 2012; Kunegis et al., 2010) and geometric (Mercado et al., 2016) means of the standard and signless graph Laplacians. We analyze the performance of the signed power mean Laplacian in a

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general Signed Stochastic Block Model. We first provide an anlysis in expectation showing that the smaller is the parameter of the signed power mean Laplacian, the less restrictive are the conditions that ensure to recover the ground truth clusters. In particular, we show that the limit cases $+\infty$ and $-\infty$ are related to the boolean operators AND and OR, respectively, in the sense that for the limit case $+\infty$ clusters are recovered only if both positive *and* negative interactions are informative, whereas for $-\infty$ clusters are recovered if positive *or* negative interactions are informative. This is consistent with related work in the context of unsigned multilayer graphs (Mercado et al., 2018). Second, we show that the eigenvalues and eigenvectors of the signed power mean Laplacian concentrate around their mean, so that our results hold also for the case where one samples from the stochastic block model. Our result extends with minor changes to the unsigned multilayer graph setting considered in (Mercado et al., 2018), where just the expected case has been studied. To our knowledge these are the first concentration results for matrix power means under any stochastic block model for signed graphs. Finally, we show that the signed power mean Laplacian compares favorably with state-of-the-art approaches through extensive numerical experiments on diverse real world datasets. All the proofs have been moved to the supplementary material.

Notation. A signed graph is a pair $G^{\pm} = (G^+, G^-)$, where $G^+ = (V, W^+)$ and $G^- = (V, W^-)$ encode positive and negative edges, respectively, with positive symmetric adjacency matrices W^+ and W^-, and a common vertex set $V = \{v_1, \dots, v_n\}$. Note that this definition allows the simultaneous presence of both positive and negative interactions between the same two nodes. This is a major difference with respect to the alternative point of view where $G^{\pm}$ is associated to a single symmetric matrix W with positive and negative entries. In this case $W = W^+ - W^-$, with $W^+_{ij} = \max\{0, W_{ij}\}$ and $W^-_{ij} = -\min\{0, W_{ij}\}$, implying that every interaction is either positive or negative, but not both at the same time. We denote by $D^+_{ii} = \sum_{j=1}^n w^+_{ij}$ and $D^-_{ii} = \sum_{j=1}^n w^-_{ij}$ the diagonal matrix of the degrees of G^+ and G^-, respectively, and $\overline{D} = D^+ + D^-$.

2. Related work

The study of clustering of signed graphs can be traced back to the theory of social balance (Cartwright & Harary, 1956; Harary, 1953; Davis, 1967), where a signed graph is called k-balanced if the set of vertices can be partitioned into k sets such that within the subsets there are only positive edges, and between them only negative.

Inspired by the notion of k-balance, different approaches for signed graph clustering have been introduced. In particular, many of them aim to extend spectral clustering to signed graphs by proposing novel signed graph Laplacians. A related approach is correlation clustering (Bansal et al., 2004). Unlike spectral clustering, where the number of clusters is fixed a-priori, correlation clustering approximates the optimal number of clusters by identifying a partition that is as close as possible to be k-balanced. In this setting, the case where the number of clusters is constrained has been considered in (Giotis & Guruswami, 2006).

We briefly introduce the standard and signless Laplacian and review different definitions of Laplacians on signed graphs. The final clustering algorithm to find k clusters is the same for all of them: compute the smallest k eigenvectors of the corresponding Laplacian, use the eigenvectors to embed the nodes into $\mathbb{R}^k$, obtain the final clustering by doing k-means in the embedding space. However, we will see below that in some cases we have to slightly deviate from this generic principle by using the $k-1$ smallest eigenvectors instead.

Laplacians of Unsigned Graphs: In the following all weight matrices are non-negative and symmetric. Given an assortative graph $G = (V, W)$, standard spectral clustering is based on the Laplacian and its normalized version defined as:

$$L = D - W \qquad L_{\text{sym}} = D^{-1/2} L D^{-1/2}$$

where $D_{ii} = \sum_{j=1}^n w_{ij}$ is the diagonal matrix of the degrees of G. Both Laplacians are symmetric positive semidefinite and the multiplicity of the eigenvalue 0 is equal to the number of connected components in G.

For disassortative graphs, i.e. when edges carry only dissimilarity information, the goal is to identify clusters such that the amount of edges between clusters is larger than the one inside clusters. Spectral clustering is extended to this setting by considering the signless Laplacian matrix and its normalized version (see e.g. (Liu, 2015; Mercado et al., 2016)), defined as:

$$Q = D + W \qquad Q_{\text{sym}} = D^{-1/2} Q D^{-1/2}$$

Both Laplacians are positive semi-definite, and the smallest eigenvalue is zero if and only if the graph has a bipartite component (Desai & Rao, 1994).

Laplacians of Signed Graphs: Signed graphs encode both positive and negative interactions. In the ideal k-balanced case positive interactions present an assortative behaviour, whereas negative interactions present a disassortative behaviour. With this in mind, several novel definitions of *signed Laplacians* have been proposed. We briefly review them for later reference.

In (Chiang et al., 2012) the balance ratio Laplacian and its normalized version are defined as:

$$L_{BR} = D^+ - W^+ + W^-, \qquad L_{BN} = \overline{D}^{-1/2} L_{BR} \overline{D}^{-1/2}$$

whereas in (Kunegis et al., 2010) the signed ratio Laplacian and its normalized version have been defined as:

$$L_{SR} = \overline{D} - W^+ + W^-, \qquad L_{SN} = \overline{D}^{-1/2} L_{SR} \overline{D}^{-1/2}$$

The signed Laplacians L_{BR} and L_{BN} need not be positive semidefinite, while the signed Laplacians L_{SR} and L_{SN} are positive semidefinite with eigenvalue zero if and only if the graph is 2-balanced.

In the context of correlation clustering, in (Saade et al., 2015) the Bethe Hessian matrix is defined as:

$$H = (\alpha - 1)I - \sqrt{\alpha}(W^+ - W^-) + \overline{D}$$

where α is the average node degree $\alpha = \frac{1}{n}\sum_{i=1}^n \overline{D}_{ii}$. The Bethe Hessian H need not be positive definite. In fact, eigenvectors with negative eigenvalues bring information of clustering structure (Saade et al., 2014).

Let $L^+ = D^+ - W^+$ and $Q^- = D^- + W^-$ be the Laplacian and signless Laplacian of G^+ and G^-, respectively. As noted in (Mercado et al., 2016), $L_{SR} = L^+ + Q^-$ i.e. it coincides with twice the arithmetic mean of L^+ and Q^-. Note that the same holds for H when the average degree α is equal to one, i.e. $H = L_{SR}$ when $\alpha = 1$. In (Mercado et al., 2016), the arithmetic mean and geometric mean of the normalized Laplacian and its signless version are used to define new Laplacians for signed graphs:

$$L_{AM} = L^+_{\text{sym}} + Q^-_{\text{sym}}, \qquad L_{GM} = L^+_{\text{sym}} \# Q^-_{\text{sym}}$$

where $A\#B = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$ is the geometric mean of A and B, $L^+_{\text{sym}} = (D^+)^{-1/2}L^+(D^+)^{-1/2}$ and $Q^-_{\text{sym}} = (D^-)^{-1/2}Q^-(D^-)^{-1/2}$. While the computation of L_{GM} is more challenging, in (Mercado et al., 2016) it is shown that the clustering assignment obtained with the geometric mean Laplacian L_{GM} outperforms all other signed Laplacians.

Both the arithmetic and the geometric means are special cases of a much richer one-parameter family of means known as power means. Based on this observation, we introduce the *Signed Power Mean Laplacian* in Section 2.2, defined via a matrix version of the family of power means which we briefly review below.

2.1. Matrix Power Means

The scalar power mean of two non-negative scalars a, b is a one-parameter family of means defined for $p \in \mathbb{R}$ as $m_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$. Particular cases are the arithmetic, geometric and harmonic means, as shown in Table 1. Moreover, the scalar power mean is monotone in the parameter p, i.e. $m_p(a,b) \leq m_q(a,b)$ when $p \leq q$ (see (Bullen, 2013) , Ch. 3, Thm. 1), which yields the well known arithmetic-geometric-harmonic mean inequality $m_{-1}(a,b) \leq m_0(a,b) \leq m_1(a,b)$. As matrices do not commute, several matrix extensions of the scalar power mean have been introduced, which typically agree if the matrices commute, see e.g. Chapter 4 in (Bhatia, 2009). We consider the following matrix extension of the scalar power mean:

Table 1 Particular cases of scalar power means

p	$m_p(a,b)$	*name*
$p \to \infty$	$\max\{a,b\}$	maximum
$p = 1$	$(a+b)/2$	arithmetic mean
$p \to 0$	$\sqrt{ab}$	geometric mean
$p = -1$	$2\left(\frac{1}{a}+\frac{1}{b}\right)^{-1}$	harmonic mean
$p \to -\infty$	$\min\{a,b\}$	minimum

Definition 1 ((Bhagwat & Subramanian, 1978)). *Let A, B be symmetric positive definite matrices, and $p \in \mathbb{R}$. The matrix power mean of A, B with exponent p is*

$$M_p(A,B) = \left(\frac{A^p + B^p}{2}\right)^{1/p}$$

where $Y^{1/p}$ is the unique positive definite solution of the matrix equation $X^p = Y$.

Please note that this definition can be extended to positive semidefinite matrices (Bhagwat & Subramanian, 1978) for $p > 0$, as $M_p(A,B)$ exists, whereas for $p < 0$ a diagonal shift is necessary to ensure that the matrices A, B are positive definite.

2.2. The Signed Power Mean Laplacian

Given a signed graph $G^\pm = (G^+, G^-)$ we define the Signed Power Mean (SPM) Laplacian L_p of $G^\pm$ as

$$L_p = M_p(L^+_{\text{sym}}, Q^-_{\text{sym}}). \qquad (1)$$

For the case $p < 0$ the matrix power mean requires positive definite matrices, hence we use in this case the matrix power mean of diagonally shifted Laplacians, i.e. $L^+_{\text{sym}} + \varepsilon I$ and $Q^-_{\text{sym}} + \varepsilon I$. Our following theoretical analysis holds for all possible shifts $\varepsilon > 0$, whereas we discuss in the supplementary material the numerical robustness with respect to ε. The clustering algorithm for identifying k clusters in signed graphs is given in Algorithm 1. Please note that for $p \geq 1$ we deviate from the usual scheme and use the first $k-1$ eigenvectors rathen than the first k. The reason is a result of the analysis in the stochastic block model in Section 3. In general, the main influence of the parameter p of the power mean is on the ordering of the eigenvalues. In Section 3 we will see that this significantly influences the performance of different instances of SPM Laplacians, in particular, the arithmetic and geometric mean discussed in (Mercado et al., 2016) are suboptimal for the recovery of the ground truth clusters. For the computation of the matrix power mean we adapt the scalable Krylov subspace-based algorithm proposed in (Mercado et al., 2018).

Algorithm 1: Spectral clustering of signed graphs with L_p

Input: Symmetric matrices W^+, W^-, number k of clusters to construct.
Output: Clusters $C_1, \ldots, C_k$.
Let $k' = k-1$ if $p \geq 1$ and $k' = k$ if $p < 1$.
Compute eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_{k'}$ corresponding to the k' smallest eigenvalues of L_p.
Set $U = (\mathbf{u}_1, \ldots, \mathbf{u}_{k'})$ and cluster the rows of U with k-means into clusters $C_1, \ldots, C_k$.

3. Stochastic Block Model Analysis of the Signed Power Mean Laplacian

In this section we analyze the signed power mean Laplacian L_p under a general Signed Stochastic Block Model. Our results here are twofold. First, we derive new conditions in expectation that guarantee that the eigenvectors corresponding to the smallest eigenvalues of L_p recover the ground truth clusters. These conditions reveal that, in this setting, the state-of-the-art signed graph matrices are suboptimal as compared to L_p for negative values of p. Second, we show that our result in expectation transfer to sampled graphs as we prove conditions that ensure that both eigenvalues and eigenvectors of L_p concentrate around their expected value with high probability. We verify our results by several experiments where the clustering performance of state-of-the-art matrices and L_p are compared on random graphs following the Signed Stochastic Block Model.

All proofs hold for an arbitrary diagonal shift $\varepsilon > 0$, whereas the shift is set to $\varepsilon = \log_{10}(1 + |p|) + 10^{-6}$ in the numerical experiments. Numerical robustness with respect to ε is discussed in the supplementary material.

The Stochastic Block Model (**SBM**) is a well-established generative model for graphs and a canonical tool for studying clustering methods (Holland et al., 1983; Rohe et al., 2011; Abbe, 2018). Graphs drawn from the SBM show a prescribed clustering structure, as the probability of an edge between two nodes depends only on the clustering membership of each node. We introduce our SBM for signed Graphs (**SSBM**): we consider k ground truth clusters $\mathcal{C}_1, \ldots, \mathcal{C}_k$, all of them of size $|\mathcal{C}| = \frac{n}{k}$, and parameters $p^+_{\mathrm{in}}, p^+_{\mathrm{out}}, p^-_{\mathrm{in}}, p^-_{\mathrm{out}} \in [0,1]$ where p^+_{in} (resp. p^-_{in}) is the probability of observing an edge inside clusters in G^+ (resp. G^-) and p^+_{out} (resp. p^-_{out}) is the probability of observing an edge between clusters in G^+ (resp. G^-). Calligraphic letters are used for the expected adjacency matrices: $\mathcal{W}^+$ and $\mathcal{W}^-$ are the expected adjacency matrix of G^+ and G^-, respectively, where $\mathcal{W}^+_{i,j} = p^+_{\mathrm{in}}$ and $\mathcal{W}^-_{i,j} = p^-_{\mathrm{in}}$ if v_i, v_j belong to the same cluster, whereas $\mathcal{W}^+_{i,j} = p^+_{\mathrm{out}}$ and $\mathcal{W}^-_{i,j} = p^-_{\mathrm{out}}$ if v_i, v_j belong to different clusters.

Other extensions of the SBM to the signed setting have been considered. Particularly relevant examples are the Labelled Stochastic Block Model (**LSBM**) (Heimlicher et al., 2012) and the Censored Block Model (**CBM**) (Abbe et al., 2014). In the context of signed graphs, both LSBM and CBM assume that an observed edge can be either positive or negative, but not both. Our SSBM, instead, allows the simultaneous presence of both positive and negative edges between the same pair of nodes, as the parameters $p^+_{\mathrm{in}}, p^+_{\mathrm{out}}, p^-_{\mathrm{in}}, p^-_{\mathrm{out}}$ in SSBM are independent. Moreover, the edge probabilities defining both the LSBM and the CBM can be recovered as special cases of the SSBM. In particular, the LSBM corresponds to the SSBM for the choices

$$p^+_{\mathrm{in}} = \overline{p}_{\mathrm{in}}\mu^+, \quad p^-_{\mathrm{in}} = \overline{p}_{\mathrm{in}}\mu^- \qquad \text{(within clusters)}$$
$$p^+_{\mathrm{out}} = \overline{p}_{\mathrm{out}}\nu^+, \ p^-_{\mathrm{out}} = \overline{p}_{\mathrm{out}}\nu^- \qquad \text{(between clusters)}$$

where $\overline{p}_{\mathrm{in}}$ and $\overline{p}_{\mathrm{out}}$ are edge probabilities within and between clusters, respectively, whereas μ^+ and $\mu^- = 1 - \mu^+$ (resp. ν^+ and $\nu^- = 1 - \nu^+$) are the probabilities of assigning a positive and negative label to an edge within (resp. between) clusters. Similarly, the CBM corresponds to the SSBM for the particular choices $\overline{p}_{\mathrm{in}} = \overline{p}_{\mathrm{out}}$, $\mu^+ = \nu^- = (1-\eta)$ and $\mu^- = \nu^+ = \eta$ where η is a noise parameter.

Our goal is to identify conditions in terms of $k, p^+_{\mathrm{in}}, p^+_{\mathrm{out}}, p^-_{\mathrm{in}}$, and p^-_{out}, such that $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are recovered by the smallest eigenvectors of the signed power mean Laplacian. Consider the following k vectors:

$$\boldsymbol{\chi}_1 = \mathbf{1}, \qquad \boldsymbol{\chi}_i = (k-1)\mathbf{1}_{\mathcal{C}_i} - \mathbf{1}_{\overline{\mathcal{C}_i}}\ .$$

$i = 2, \ldots, k$. The node embedding given by $\{\boldsymbol{\chi}_i\}_{i=1}^k$ is informative in the sense that applying k-means on $\{\boldsymbol{\chi}_i\}_{i=1}^k$ trivially recovers the ground truth clusters $\mathcal{C}_1, \ldots, \mathcal{C}_k$ as all nodes of a cluster are mapped to the same point. Note that the constant vector $\boldsymbol{\chi}_1$ could be omitted as it does not add clustering information. We derive conditions for the SSBM such that $\{\boldsymbol{\chi}_i\}_{i=1}^k$ are the smallest eigenvectors of the signed power mean Laplacian in expectation.

Theorem 1. *Let $\mathcal{L}_p = M_p(\mathcal{L}^+_{\mathrm{sym}}, \mathcal{Q}^-_{\mathrm{sym}})$ and let $\varepsilon > 0$ be the diagonal shift.*

- *If $p \geq 1$, then $\{\boldsymbol{\chi}_i\}_{i=2}^k$ correspond to the $(k$-$1)$-smallest eigenvalues of $\mathcal{L}_p$ if and only if $m_p(\rho^+_\varepsilon, \rho^-_\varepsilon) < 1 + \varepsilon$;*
- *If $p < 1$, then $\{\boldsymbol{\chi}_i\}_{i=1}^k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_p$ if and only if $m_p(\rho^+_\varepsilon, \rho^-_\varepsilon) < 1 + \varepsilon$;*

with $\rho^+_\varepsilon = 1 - (p^+_{\mathrm{in}} - p^+_{\mathrm{out}})/(p^+_{\mathrm{in}} + (k-1)p^+_{\mathrm{out}}) + \varepsilon$ and $\rho^-_\varepsilon = 1 + (p^-_{\mathrm{in}} - p^-_{\mathrm{out}})/(p^-_{\mathrm{in}} + (k-1)p^-_{\mathrm{out}}) + \varepsilon$.

Note that Theorem 1 is the reason why Alg. 1 uses only the first $k-1$ eigenvectors for $p \geq 1$. The problem is that the constant eigenvector need not be among the first k eigenvectors in the SSBM for $p \geq 1$. However, as it is constant and thus uninformative in the embedding, this does not lead to any loss of information. The following Corollary shows that the limit cases of L_p are related to the boolean operators `AND` and `OR`.

Corollary 1. *Let $\mathcal{L}_p = M_p(\mathcal{L}^+_{\mathrm{sym}}, \mathcal{Q}^-_{\mathrm{sym}})$.*

- *$\{\boldsymbol{\chi}_i\}_{i=2}^k$ correspond to the $(k$-$1)$-smallest eigenvalues of*

$\mathcal{L}_\infty$ *if and only if* $p^+_{\text{in}} > p^+_{\text{out}}$ ***and*** $p^-_{\text{in}} < p^-_{\text{out}}$,

- $\{\chi_i\}_{i=1}^k$ *correspond to the* k*-smallest eigenvalues of* $\mathcal{L}_{-\infty}$ *if and only if* $p^+_{\text{in}} > p^+_{\text{out}}$ ***or*** $p^-_{\text{in}} < p^-_{\text{out}}$.

The conditions for $\mathcal{L}_\infty$ are the most conservative ones, as they require that G^+ and G^- are informative, i.e. G^+ has to be assortative and G^- disassortative. Under these conditions every clustering method for signed graphs should be able to identify the ground truth clusters in expectation. On the other hand, the less restrictive conditions for the recovery of the ground truth clusters correspond to the limit case $\mathcal{L}_{-\infty}$. If G^+ or G^- are informative, then the ground truth clusters are recovered, that is, $\mathcal{L}_{-\infty}$ only requires that G^+ is assortative or G^- is disassortative. In particular, the following corollary shows that smaller values of p require less restrictive conditions to ensure the identification of the informative eigenvectors.

Corollary 2. *Let* $q \leq p$*. If* $\{\chi_i\}_{i=\theta(p)}^k$ *correspond to the* k*-smallest eigenvalues of* $\mathcal{L}_p$*, then* $\{\chi_i\}_{i=\theta(q)}^k$ *correspond to the* k*-smallest eigenvalues of* $\mathcal{L}_q$*, where* $\theta(x) = 1$ *if* $x \leq 0$ *and* $\theta(x) = 2$ *if* $x > 0$*.*

To better understand the different conditions we have derived, we visualize them in Fig. 1, where the x-axis corresponds to how assortative G^+ is, while the y-axis corresponds to how disassortative G^- is. The conditions of the limit case $\mathcal{L}_\infty$, i.e. the case where G^+ and G^- have to be informative, correspond to the upper-right region, dark blue region in Fig. 1c, and correspond to the 25% of all possible configurations of the SBM. The conditions for the limit case $\mathcal{L}_{-\infty}$, i.e. the case where G^+ or G^- has to be informative, instead correspond to all possible configurations of the SBM except for the bottom-left region. This is depicted in Fig. 1b and corresponds to the 75% of all possible configurations under the SBM.

In Fig. 2 we present the corresponding conditions for recovery in expectation for the cases $p \in \{-10, -1, 0, 1, 10\}$. We can visually verify that the larger the value of p the smaller is the region where the conditions of Theorem 1 hold. In particular, one can compare the change of conditions as one moves from the signed harmonic ($\mathcal{L}_{-1}$), geometric ($\mathcal{L}_0$), to the arithmetic ($\mathcal{L}_1$) mean Laplacians verifying the ordering described in Corollary 2. Moreover, we clearly observe that $\mathcal{L}_{-10}$ and $\mathcal{L}_{10}$ are already quite close to the conditions necessary for the limit cases $\mathcal{L}_{-\infty}$ and $\mathcal{L}_\infty$, respectively.

In the middle row of Fig. 2 we show the average clustering error for each power mean Laplacian when sampling 50 times from the SSBM following the diagram presented in Fig. 1a and fixing the sparsity of G^+ and G^- by setting $p^+_{\text{in}} + p^+_{\text{out}} = 0.1$ and $p^-_{\text{in}} + p^-_{\text{out}} = 0.1$ with two clusters each of size 100. We observe that the areas with low clustering error qualitatively match the regions where in expectation we have recovery of the clusters. However, due to the sampling which can make one of the graphs G^+ and G^- quite sparse and as we just consider graphs with 200 nodes, the area of low clustering error is smaller in comparison to the region of guaranteed recovery in expectation due to the sampling variance in the stochastic block model.

In the bottom row of Fig. 2 we show the clustering error for the state of the art methods L_{GM}, L_{SN}, L_{BM} and H. We can see that L_{GM} presents a similar performance as the signed power mean Laplacian L_0. The next Theorem shows that the geometric mean Laplacian $\mathcal{L}_{GM}$ and the limit $p \to 0$ of the signed power mean Laplacian agree in expectation for the SSBM. This implies via Corollary 2 that this operator is inferior to the signed power mean Laplacian for $p < 0$. This is why we use in the experiments on real world graphs later on always $p < 0$.

Theorem 2. *Let* $\mathcal{L}_{GM} = \mathcal{L}^+_{\text{sym}} \# \mathcal{Q}^-_{\text{sym}}$ *and* $\mathcal{L}_0$ *be the signed power mean Laplacian with* $p \to 0$ *of the expected signed graph. Then,* $\mathcal{L}_0 = \mathcal{L}_{GM}$*.*

In the bottom row of Fig. 2 we can observe that L_{SN}, L_{BN} and H present a similar behaviour to the arithmetic mean Laplacian L_1. A quick computation shows that for the case where both G^+, G^- have the same node degree in expectation, the conditions of Theorem 1 for $\mathcal{L}_1$ reduce to $p^-_{\text{in}} + p^+_{\text{out}} < p^+_{\text{in}} + p^-_{\text{out}}$. It turns out that this condition is also required by $\mathcal{L}_{SN}, \mathcal{L}_{BN}$ and $\mathcal{H}$, as the following shows.

Theorem 3 ((Mercado et al., 2016))**.** *Let* $\mathcal{L}_{BN}$ *and* $\mathcal{L}_{SN}$ *be the balanced normalized Laplacian and signed normalized Laplacian of the expected signed graph. The following statements are equivalent:*

- $\{\chi_i\}_{i=1}^k$ *are the eigenvectors corresponding to the* k*-smallest eigenvalues of* $\mathcal{L}_{BN}$*.*
- $\{\chi_i\}_{i=1}^k$ *are the eigenvectors corresponding to the* k*-smallest eigenvalues of* $\mathcal{L}_{SN}$*.*
- *inequalities* $p^-_{\text{in}} + (k-1)p^-_{\text{out}} < p^+_{\text{in}} + (k-1)p^+_{\text{out}}$ *and* $p^-_{\text{in}} + p^+_{\text{out}} < p^+_{\text{in}} + p^-_{\text{out}}$ *hold.*

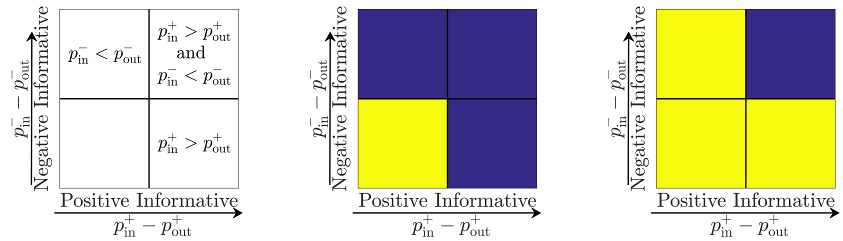
Finally, we present conditions in expectation for the Bethe Hessian to identify the ground truth clustering.

Theorem 4. *Let* $\mathcal{H}$ *be the Bethe Hessian of the expected signed graph. Then* $\{\chi_i\}_{i=2}^k$ *are the eigenvectors corresponding to the* $(k-1)$*-smallest negative eigenvalues of* $\mathcal{H}$ *if and only if the following conditions hold:*

1. $\max\{0, \frac{2(d^+ + d^-) - 1}{\sqrt{d^+ + d^- |\mathcal{C}|}}\} < (p^+_{\text{in}} - p^+_{\text{out}}) - (p^-_{\text{in}} - p^-_{\text{out}})$
2. $p^+_{\text{out}} < p^-_{\text{out}}$

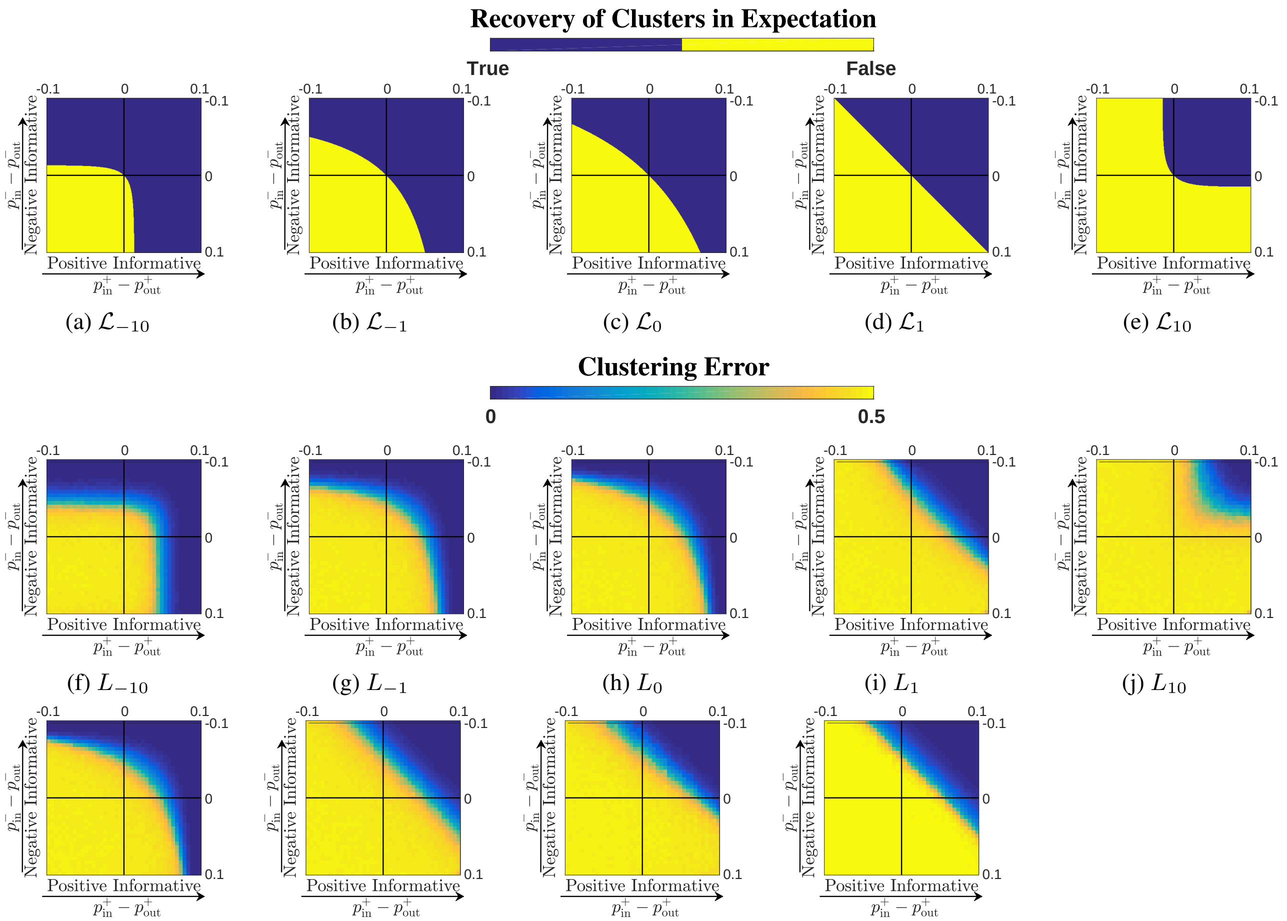
Moreover, for the limit case $|V| \to \infty$ *the first condition reduces to* $p^-_{\text{in}} + p^+_{\text{out}} < p^+_{\text{in}} + p^-_{\text{out}}$*.*

Please see the supplementary material for a further analysis in expectation. We can observe that the first condition in Theorem 4 is related to conditions of $\mathcal{L}_1$ and $\mathcal{L}_{SN}, \mathcal{L}_{BN}$ through the inequality $p^-_{\text{in}} + p^+_{\text{out}} < p^+_{\text{in}} + p^-_{\text{out}}$. This explains



(a) SBM Diagram (b) $L_{-\infty}$ (OR) (c) L_{∞} (AND)

Figure 1: Stochastic Block Model (SBM) for signed graphs. From left to right: Fig. 1a SBM Diagram. Fig. 1b SBM for $L_{-\infty}$(OR), Fig. 1c SBM for L_{∞}(AND). according to Corollary 1.



(a) $\mathcal{L}_{-10}$ (b) $\mathcal{L}_{-1}$ (c) $\mathcal{L}_{0}$ (d) $\mathcal{L}_{1}$ (e) $\mathcal{L}_{10}$

(f) L_{-10} (g) L_{-1} (h) L_{0} (i) L_{1} (j) L_{10}

(k) L_{GM} (l) L_{SN} (m) L_{BN} (n) H

Figure 2: Performance visualization for two clusters for different parameters of the SBM. **Top row**: In dark blue the settings where the signed power mean Laplacians $\mathcal{L}_p$ identify the ground truth clusters in expectation for the SBM, see Theorem 1, whereas yellow indicates failure. **Middle/Bottom row**: average clustering error (dark blue: small error, yellow: large error) of the signed power mean Laplacian L_p and $L_{GM}, L_{SN}, L_{BN}, H$ for 50 samples from the SBM.

why the performance of the Bethe Hessian H resembles the one of arithmetic Laplacians L_{SN}, L_{BN}, L_1. A more detailed comparison between the conditions of Theorems 1, 3 and 4 is detailed in the supplementary material.

Note that our analysis in expectation considers the dense regime where the average degree increases with the number of nodes and hence our results in expectation are verified under the SSBM setting here considered, showing that L_{SN}, L_{BN}, L_1, H have a similar performance. However, in the case of sparse graphs, it is known that the Bethe Hessian is asymptotically optimal in the information-theoretic transition limit (Saade et al., 2014; 2015). Please see the supplementary material for an evaluation under the CBM.

We now zoom in on a particular setting of Fig. 2. Namely, the case where G^+ (resp.G^-) is fixed to be informative, whereas the remaining graph transitions from informative to

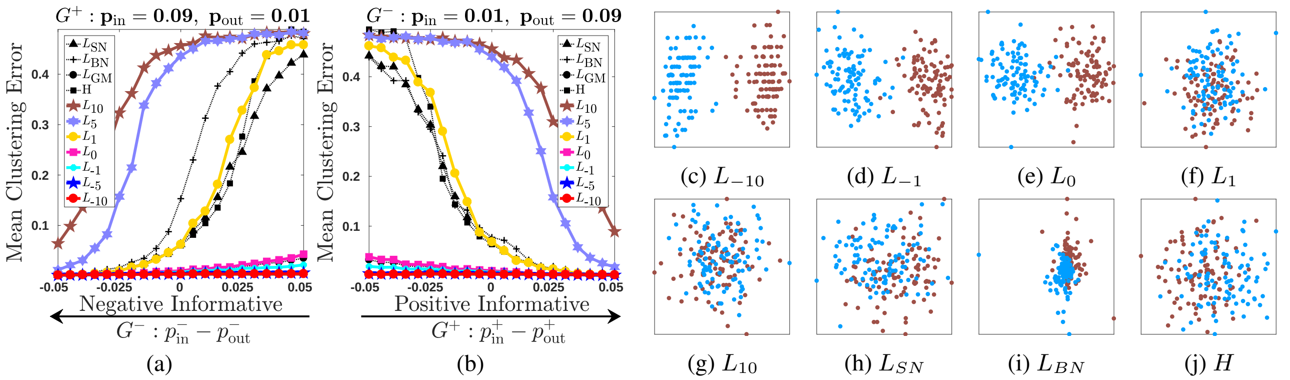


Figure 3: **Left**: Mean clustering error under the SSBM, with two clusters of size 100 and 50 runs. In Fig. 3a: G^+ is informative, i.e. assortative with $p_{\text{in}}^+ = 0.09$ and $p_{\text{out}}^+ = 0.01$. In Fig. 3b: G^- is informative, i.e. disassortative with $p_{\text{in}}^- = 0.01$ and $p_{\text{out}}^- = 0.09$. **Right**: Node embeddings induced by eigenvectors of different signed Laplacians for a random graph drawn from SSBM for 2 clusters of size 100, $p_{\text{in}}^+ = 0.025, p_{\text{out}}^+ = 0.075, p_{\text{in}}^- = 0.01, p_{\text{out}}^- = 0.09$.

uninformative. The corresponding results are in Fig. 3. In Fig. 3a we consider the case where G^+ is informative with parameters $p_{\text{in}}^+ = 0.09$ and $p_{\text{out}}^+ = 0.01$ (this corresponds to $p_{\text{in}}^+ - p_{\text{out}}^+ = 0.08$ in Fig. 2), and G^- goes from being informative ($p_{\text{in}}^- < p_{\text{out}}^-$) to non-informative ($p_{\text{in}}^- \geq p_{\text{out}}^-$). We confirm that the power mean Laplacian L_p presents smaller clustering errors for smaller values of p. Moreover, it is clear that in the case $p < 0$, L_p is able to recover clusters even in the case where G^- is not informative, whereas for $p > 0$, L_p requires both G^+ and G^- to be informative. We observe that the smallest (resp. largest) clustering errors correspond to L_{-10} (resp. L_{10}), corroborating Corollary 2. Further, we can observe that L_{GM} and L_0 have a similar performance, as well as L_{SN}, L_{BN}, L_1, H, as observed before, confirming Theorem 2 and Theorem 4, respectively. In Fig. 3b similar observations hold for the case where G^- is informative with parameters $p_{\text{in}}^- = 0.01$ and $p_{\text{out}}^- = 0.09$ (this corresponds to $p_{\text{in}}^- - p_{\text{out}}^- = -0.08$ in Fig. 2), and G^+ goes from being non-informative ($p_{\text{in}}^+ \leq p_{\text{out}}^+$) to informative ($p_{\text{in}}^+ > p_{\text{out}}^+$). Within this setting we present the eigenvector-based node embeddings of each method for the case $p_{\text{in}}^+ = 0.025, p_{\text{out}}^+ = 0.075, p_{\text{in}}^- = 0.01, p_{\text{out}}^- = 0.09$, in right hand side of Fig. 3. For L_{-10}, L_{-1}, L_0 the embeddings split the clusters properly, whereas remaining embeddings are not informative, verifying the effectivity of L_p with $p < 0$.

3.1. Consistency of the Signed Power Mean Laplacian for the Stochastic Block Model

In this section we prove two novel concentration bounds for signed power mean Laplacians of signed graphs drawn from the SSBM. The bounds show that, for large graphs, our previous results in expectation transfer to sampled graphs with high probability. We first show in Theorem 5 that L_p is close to $\mathcal{L}_p$. Then, in Theorem 6, we show that eigenvalues and eigenvectors of L_p are close to those of $\mathcal{L}_p$. We derive this result by tracing back the consistency of the matrix power mean to the consistency of the standard and signless Laplacian established in (Chung & Radcliffe, 2011).

The consistency of spectral clustering on unsigned graphs for the SBM has been studied in (Lei & Rinaldo, 2015; Sarkar & Bickel, 2015; Rohe et al., 2011) and more recently consistency of several variants of spectral clustering has been shown (Qin & Rohe, 2013; Joseph & Yu, 2016; Chaudhuri et al., 2012; Le et al.; Fasino & Tudisco, 2018; Davis & Sethuraman, 2018). Moreover, while the case of multilayer graphs under the SBM has been previously analyzed (Han et al., 2015; Heimlicher et al., 2012; Jog & Loh, 2015; Paul & Chen, 2017; Xu et al., 2014; 2017; Yun & Proutiere, 2016), there are no consistency results for matrix power means for multilayer graphs as studied in (Mercado et al., 2018). While our main emphasis is on the analysis of the SPM Laplacian, our proofs are general enough to cover also the consistency of the matrix power means for unsigned multilayer graphs (Mercado et al., 2018). In Thm. 5 we show that the SPM Laplacian L_p for the SSBM is concentrated around $\mathcal{L}_p$, with high probability for large n. The following results hold for general shifts ε.

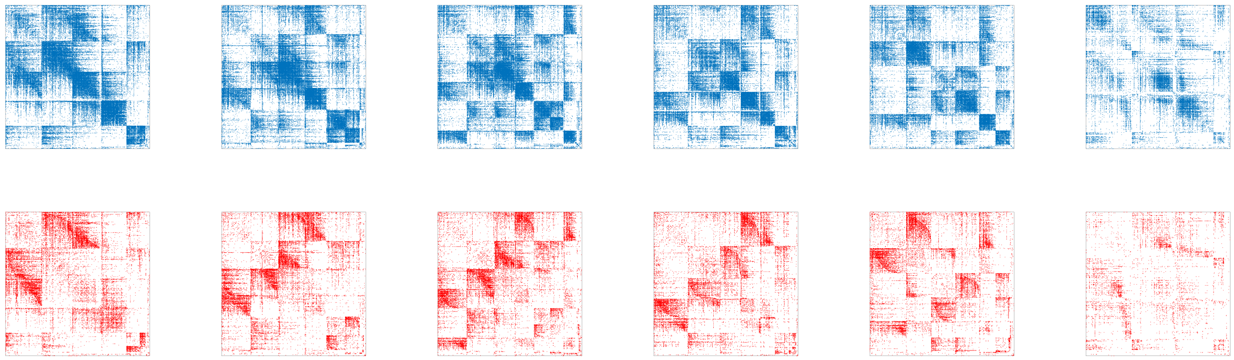
Theorem 5. *Let p be a non-zero integer, let*

$$C_p = \begin{cases} (2p)^{1/p}(2+\varepsilon)^{1-1/p} & p \geq 1 \\ |2p|^{1/|p|}\,\varepsilon^{-(3+1/|p|)} & p \leq -1 \end{cases}$$

and choose $\epsilon > 0$. If $\frac{n}{k}(p_{\text{in}}^+ + (k-1)p_{\text{out}}^+) > 3\ln(8n/\epsilon)$, and $\frac{n}{k}(p_{\text{in}}^- + (k-1)p_{\text{out}}^-) > 3\ln(8n/\epsilon)$, then with probability at least $1-\epsilon$, we have

$$\|L_p - \mathcal{L}_p\| \leq C_p\, m_{|p|}^{1/|p|}\left(\sqrt{\frac{3\ln(8n/\epsilon)}{\frac{n}{k}(p_{\text{in}}^+ + (k-1)p_{\text{out}}^+)}}, \sqrt{\frac{3\ln(8n/\epsilon)}{\frac{n}{k}(p_{\text{in}}^- + (k-1)p_{\text{out}}^-)}}\right)$$

In Thm 5 we take the spectral norm. A more general version of Theorem 5 for the inhomogeneous Erdős-Rényi model, where edges are formed independently with probabilities p_{ij}^+, p_{ij}^- is given in the supplementary material. Theorem 5 builds on top of concentration results of (Chung & Radcliffe, 2011) proven for the unsigned case $\left\|L_{\text{sym}}^+ - \mathcal{L}_{\text{sym}}^+\right\|$. We



(a) L_{-10} (b) L_{-5} (c) L_{-2} (d) L_{-1} (e) L_0 (f) L_1

Figure 4: Adjacency matrices W^+ and W^- sorted through clustering of Wikipedia Elections dataset by the proposed Signed Power mean Laplacians $L_{-10}, L_{-5}, L_{-2}, L_{-1}, L_0, L_1$. **Top row**: zoom-in visualization of positive edges W^+. **Bottom row**: zoom-in visualization of negative edges W^-. See supplementary material for more details.

can see that the deviation of L_p from $\mathcal{L}_p$ depends on the power mean of the individual deviations of L^+_{sym} and Q^-_{sym} from $\mathcal{L}^+_{\mathrm{sym}}$ and $\mathcal{Q}^-_{\mathrm{sym}}$, respectively. Note that the larger the size n of the graph is, the stronger is the concentration of L_p around $\mathcal{L}_p$.

The next Theorem shows that the eigenvectors corresponding to the smallest eigenvalues of L_p are close to the corresponding eigenvectors of $\mathcal{L}_p$. This is a key result showing consistency of our spectral clustering technique with L_p for signed graphs drawn from the SSBM.

Theorem 6. *Let $p \neq 0$ be an integer. Let $V_k, \mathcal{V}_k \in \mathbb{R}^{n\times k}$ be orthonormal matrices whose columns are the eigenvectors of the k smallest eigenvalues of L_p and $\mathcal{L}_p$, respectively. Let ρ^+_ε, ρ^-_ε and C_p be defined as in Theorems 1 and 5, respectively. Define $\widetilde{k} = k-1$, if $p \geq 1$, and $\widetilde{k} = k$, if $p \leq -1$ and choose $\epsilon > 0$.*
If $m_p(\rho^+_\varepsilon, \rho^-_\varepsilon) < 1+\varepsilon$, $\delta^+ := \frac{n}{k}(p^+_{\mathrm{in}} + (k-1)p^+_{\mathrm{out}}) > 3\ln(8n/\epsilon)$, and $\delta^- := \frac{n}{k}(p^-_{\mathrm{in}} + (k-1)p^-_{\mathrm{out}}) > 3\ln(8n/\epsilon)$, then there exists an orthogonal matrix $O_{\widetilde{k}} \in \mathbb{R}^{\widetilde{k}\times\widetilde{k}}$ such that, with probability at least $1-\epsilon$, we have

$$\|V_{\widetilde{k}} - \mathcal{V}_{\widetilde{k}} O_{\widetilde{k}}\| \leq \frac{\sqrt{8\widetilde{k}} C_p \, m_{|p|}^{1/|p|}\Big(\sqrt{\frac{3\ln(8n/\epsilon)}{\delta^+}}, \sqrt{\frac{3\ln(8n/\epsilon)}{\delta^-}}\Big)}{(1+\varepsilon) - m_p(\rho^+_\varepsilon, \rho^-_\varepsilon)}$$

Note that the main difference compared to Thm. 5 is the spectral gap $\gamma_p = (1+\varepsilon) - m_p(\rho^+_\varepsilon, \rho^-_\varepsilon)$ of $\mathcal{L}_p$, which is the difference of the eigenvalues corresponding to the informative versus non-informative eigenvectors of $\mathcal{L}_p$. Thus the stronger the clustering structure the tighter is the concentration of the eigenvectors. Moreover, from the monotonicity of m_p we have $\gamma_p \geq \gamma_q$ for $p < q$, and thus for $p \leq -1$ the spectral gap increases with $|p|$, ensuring a stronger concentration of eigenvectors for smaller values of p.

4. Experiments on Wikipedia-Elections

We now evaluate the Signed Power Mean Laplacian L_p with $p \in \{-10, -5, -2, -1, 0, 1\}$ on Wikipedia-Elections dataset (Leskovec & Krevl, 2014). In this dataset each node represents an editor requesting to become administrator and positive (resp. negative) edges represent supporting (resp. against) votes to the corresponding admin candidate.

While (Chiang et al., 2012) conjectured that this dataset has no clustering structure, recent works (Mercado et al., 2016; Cucuringu et al., 2019) have shown that indeed there is clustering structure. As noted in (Mercado et al., 2016), using the geometric mean Laplacian L_{GM} and looking for k clusters unveils the presence of a large non-informative cluster and $k-1$ remaining smaller clusters which show relevant clustering structure.

Our results verify these recent findings. We set the number of clusters to identify to $k = 30$ and in Fig. 4 we portray the portion of adjacency matrices of positive and negative edges W^+ and W^- corresponding to $k-1$ clusters sorted according to the corresponding identified clusters. We can see that when $p \leq 0$ the Signed Power Mean Laplacian L_p identifies clustering stucture, whereas this structure is overlooked by the arithmetic mean case $p = 1$. Moreover, we can see that different powers identify slightly different clusters: this happens as this dataset does not necessarily follow the Signed Stochastic Block Model, and hence we do not fully retrieve the same behaviour studied in Section 3.

Further experiments on UCI datasets are available in the supplementary material, suggesting that the L_{GM} together with L_p is a reasonable option under different settings.

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