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SIMPLICIAL CONVEXITY IN VECTOR SPACES

BY

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All considerations which follow will be made in a real vector space. While the principal definitions will be given in an arbitrary one, the main results remain true only in vector spaces whose dimension is finite.

We shall introduce a new notion, namely the simplicial convexity, more general than convexity, and study some of its relations with the usual convexity. Also, we shall introduce some numbers associated to a simplicial convex set and establish several properties of these numbers.

§ 0. Introduction

We give here a number of definitions, among them the last seven being new. The other are given to facilitate the reader's task, as well as the two first lemmas. For initiated readers, only the definitions 7—13 are indispensable in § 0.

Let us consider a vector space $\mathcal{V}$. We use the notations : $A - B$ for $A \cap \complement B$, when $A, B \subset \mathcal{V}$, and $[r]$ for the integral part of the real positive number r. Denote by $\mathcal{E}(M)$ the convex cover of $M \subset \mathcal{V}$ and $\mathcal{E}(x_1, \ldots, x_s) = \mathcal{E}(\{x_1, \ldots, x_s\})$, where $x_1 \ldots, x_s \in \mathcal{V}$. A set

$$\mathcal{E}(x_1, \ldots, x_s) = \left\{ \sum_{i=1}^{s} \lambda_i x_i ;\ \lambda_i \text{ real},\ \lambda_i \geqslant 0,\ \sum_{i=1}^{s} \lambda_i = 1 \right\},$$

where $\{x_j - x_1;\ 2 \leqslant j \leqslant s\}$ is linear independent, is called *simplex* with vertices $x_1, \ldots, x_s \in \mathcal{V}$.

Although boundedness and closure are respectively metric and topological notions, they both can be defined for convex sets in our arbitrary vector space.

Definition 1. *Consider the convex set C and the point $x \in \mathcal{V}$. If there exist $y \in C$ and $z \in V$, where y can coincide with x, $z \neq x$ and V*

is the smallest linear manifold containing C, such that

$$(\mathcal{E}(x, y) - C) \cup (\mathcal{E}(x, z) \cap C) = \{x\},$$

then x is a boundary point of C and the set of all these points is the frontier of C.

Observe that each boundary point x lies on V, since

$$\mathcal{E}(x, y) - C \subset \{x\}$$

implies $\mathcal{E}(x, y) \subset V$.

Definition 2. *The interior of a convex set C is the set of all the points of C which are not boundary points.*

Definition 3. *The convex set C is said to be closed if C includes its frontier.*

Definition 4. *A convex set C is said to be bounded if there exist no semilines included in C.*

Definition 5. *A point x is said to be an extreme point of a convex set C if C — {x} is convex, or — equivalently — if*

$$x \in C - \bigcup_{y, z \in C} (\mathcal{E}(y, z) - \{y, z\}).$$

Definition 6. *A set $M \subset \mathcal{V}$ is called convexly connected if there is no hyperplane Π such that $\Pi \cap M = \emptyset$ and M contains points in both the open half-spaces determined by Π; a set M' possesses m convexly connected components if there are m convexly connected sets whose union is M', but the union of any m' convexly connected sets ($m' < m$) does not equal M'.*

The next well-known result is obvious in a finite-dimensional vector space, because one can use its Euclidean topology in which the two concepts about frontier coincide. But it holds too in an arbitrary vector space.

Lemma 1. *A bounded closed convex set is the convex cover of its frontier.*

Let us remark that this lemma remains true for an arbitrary closed convex set different from a linear manifold or a half-linear manifold.

In order to respect the unity of exposition, we also give the following well-known lemma. We shall prove it in a simple direct manner *).

Lemma 2 (Krein-Milman). *A closed bounded convex set in a finite-dimensional vector space is the convex cover of its extreme points.*

Proof **). Let C be the given convex set, B its frontier and E_C the set of all extreme points of C. Obviously, $\mathcal{E}(E_C) \subset C$. Prove that $C \subset \mathcal{E}(E_C)$. Suppose that this inclusion is valid in an $(n-1)$-dimensional vector space and prove it in an n-dimensional one.

By a consequence of Hahn-Banach theorem, there exists, even in an arbitrary vector space, a supporting hyperplane at each boundary point of a convex set. Let $x \in B$ and H be a supporting hyperplane at x. The intersection $H \cap C$ is, obviously, a convex bounded set. Prove that

*) In [2] the proof of this lemma is indicated to use Carathéodory's theorem.

**) This proof has been given in a seminar on convex sets at Bucharest University, 1964.

it is closed. Let x' be a boundary point of $H \cap C$. There are $y \in H \cap C$ and $z \neq x'$ in H such that

$$(\mathcal{E}(x', y) - C) \cup (\mathcal{E}(x', z) \cap C) = \{x'\}.$$

Now, we see that x' is a boundary point of C, hence $x' \in C$. But the frontier of $H \cap C$ lies in H, hence $x' \in H \cap C$.

By the induction hypothesis,

$$x \in C \cap H \subset \mathcal{E}(E_{C \cap H}).$$

Prove that $E_{C\cap H} \subset E_C$. Let $y \in E_{C \cap H}$. Suppose there exist $z, z' \in C$ such that $z \neq y$; $z' \neq y$ and $y \in \mathcal{E}(z, z')$. The point z belongs to H if and only if z' belongs also to H, because the line through z and z' either lies on H or intersects it in a single point, y. It results from $y \in E_{C\cap H}$ that $z, z' \notin H$. Then H cuts $\mathcal{E}(z, z') \subset C$, which is impossible, H being a supporting hyperplane. Thus $y \in E_C$, whence $\mathcal{E}(E_{C\cap H}) \subset \mathcal{E}(E_C)$. Hence $x \in \mathcal{E}(E_C)$ and $B \subset \mathcal{E}(E_C)$.

Now, following lemma 1,

$$C = \mathcal{E}(B) \subset \mathcal{E}(E_C),$$

hence $C = \mathcal{E}(E_C)$.

D e f i n i t i o n 7. *The function $\mathcal{S}_l$, called the l-simplicial convex cover is defined in the family of all the subsets of $\mathcal{V}$ by*

$$\mathcal{S}_l(M) = \{\bigcup \mathcal{E}(p_1, \ldots, p_i);\ p_j \in M,\quad 1 \leqslant j \leqslant i,\quad 1 \leqslant i \leqslant l\}$$

($\mathcal{E}(p_1, \ldots, p_i)$ are simplexes), for arbitrary set $M \subset \mathcal{V}$ and natural number $l \geqslant 2$.

One easily proves that this function possesses the properties that $\mathcal{S}_{mn} = \mathcal{S}_m \circ \mathcal{S}_n$ and $\mathcal{S}_m(M) \subset \mathcal{E}(M)$, whence, also,

$$\mathcal{S}_{m^p} = \underbrace{\mathcal{S}_m \circ \mathcal{S}_m \circ \ldots \circ \mathcal{S}_m}_{p \text{ times}} = \mathcal{S}_m^p$$

and

$$\mathcal{S}_m^p(M) \subset \mathcal{E}(M)$$

for natural numbers $m, n, p \geqslant 2$ and arbitrary $M \subset \mathcal{V}$.

$\mathcal{S}_l(M)$ is an increasing fonction of l; it is also an increasing fonction of M relative to inclusion ordering.

Let us denote $\mathcal{S}_m(x_1, \ldots, x_s) = \mathcal{S}_m(\{x_1, \ldots, x_s\})$, where $x_1, \ldots, x_s \in \mathcal{V}$.

D e f i n i t i o n 8. *A set $K \subset \mathcal{V}$ is said to be l-simplicial convex if there exists a set $M \subset V$ such that $K = \mathcal{S}_l(M)$.*

D e f i n i t i o n 9. *The l-order of an l-simplicial convex set K (the l-simplicial convexity order) is*

$$\omega_l(K) = \sup_M \min\ \{k;\ \mathcal{S}_l^k(M) = K\}.$$

D e f i n i t i o n 10. *A set K is said to be simplicial convex if there exists a number l such that K is l-simplicial convex.*

D e f i n i t i o n 11. *The degree of a simplicial convex set* K *is* $\delta(K) = \min\{l;\ K$ is l-simplicial convex$\}$.

D e f i n i t i o n 12. *The order of a simplicial convex set* K *(the simplicial convexity order) is*

$$\Omega(K) = \sup_{l} \omega_l(K).$$

By convenience, we set ∞ the order of a simplicial convex set, whose l-order is ∞ for at least one number l.

D e f i n i t i o n 13. *The power of a simplicial convex set* K *of finite order is*

$$\Delta(K) = \min\{l;\ \Omega(K) = \omega_l(K)\}.$$

§ 1. Relations between convex cover and l-simplicial convex cover

We shall state here two theorems important in the study of simplicial convexity. They both give relations between the usual convex cover and the l-simplicial convex cover in finite-dimensional vector spaces. They can be found as corollaries in [1], but we shall prove directly the first of them because it is peculiarly fundamental.

T h e o r e m 1. *If* $\mathcal{V}$ *is an* n*-dimensional vector space and* $M \subset \mathcal{V}$, *then*

$$\mathcal{E}(M) = \mathcal{S}_l^{[\log_l n]+1}(M).$$

P r o o f. Clearly, $\mathcal{S}_l^m(M) \subset \mathcal{E}(M)$ for every m. Prove that

$$\mathcal{E}(M) \subset \mathcal{S}_l^{[\log_l n]+1}(M).$$

If $m = [\log_l n] + 1$, then $m - 1 \leqslant \log_l n < m$ and $l^{m-1} \leqslant n < l^m$.

For $n = m = 1$, the inclusion is obviously valid. Assume inductively that it is true for the dimensions n satisfying the above inequalities i.e. $m = [\log_l n] + 1$ and prove it for n satisfying $m = [\log_l n]$.

Let $x \in \mathcal{E}(M)$. By Carathéodory' s theorem [2], there is a simplex with vertices $x_1, \ldots, x_s \in M$ $(s \leqslant n + 1)$, such that $x \in \mathcal{E}(x_1, \ldots, x_s)$. If $s \leqslant l^m$, then the simplex is contained in an $(l^m - 1)$-dimensional linear manifold and by the induction hypothesis

$$\mathcal{E}(x_1, \ldots, x_s) \subset \mathcal{S}_l^{[\log_l(l^m-1)]+1}(M) = \mathcal{S}_l^m(M) \subset \mathcal{S}_l^{[\log_l n]+1}(M).$$

Suppose now that $l^m + 1 \leqslant s \leqslant n + 1$. We can write

$$x = \sum_{i=1}^{s} \lambda_i x_i = \sum_{j=0}^{p-1} \sum_{i=jl^m+1}^{(j+1)l^m} \lambda_i x_i + \sum_{j=pl^m+1}^{s} \lambda_i x_i,$$

where $\sum_{i=1}^{s} \lambda_i = 1$, $\lambda_i \geqslant 0$ and $p = \left[\frac{s}{l^m}\right] \leqslant l$, the last sum vanishing by convenience if $\frac{s}{l^m}$ is an integer. Denote

$$\mu_j = \sum_{i=jl^m+1}^{(j+1)l^m} \lambda_i \quad ; \quad \mu_p = \sum_{i=pl^m+1}^{s} \lambda_i \quad ;$$

$$y_j = \sum_{i=jl^m+1}^{(j+1)l^m} \frac{\lambda_i}{\mu_j} x_i ; \quad y_p = \sum_{i=pl^m+1}^{s} \frac{\lambda_i}{\mu_p} x_i .$$

Then

$$x = \sum_{j=0}^{p} \mu_j \, y_j$$

and this sum contains at most l terms because, if $p = l$, then $s \geqslant l^{m+1}$; hence $n + 1 \geqslant l^{m+1}$ and since $n < l^{m+1}$, the equality $n + 1 = l^{m+1}$ holds, thus s and l^{m+1} are equal, whence l^m devides s and $\mu_p = 0$. Since $\mu_j \geqslant 0$ and

$$\sum_{j=0}^{p} \mu_j = \sum_{i=1}^{s} \lambda_i = 1,$$

the point x belongs to a simplex with vertices $y'_1, \ldots, y'_p$ $(p \leqslant l)$. Every vertex y'_j belongs to a simplex with at most l^m vertices in M; by the induction hypothesis, $y'_j \in \mathcal{S}_l^m(M)$. It follows that

$$x \in \mathcal{S}_l^{m+1}(M) = \mathcal{S}_l^{[\log_l n]+1}(M).$$

R e m a r k 1. *The exponent* $[\log_l n] + 1$ *of* $\mathcal{S}_l$ *in theorem* 1 *is the best possible, i.e. in general* $\mathcal{E}(M) \neq \mathcal{S}_l^q(M)$ *with* $q \leqslant [\log_l n]$.

The proof will be given by a suitable example. Let M be a set of $n + 1$ vertices of a simplex. Then $\mathcal{S}_l(M)$ is the union of the subsimplexes with at most l vertices. Suppose that $\mathcal{S}_l^\nu(M)$ is included in the union of the subsimplexes with at most l^ν vertices and prove that this inclusion is true for $\nu + 1$. Indeed,

$$\mathcal{S}_l^{\nu+1}(M) = \mathcal{S}_l(\mathcal{S}_l^\nu(M)) = \{\bigcup \mathcal{E}(p_1, \ldots, p_i) ; \; p_j \in \mathcal{S}_l^\nu(M),$$
$$1 \leqslant j \leqslant i, \quad 1 \leqslant i \leqslant l\}.$$

Consider a simplex $\mathcal{E}(p_1, \ldots, p_i)$ of this union. Every vertex p_j belongs to a subsimplex S_j with at most l^ν vertices $(j \leqslant i)$. Then

$$\mathcal{E}(p_1, \ldots, p_i) \subset \mathcal{E}\left(\bigcup_{j=1}^{i} S_j\right),$$

where the second convex cover is a simplex with at most $l^{\nu+1}$ vertices. Hence $\mathcal{S}_l^q(M)$ is included in (even equal to) the union of the subsimplexes with at most l^q vertices. But from $l^q \leqslant n$, it results that $\mathcal{S}_l^q(M)$ is included in the frontier of $\mathcal{E}(M)$, hence $\mathcal{E}(M)$ strictly includes $\mathcal{S}_l^q(M)$.

Theorem 2. *If $\mathcal{V}$ is an n-dimensional vector space and $M \subset \mathcal{V}$ has at most n components* *) *or is compact* *) *and has at most n convexly connected components, then*

$$\mathcal{E}(M) = \mathcal{S}_l^{[\log_l(n-1)]+1}(M).$$

Proof. To prove the inclusion

$$\mathcal{E}(M) \subset \mathcal{S}_l^{[\log_l(n-1)]+1}(M),$$

we use again Carathéodory's theorem and a result of O. Hanner and H. Rådström [3], which state that each point $x \in \mathcal{E}(M)$ belongs to a simplex with vertices $x_1, \ldots, x_s \in M$ $(s \leqslant n)$. One obtains now by an analogous way (changing everywhere n by $n-1$), the desired inclusion.

Remark 2. *The exponent* $[\log_l(n-1)]+1$ *of* $\mathcal{S}_l$ *in theorem 2 is the best possible, i.e. in general* $\mathcal{E}(M) \neq \mathcal{S}_l^q(M)$ *with* $q \leqslant [\log_l(n-1)]$ *under the hypothesis of theorem 2.*

Again a suitable example will be given: Let $x_1, \ldots, x_{n+1}$ be the vertices of a simplex and

$$M = \bigcup_{s=1}^{n} \mathcal{E}(x_s, x_{n+1}).$$

Then

$$\mathcal{S}_l(M) = (\mathcal{E}(x_1, \ldots, x_{n+1}) - \mathcal{E}(x_1, \ldots, x_n)) \cup \{\bigcup \mathcal{E}(x_{i_1}, \ldots, x_{i_l}); \; 1 \leqslant i_j \leqslant n, \; 1 \leqslant j \leqslant l\}.$$

Suppose that $\mathcal{S}_l^\nu(M)$ contains no interior points of $\mathcal{E}(x_1, \ldots, x_n)$ reative to the hyperplane H containing $x_1, \ldots, x_n$ (see definitions 1 and 2) if $\mathcal{S}_l^\nu(x_1, \ldots, x_n)$ does not meet the interior of $\mathcal{E}(x_1, \ldots, x_n)$ and prove that this is true for $\nu+1$ instead of ν.

$$\mathcal{S}_l^{\nu+1}(M) = \mathcal{S}_l(\mathcal{S}_l^\nu(M)) = \{\bigcup \mathcal{E}(p_1, \ldots, p_i); \; p_j \in \mathcal{S}_l^\nu(x_1, \ldots, x_n), \; 1 \leqslant j \leqslant i, \; 1 \leqslant i \leqslant l\} \cup \{\bigcup \mathcal{E}(p_1, \ldots, p_i);$$
$$p_{j_1} \in \mathcal{S}_l^\nu(x_1, \ldots, x_n), p_{j_2} \in \mathcal{S}_l^\nu(M) - H, 1 \leqslant j_1 \leqslant k, \; k < j_2 \leqslant i, \; 1 \leqslant k < i, \; 1 \leqslant i \leqslant l\}.$$

But the union of the simplexes $\mathcal{E}(p_1, \ldots, p_i)$, the vertices $p_1, \ldots, p_i$ varying in $\mathcal{S}_l^\nu(x_1, \ldots, x_n)$ and i between 1 and l, is $\mathcal{S}_l^{\nu+1}(x_1, \ldots, x_n)$ and

$$H \cap \{\bigcup \mathcal{E}(p_1, \ldots, p_i); \; p_{j_1} \in \mathcal{S}_l^\nu(x_1, \ldots, x_n), \; p_{j_2} \in \mathcal{S}_l^\nu(M) - H, \; 1 \leqslant j_1 \leqslant k, \; k < j_2 \leqslant i, \; 1 \leqslant k < i, \; 1 \leqslant i \leqslant l\} \subset \mathcal{S}_{l-1}(\mathcal{S}_l^\nu(x_1, \ldots, x_n)) \subset \mathcal{S}_l^{\nu+1}(x_1, \ldots, x_n).$$

Therefore

$$H \cap \mathcal{S}_l^{\nu+1}(M) = \mathcal{S}_l^{\nu+1}(x_1, \ldots, x_n)$$

and if $\mathcal{S}_l^{\nu+1}(x_1, \ldots, x_n)$ does not intersect the interior of $\mathcal{E}(x_1, \ldots, x_n)$, then $\mathcal{S}_l^{\nu+1}(M)$ does not too.

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*) — In the usual topology of the n-dimensional vector space $\mathcal{V}$.

In the proof of remark 1, it is established that if $\mathcal{W}$ is n-dimensional, S is a simplex with $n+1$ vertices in $\mathcal{W}$ and $q \leqslant [\log_l n]$, then $\mathcal{S}_l^q(S)$ is included in the frontier of S. Consequently, if $q \leqslant [\log_l (n-1)]$, then $\mathcal{S}_l^q(x_1,\ldots, x_n)$ does not contain interior points of $\mathcal{E}(x_1,\ldots, x_n)$. Hence $\mathcal{S}_l^q(M)$ is strictly included in $\mathcal{E}(M) = \mathcal{E}(x_1,\ldots, x_{n+1})$.

§ 2. The l-simplicial convexity order

We shall give in this paragraph some upper bounds for l-simplicial convexity order in finite-dimensional vector spaces and prove the existence of simplicial convex sets of infinite order in arbitrary spaces.

Theorem 3. *The l-order of an l-simplicial convex set in an n-dimensional vector space is at most* $[\log_l n]+1$.

Proof. Prove that, if K is the given set,

$$\min \{k;\ \mathcal{S}_l^k(M) = K\} \leqslant [\log_l n]+1$$

for arbitrary M. Consider a number k such that $\mathcal{S}_l^k(M) = K$. If $k \leqslant [\log_l n]+1$, then clearly the above inequality holds. If $k > [\log_l n]+1$, then, simultaneously,

$$\mathcal{S}_l^k(M) \subset \mathcal{E}(M)$$

and

$$\mathcal{S}_l^k(M) \supset \mathcal{S}_l^{[\log_l n]+1}(M) = \mathcal{E}(M),$$

according to theorem 1. Hence $\mathcal{S}_l^k(M) = \mathcal{E}(M)$, that is

$$K = \mathcal{S}_l^{[\log_l n]+1}(M)$$

and the inequality is proved. Since M is arbitrary,

$$\omega_l(K) \leqslant [\log_s n]+1$$

which concludes the theorem.

Theorem 4. *In an n-dimensional vector space, a bounded closed convex set, whose extreme points form a set which has at most n components or is closed and has at most n convexly connected components, is l-simplicial convex of l-order at most* $[\log_l (n-1)]+1$.

Proof. One sees immediately that a convex set is also l-simplicial convex (see begining of § 3). Let M be such that $\mathcal{S}_l^k(M) = K$, where k is a natural number and K the given set. Then $\mathcal{S}_l^m(M) = K$ for each $m \geqslant k$. The set E_K of all extreme points of K must be included in M. Suppose it is not true : there is a point $y \in E_K - M$. Suppose that $y \notin \mathcal{S}_l^{\nu-1}(M)$ and prove that $y \notin \mathcal{S}_l^{\nu}(M)$ $(\nu \geqslant 1,\ \mathcal{S}_l^0(M) = M)$. If

$$y \in \mathcal{S}_l^{\nu}(M) = \mathcal{S}_l(\mathcal{S}_l^{\nu-1}(M)),$$

then y belongs to a simplex with vertices $x_1, \ldots, x_s \in \mathcal{S}_l^{\nu-1}(M)$ $(s \leqslant l)$. Hence $y \in \mathcal{E}(x_1, \ldots, x_s) - \{x_1, \ldots, x_s\}$ and there are $v, w \in \mathcal{E}(x_1, \ldots, x_s)$, such that $v \neq y$, $w \neq y$ and $y \in \mathcal{E}(v,w)$. Thus y would belong to the interior of a segment with end points in $\mathcal{E}(x_1, \ldots, x_s) \subset \mathcal{S}_l^{\nu}(M) \subset K$, which is not possible owing to the definition of the extreme points. We proved that $y \notin \mathcal{S}_l^{\nu}(M)$. Hence $y \notin \mathcal{S}_l^{k}(M) = K$; this contradiction shows that $E_K \subset M$, whence

$$\mathcal{S}_l^{[\log_l(n-1)]+1}(M) \supset \mathcal{S}_l^{[\log_l(n-1)]+1}(E_K).$$

Also

$$\mathcal{S}_l^{[\log_l(n-1)+1]}(E_K) = \mathcal{E}(E_K)$$

by theorem 2, $\mathcal{E}(E_K) = K$ by lemma 2 and

$$\mathcal{S}_l^{[\log_l(n-1)]+1}(M) \subset K.$$

Hence

$$\mathcal{S}_l^{[\log_l(n-1)]+1}(M) = K,$$

$$\min \{k;\ \mathcal{S}_l^{k}(M) = K\} \leqslant [\log_l(n-1)] + 1$$

and the theorem is prov ed.

We remark that in theorems 2 and 4, compactness and convexly connectedness on the one hand and usual connectedness on the other hand are different assumptions. However, in theorem 4 they are equivalent for $n = 2$.

An example of an l-simplicial convex set of l-order ∞ is given by the convex cover of an infinite linear independent family of vectors $\{x_\lambda\}_{\lambda \in \Lambda}$ in an infinite-dimensional vector space $\mathcal{V}$.

For, consider a number N arbitrarily great and

$$M_N = (\mathcal{E}(\{x_\lambda\}_{\lambda \in \Lambda}) - \mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})) \cup \{x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}\},$$

where $\lambda_i \in \Lambda$ for $i \leqslant l^{N-1} + 1$. Prove that

$$\min \{k;\ \mathcal{S}_l^{k}(M_N) = \mathcal{E}(\{x_\lambda\}_{\lambda \in \Lambda})\} = N \qquad (*)$$

By an analogous way as that used to prove remark 2, we obtain that if $\mathcal{S}_l^{\nu}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})$ does not meet the interior of $\mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})$, then $\mathcal{S}_l^{\nu}(M_N)$ does not too. The proof of remark 1 shows that $\mathcal{S}_l^{N-1}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})$ is included in the frontier of $\mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})$, whence $\mathcal{S}_l^{N-1}(M_N)$ does not include the interior of $\mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}})$. But, following theorem 1,

$$\mathcal{S}_l^{N}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}) = \mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}),$$

the dimension of the linear manifold defined by $x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}$ being l^{N-1}. The inclusion $\{x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}\} \subset M_N$ implies

$$\mathcal{S}_l^{N}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}) \subset \mathcal{S}_l^{N}(M_N).$$

Since $M_N \subset \mathcal{S}_l^N(M_N)$ trivially,

$$\mathcal{E}(\{x_\lambda\}_{\lambda\in\Lambda}) = M_N \cup \mathcal{E}(x_{\lambda_1}, \ldots, x_{\lambda_{l^{N-1}+1}}) \subset \mathcal{S}_l^N(M_N).$$

Evidently,

$$\mathcal{S}_l^N(M_N) \subset \mathcal{E}(M_N) = \mathcal{E}(\{x_\lambda\}_{\lambda\in\Lambda});$$

then $\mathcal{E}(\{x_\lambda\}_{\lambda\in\Lambda}) = \mathcal{S}_l^N(M_N)$ and this is not valid for $\mathcal{S}_l^{N-1}$ instead of $\mathcal{S}_l^N$. Thus (*) is proved.

Since we can choose for each natural number N a convenient set M_N such that (*) holds,

$$\omega_l(\{x_\lambda\}_{\lambda\in\Lambda}) = \sup_M \min\{k;\ \mathcal{S}_l^k(M) = \mathcal{E}(\{x_\lambda\}_{\lambda\in\Lambda})\} = \infty.$$

It results also that $\Omega(\{x_\lambda\}_{\lambda\in\Lambda}) = \infty$, which proves the existence of simplicial convex sets of infinite order.

Theorem 5. *The l-order of a non-convex l-simplicial convex set K equals*

$$\sup_M \{k;\ \mathcal{S}_l^k(M) = K\}$$

in a finite-dimensional vector space.

Proof. Prove that

$$\{k;\ \mathcal{S}_l^k(M) = K\}$$

contains at most one number, for arbitrary M. Suppose that

$$\mathcal{S}_l^{k_M}(M) = \mathcal{S}_l^j(M) = K,$$

where

$$j > k_M = \min\{k;\ \mathcal{S}_l^k(M) = K\}.$$

Then

$$\mathcal{S}_l^{k_M}(M) \subset \mathcal{S}_l^{k_M+1}(M) \subset \mathcal{S}_l^j(M)$$

implies $\mathcal{S}_l^{k_M+1}(M) = K$. Now, if $\mathcal{S}_l^p(M) = K$ for a number $p > k_M$, then

$$\mathcal{S}_l^{p+1}(M) = \mathcal{S}_l(\mathcal{S}_l^p(M)) = \mathcal{S}_l(K) = \mathcal{S}_l(\mathcal{S}_l^{k_M}(M)) = \mathcal{S}_l^{k_M+1}(M) = K,$$

whence $\mathcal{S}_l^p(M) = K$, for all $p \geqslant k_M$. By theorem 3,

$$[\log_l n] + 1 \geqslant k_M,$$

where n is the dimension of the space, whence

$$\mathcal{S}_l^{[\log_l n]+1}(M) = K.$$

By theorem 1,

$$\mathcal{E}(M) = \mathcal{S}_l^{[\log_l n]+1}(M),$$

hence $\mathcal{E}(M) = K$ and K is convex. This contradiction shows that either

$$\{k\,;\ \mathcal{S}_l^k(M) = K\} = \emptyset$$

or

$$\{k\,;\ \mathcal{S}_l^k(M) = K\} = \{k_M\},$$

hence

$$\sup_M \{k\,;\ \mathcal{S}_l^k(M) = K\} = \sup_M k_M = \omega_l(K),$$

which proves the theorem.

Theorem 6. *The l-order of a non-convex l-simplicial convex set in an n-dimensional vector space is at most* $[\log_l n]$.

Proof. Let K be the given set. By theorem 3, $\omega_l(K) \leqslant [\log_l n] + 1$. Suppose that $\omega_l(K) = [\log_l n] + 1$. Then there exists a set M such that

$$\mathcal{S}_l^{[\log_l n]+1}(M) = K.$$

By theorem 1,

$$\mathcal{S}_l^{[\log_l n]+1}(M) = \mathcal{E}(M),$$

hence $K = \mathcal{E}(M)$ is convex, absurd. It follows that

$$\omega_l(K) \leqslant [\log_l n].$$

§ 3. The degree and projections of simplicial convex sets

The notion of simplicial convexity is more general than that of convexity, because every convex set is l-simplicial convex for arbitrary l, being its l-simplicial convex cover, but there are simplicial convex sets which are not convex, like, for instance, the frontier of a non-degenerate tetrahedron. This example and several other use frontiers of convex sets. We will point out that in general a simplicial convex set is neither included in the frontier of its convex cover, nor includes it. For example, the union of the segments joining the vertices and the centroid of a nondegenerate tetrahedron satisfies these demands.

Obviously, a convex set is simplicial convex of degree 2, the least possible degree. One can think that $\delta(K) = 2$ for every simplicial convex set K. This is disproved by the following example:

Let us consider a simplex with 5 vertices $x_1, \ldots, x_5$ in a 4-dimensional vector space. Obviously, $K = \mathcal{S}_3(x_1, \ldots, x_5)$ is 3-simplicial convex. Prove that K is not 2-simplicial convex. Suppose, on the contrary, that there is M such that $\mathcal{S}_2(M) = K$. Because

$$\mathcal{S}_2^2(M) = \mathcal{S}_2(\mathcal{S}_2(M)) = \mathcal{S}_2(K) = \mathcal{E}(x_1, \ldots, x_5)$$

and according to a result of the first part of the proof of theorem 4,

$$x_1, \ldots, x_5 \in E_{\mathcal{E}(x_1, \ldots, x_5)} \subset M.$$

But $\{x_1,\ldots, x_5\} \neq M$, because $\mathcal{S}_2(x_1,\ldots, x_5)$ is the union of edges of the simplex. Let $y \in M - \{x_1,\ldots, x_5\}$. If y is an interior point of a triangle of K, say $\mathcal{E}(x_1, x_2, x_3)$, then $\mathcal{E}(y, x_4) \not\subset K$, absurd. If y belongs to a segment of K, say $\mathcal{E}(x_1, x_2)$, then

$$M \cap \mathcal{E}(x_3, x_4, x_5) = \{x_3, x_4, x_5\}.$$

For, suppose that there is a point

$$z \in M \cap \mathcal{E}(x_3, x_4, x_5) - \{x_3, x_4, x_5\}.$$

We have established that z does not belong to the interior of $\mathcal{E}(x_3, x_4, x_5)$. If z lies on its frontier, say $z \in \mathcal{E}(x_3, x_4)$, then $\mathcal{E}(y, z)$ meets the interior of $\mathcal{E}(x_1,\ldots, x_4)$, absurd again.

The equality that we have proved shows that $\mathcal{S}_2(M)$ does not contain the interior of $\mathcal{E}(x_3, x_4, x_5)$; this contradiction disproves the existence of M such that $\mathcal{S}_2(M) = K$. Hence K is not 2-simplicial convex and $\delta(K) = 3$.

Euclidean spaces will be considered in theorems 7 and 8.

Theorem 7. *The projection of an l-simplicial convex set on a linear manifold is also l-simplicial convex.*

Proof. Let K be the given set, V a linear manifold and M a set such that $\mathcal{S}_l(M) = K$. Prove that the projections K' and M' of K and M on V satisfy $\mathcal{S}_l(M') = K'$.

Let $x' \in \mathcal{S}_l(M')$; x' belongs to a simplex $\mathcal{E}(x'_1,\ldots, x'_h)$, where $x'_i \in M'$ $(1 \leqslant i \leqslant h)$ and $h \leqslant l$. Then x' belongs to the projection of a simplex $\mathcal{E}(x_1,\ldots, x_h)$ with $x_i \in M$ $(1 \leqslant i \leqslant h)$, therefore it belongs to the projection of $\mathcal{S}_l(M)$ on V. Hence $x' \in K'$.

Let $x' \in K'$. This point x' is the projection of a point $x \in K = \mathcal{S}_l(M)$. Then x belongs to a simplex $\mathcal{E}(x_1,\ldots, x_h)$, where $x_i \in M$ $(1 \leqslant i \leqslant h)$ and $h \leqslant l$. Thus x' belongs to the polytope $\mathcal{E}(x'_1,\ldots, x'_h)$, whose vertices are the projections of those of $\mathcal{E}(x_1,\ldots, x_h)$. Since $\mathcal{E}(x'_1,\ldots, x'_h)$ is a union of simplexes with at most h vertices, then x' belongs to a simplex $\mathcal{E}(x'_{n_1},\ldots, x'_{n_k})$, where $n_i < n_j$ for $i < j$ and $n_k \leqslant h$, whence $k \leqslant h$. Because $x'_{n_i} \in M'$ $(1 \leqslant i \leqslant k)$, $x' \in \mathcal{S}_k(M') \subset \mathcal{S}_l(M')$.

Hence $K' = \mathcal{S}_l(M')$ and the proof of the theorem is complete.

However, nothing can be said about the l-simplicial convexity order of projections related to that of the given set. Indeed,

1) If we consider an l-simplicial convex set K with $\omega_l(K) \geqslant 2$ that we project on an $(l^{\omega_l(K)-1} - 1)$-dimensional linear manifold, then the projection K' has, by theorem 3, the l-simplicial convexity order $\omega_l(K') < \omega_l(K)$.

2) All projections of a non-degenerate tetrahedron, whose 2-simplicial convexity order is 2, on planes are triangles or quadrilaters, both of 2-order 2.

3) Consider in a ball (solid sphere) of the 3-dimensional Euclidean space three points a,b,c that do not belong to a line. The set of all the points of the ball whose projections on the plane through a,b,c belong to $\mathcal{E}(a,b,c)$ is a convex set of 2-simplicial convexity order 1. Simultaneously, its projection on the plane through a,b,c has 2-simplicial convexity order 2.

T h e o r e m 8. *The projection K' of a simplicial convex set K on a linear manifold is also simplicial convex and $\delta(K') \leqslant \delta(K)$.*

P r o o f. The simplicial convexity of K' and the inequality are obvious because the k-simplicial convexity of K implies, by theorem 7, the k-simplicial convexity of K'.

An example proving the possibility of strict inequality between the degree of a set and that of its projection can be obtained by taking the set $K = \mathcal{S}_3(x_1, \ldots, x_5)$ used above, $x_1, \ldots, x_5$ being the vertices of a simplex in a 4-dimensional vector space. Since K includes the frontiers of all the subtetrahedrons, the projection K' of K on a plane is a convex polygon with 3, 4 or 5 vertices. Thus $\delta(K') = 2$ and $\delta(K) = 3$.

Theorems 7 and 8 are also valid for central instead of orthogonal projections and, more general, for linear operators in vector spaces.

One might expect that a cone C based on an l-simplicial convex set C' equals or includes an l-simplicial convex set. It is not generally true. One can only say that, if $\delta(C') = 2$, then C includes a simplicial convex set of degree 2.

T h e o r e m 9. *The degree of a simplicial convex set is a prime number.*

P r o o f. Let us consider $k,l,m \geqslant 2$ satisfying $k = ml$; k-simplicial convexity implies l-simplicial convexity, following theorem 10, which will be established in § 4. Suppose that $\delta(K) = pq$, where p,q are integers and $2 \leqslant p \leqslant \left[\frac{\delta(K)}{2}\right]$. Then K is p-simplicial convex, which is absurd, because $p < \delta(K)$.

§ 4. The simplicial convexity order and power

T h e o r e m 10. *If k is a multiple of l, then k-simplicial convexity implies l-simplicial convexity and $\omega_k \leqslant \omega_l$.*

P r o o f. Let K be a k-simplicial convex set and M be such that $\mathcal{S}_k^{\omega_k(K)}(M) = K$. If $k = ml$, then

$$\mathcal{S}_l^{\omega_k(K)}(\mathcal{S}_m^{\omega_k(K)}(M)) = K,$$

whence K is l-simplicial convex and $\omega_l(K) \geqslant \omega_k(K)$.

T h e o r e m 11. *The order of a non-convex simplicial convex set K in an n-dimensional vector space is at most $[\log_{\delta(K)} n]$.*

P r o o f. Let K be l-simplicial convex. Then, by theorem 6, $\omega_l(K) \leqslant [\log_l n]$. Since $l \geqslant \delta(K)$, $\log_l n \leqslant \log_{\delta(K)} n$, whence $\omega_l(K) \leqslant [\log_{\delta(K)} n]$, for arbitrary l; hence

$$\Omega(K) \leqslant [\log_{\delta(K)} n].$$

Let $\mathcal{V}$ be n-dimensional and K convex; it results from theorem 3 that

$$\Omega(K) \leqslant [\log_2 n] + 1.$$

If K is, in addition, bounded and closed and E_K has at most n components or is closed and has at most n convexly connected components, then

$$\Omega (K) \leqslant [\log_2 (n-1)] + 1,$$

owing to theorem 4.

Theorem 12. *The power of a simplicial convex set K of finite order is a prime number and $\Delta (K) \geqslant \delta (K)$.*

Proof. Owing to theorem 10, if $\Delta(K) = pq$, where $p, q \geqslant 2$ are integers, then $\Delta (K)$-simplicial convexity implies p-simplicial convexity and $\omega_{\Delta(K)} (K) \leqslant \omega_p (K)$. Therefore $\Omega (K) \leqslant \omega_p (K)$, which is in fact an equality, and $p \geqslant \Delta (K)$ by the definition of $\Delta (K)$; absurd because

$$p \leqslant \left[\frac{\Delta (K)}{2}\right] < \Delta (K).$$

Hence $\Delta (K)$ is a prime. Since K is $\Delta (K)$-simplicial convex, evidently $\Delta (K) \geqslant \delta (K)$.

§ 5. Final remarks

In § 0 we have introduced some integer numbers characterizing the simplicial convex sets. It can be difficult to calculate them, even for simple sets. This fact points out the importance of each general result about these integers and the relations between them.

The vector spaces used in this paper can be changed by other spaces. For instance, all the definitions in § 0 can be conveniently modified if the space in which one works is the hypersurface of a hypersphere. But no more references to Carathéodory's theorem can be made and almost all the results of this paper, especially those concerning dimension do not remain true.

One can conjecture that ω_l is generally a decreasing function of l. Then $\Omega (K)$ would equal $\omega_{\delta(K)} (K)$ for non-convex K and the inequality of theorem 11 would be trivially implied by theorem 6. Then even the introduction of the simplicial convexity order would be banal, because, for all K, it would equal the $\delta (K)$-simplicial convexity order. Also, power and degree would always coincide.

This conjecture will be disproved in [4].

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REFERENCES

[1] W. Bonnice and V. L. Klee, „*The Generation of Convex Hulls*", Math. Annalen 152 (1963).
[2] H. G. Eggleston, „*Convexity*", Cambridge University Press, 1958.
[3] O. Hanner and H. Rådström, „*A Generalization of a Theorem of Fenchel*", Proc. Amer. Math. Soc. 2 (1951).
[4] T. Zamfirescu, „*On l-simplicial Convexity in Vector Spaces*", to appear in Pacif. J. Math.