Some formulæ in the analytic theory of numbers

Messenger of Mathematics, XLV, 1916, 81 – 84

I have found the following formulæ incidentally in the course of other investigations. None of them seem to be of particular importance, nor does their proof involve the use of any new ideas, but some of them are so curious that they seem to be worth printing. I denote by $d(x)$ the number of divisors of x, if x is an integer, and zero otherwise, and by $\zeta(s)$ the Riemann Zeta-function.

$$\text{(A)} \qquad \frac{\zeta^4(s)}{\zeta(2s)} = 1^{-s}d^2(1) + 2^{-s}d^2(2) + 3^{-s}d^2(3) + \cdots, \tag{1}$$

$$\frac{\eta^4(s)}{(1-2^{-2s})\zeta(2s)} = 1^{-s}d^2(1) - 3^{-s}d^2(3) + 5^{-s}d^2(5) - \cdots, \tag{2}$$

where

$$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots .$$

$$\text{(B)} \qquad d^2(1) + d^2(2) + d^2(3) + \ldots + d^2(n)$$

$$= An(\log n)^3 + Bn(\log n)^2 + Cn\log n + Dn + O(n^{\frac{3}{5}+\epsilon}),^* \tag{3}$$

where

$$A = \frac{1}{\pi^2},\ B = \frac{12\gamma - 3}{\pi^2} - \frac{36}{\pi^4}\zeta'(2),$$

γ is Euler's constant, C, D more complicated constants, and ϵ any positive number.

$$\text{(C)} \qquad d^3\left(\frac{n}{1}\right) + d^3\left(\frac{n}{3}\right) + d^3\left(\frac{n}{2}\right) + \cdots$$

$$= \left\{d\left(\frac{n}{1}\right) + d\left(\frac{n}{2}\right) + d\left(\frac{n}{3}\right) + \cdots\right\}^2,^\dagger \tag{4}$$

$$\sum_1^\infty n^{-s}d^r(n) = \{\zeta(s)\}^{2^r}\phi(s), \tag{5}$$

where $\phi(s)$ is absolutely convergent for $R(s) > \frac{1}{2}$, and in particular

$$\sum_1^\infty \frac{1}{n^s d(n)} = \prod_p \left\{p^s \log\left(\frac{1}{1-p^{-s}}\right)\right\} = \sqrt{\{\zeta(s)\}}\phi(s). \tag{6}$$

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*If we assume the Riemann hypothesis, the error term here is of the form $O(n^{\frac{1}{2}+\epsilon})$.

†Mr Hardy has pointed out to me that this formula has been given already by Liouville, *Journal de Mathématiques*, Ser.2, Vol. II (1857), p.393.

$$\text{(D)} \qquad \frac{1}{d(1)}+\frac{1}{d(2)}+\frac{1}{d(3)}+\cdots+\frac{1}{d(n)}$$
$$= n\left\{\frac{A_1}{(\log n)^{\frac{1}{2}}}+\frac{A_2}{(\log n)^{\frac{3}{2}}}+\cdots+\frac{A_r}{(\log n)^{r-\frac{1}{2}}}+O\frac{1}{(\log n)^{r+\frac{1}{2}}}\right\}, \tag{7}$$

where

$$A_1=\frac{1}{\sqrt{\pi}}\prod_p\left\{\sqrt{(p^2-p)}\log\left(\frac{p}{p-1}\right)\right\}$$

and $A_2, A_3, \ldots A_r$ are more complicated constants.
More generally

$$d^s(1)+d^s(2)+d^s(3)+\ldots+d^s(n)$$
$$= n\{A_1(\log n)^{2^s-1}+A_2(\log n)^{2^s-2}+\ldots+A_{2^s}\}+O(n^{\frac{1}{2}+\epsilon}),^{*} \tag{8}$$

if 2^s is an integer, and

$$d^s(1)+d^s(2)+d^s(3)+\ldots+d^s(n)$$
$$= n\left\{A_1(\log n)^{2^s-1}+A_2(\log n)^{2^s-2}+\ldots+\frac{A_{r+2^s}}{(\log n)^r}+O\left[\frac{1}{(\log n)^{r+1}}\right]\right\}, \tag{9}$$

if 2^s is not an integer, the A's being constants.

$$(E) \qquad d(1)d(2)d(3)\ldots d(n)=2^{n(\log\log n+C)+\phi(n)}, \tag{10}$$

where

$$C=\gamma+\sum_2^{\infty}\left\{\log_2\left(1+\frac{1}{\nu}\right)-\frac{1}{\nu}\right\}(2^{-\nu}+3^{-\nu}+5^{-\nu}+\ldots).$$

Here $2, 3, 5, \ldots$ are the primes and

$$\frac{\phi(n)}{n}=\frac{\gamma-1}{\log n}+\frac{1!}{(\log n)^2}(\gamma+\gamma_1-1)+\frac{2!}{(\log n)^3}(\gamma+\gamma_1+\gamma_2-1)+\ldots$$
$$+\frac{(r-1)!}{(\log n)^r}(\gamma+\gamma_1+\gamma_2+\ldots+\gamma_{r-1}-1)+O\left\{\frac{1}{(\log n)^{r+1}}\right\},$$

where

$$\zeta(1+s)=\frac{1}{s}+\gamma-\gamma_1 s+\gamma_2 s^2-\gamma_3 s^3+\ldots$$

or

$$r!\gamma_r=\lim_{\nu\to\infty}\left\{(\log 1)^r+\frac{1}{2}(\log 2)^r+\ldots+\frac{1}{\nu}(\log\nu)^r-\frac{1}{r+1}(\log\nu)^{r+1}\right\}.$$

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*Assuming the Riemann hypothesis.

$$\text{(F)} \qquad d(uv) = \sum_1^\infty \mu(n) d\left(\frac{u}{n}\right) d\left(\frac{v}{n}\right) = \sum \mu(\delta) d\left(\frac{u}{\delta}\right) d\left(\frac{v}{\delta}\right), \tag{11}$$

where δ is a common factor of u and v, and

$$\frac{1}{\zeta(s)} = \sum_1^\infty \frac{\mu(n)}{n^s}.$$

$$\text{(G)} \qquad \text{If} \quad D_v(n) = d(v) + d(2v) + \ldots + d(nv),$$

we have

$$D_v(n) = \sum \mu(\delta) d\left(\frac{v}{\delta}\right) D_1\left(\frac{n}{\delta}\right), \tag{12}$$

where δ is a divisor of v, and

$$D_v(n) = \alpha(v) n(\log n + 2\gamma - 1) + \beta(v) n + \Delta_v(n), \tag{13}$$

where

$$\sum_1^\infty \frac{\alpha(\nu)}{\nu^s} = \frac{\zeta^2(s)}{\zeta(1+s)}, \sum_1^\infty \frac{\beta(\nu)}{\nu^s} = -\frac{\zeta^2(s)\zeta'(1+s)}{\zeta^2(1+s)},$$

and

$$\Delta_v(n) = O(n^{\frac{1}{3}} \log n) \text{ *}$$

$$\begin{aligned} (H) \qquad & d(v+c) + d(2v+c) + d(3v+c) + \ldots + d(nv+c) \\ & = \alpha_c(v) n(\log n + 2\gamma - 1) + \beta_c(v) n \Delta_{v,c}(n), \end{aligned} \tag{14}$$

where

$$\sum_1^\infty \frac{\alpha_c(\nu)}{\nu^s} = \frac{\zeta(s)\sigma_{-s}(|c|)}{\zeta(1+s)},$$

$$\sum_1^\infty \frac{\beta_c(\nu)}{\nu^s} = \frac{\zeta(s)\sigma_{-s}(|c|)}{\zeta(1+s)} \left\{ \frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1+s)}{\zeta(1+s)} + \frac{\sigma_{-s}{}'(|c|)}{\sigma_{-s}(|c|)} \right\},$$

$\sigma_s(n)$ being the sum of the sth powers of the divisors of n and $\sigma'_s(n)$ the derivative of $\sigma_s(n)$ with respect to s, and

$$\Delta_{v,c}(n) = O(n^{\frac{1}{3}} \log n).\dagger$$

(I) The formulæ(1) and (2) are special cases of

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

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*It seems not unlikely that $\Delta_v(n)$ is of the form $O(n^{\frac{1}{4}+\epsilon})$. Mr Hardy has recently shewn that $\Delta_1(n)$ is not of the form $o\{(n\log n)^{\frac{1}{4}} \log\log n\}$. The same is true in this case also.

†It is very likely that the order of $\Delta_{v,c}(n)$ is the same as that of $\Delta_1(n)$.

$$= 1^{-s}\sigma_a(1)\sigma_b(1) + 2^{-s}\sigma_a(2)\sigma_b(2) + 3^{-s}\sigma_a(3)\sigma_b(3) + \cdots; \tag{15}$$

$$\frac{\eta(s)\eta(s-a)\eta(s-b)\eta(s-a-b)}{(1-2^{-2s+a+b})\zeta(2s-a-b)}$$
$$= 1^{-s}\sigma_a(1)\sigma_b(1) - 3^{-s}\sigma_a(3)\sigma_b(3) + 5^{-s}\sigma_a(5)\sigma_b(5) - \cdots \tag{16}$$

It is possible to find an approximate formula for the general sum

$$\sigma_a(1)\sigma_b(1) + \sigma_a(2)\sigma_b(2) + \ldots + \sigma_a(n)\sigma_b(n). \tag{17}$$

The general formula is complicated, The most interesting cases are $a = 0, b = 0$, when the formula is (3); $a = 0, b = 1$, when it is

$$\frac{\pi^4 n^2}{72\zeta(3)}(\log n + 2c) + nE(n), \tag{18}$$

where

$$c = \gamma - \frac{1}{4} + \frac{\zeta'(2)}{\zeta(2)} - \frac{\zeta'(3)}{\zeta(3)},$$

and the order of $E(n)$ is the same as that of $\Delta_1(n)$; and $a = 1, b = 1$, when it is

$$\frac{5}{6}n^3\zeta(3) + E(n), \tag{19}$$

where

$$E(n) = O\{n^2(\log n)^2\},\ E(n) \neq o(n^2 \log n).$$

(J) If $s > 0$, then

$$\sigma_s(1)\sigma_s(2)\sigma_s(3)\sigma_s(4)\ldots\sigma_s(n) = \theta c^n (n!)^s, \tag{20}$$

where

$$1 > \theta > (1-2^{-s})(1-3^{-s})(1-5^{-s})\ldots(1-\varpi^{-s}),$$

ϖ is the greatest prime not exceeding n, and

$$c = \prod_p \left\{ \left(\frac{p^{2s}-1}{p^{2s}-p^s}\right)^{1/p} \left(\frac{p^{3s}-1}{p^{3s}-p^s}\right)^{1/p^2} \left(\frac{p^{4s}-1}{p^{4s}-p^s}\right)^{1/p^3} \cdots \right\}.$$

(K) If $(\frac{1}{2} + q + q^4 + q^9 + q^{16} + \cdots)^2 = \frac{1}{4} + \sum_1^\infty r(n)q^n,$
so that

$$\zeta(s)\eta(s) = \sum_1^\infty r(n)n^{-s},$$

then

$$\frac{\zeta^2(s)\eta^2(s)}{(1+2^{-s})\zeta(2s)} = 1^{-s}r^2(1) + 2^{-s}r^2(2) + 3^{-s}r^2(3) + \cdots \tag{21}$$

$$r^2(1) + r^2(2) + r^2(3) + \cdots + r^2(n) = \frac{n}{4}(\log n + C) + O(n^{\frac{3}{5}+\epsilon}), \tag{22}$$

where

$$C = 4\gamma - 1 + \frac{1}{3}\log 2 - \log \pi + 4\log \Gamma(\frac{3}{4}) - \frac{12}{\pi^2}\zeta'(2).$$

These formulæ are analogous to (1) and (3).