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ÉDITIONS DE L'ACADÉMIE DE LA RÉPUBLIQUE POPULAIRE ROUMAINE

ON THE FUNDAMENTAL LEMMAS OF THE CALCULUS OF VARIATIONS

BY

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If the hypothesis of a theorem contains a class of arbitrary elements, then the restriction of this class is, in fact, a generalization of the theorem.

The following discussion will lead to such extensions of the well-known lemmas especially used in the Calculus of Variations and in the Theory of Distributions.

First, we shall notice that the word "domain" will be used for an open and connected set together with part of its boundary.

Let $\mathcal{D}$ be a domain of the Euclidian n-dimensional space R^n. We shall recall that a real function f of n variables is said to be of class C^m $(0 \leqslant m \leqslant \infty)$ on $\mathcal{D}$, if f possesses all its partial derivatives up to and including those of the m-th order, which are continuous on $\mathcal{D}$. We also add that a function $f: \mathcal{D} \to R$ will be said to be of class D^m $(0 \leqslant m \leqslant \infty)$ if $\mathcal{D}$ can be divided into a finite set of subdomains $\mathcal{D}_i \left(\bigcap_{i=1}^{p} \mathcal{D}_i = 0, \bigcup_{i=1}^{p} \mathcal{D}_i = \mathcal{D}\right)$ on each of which f is of class C^m and if, for and only for $m \neq 0$, f is continuous on $\mathcal{D}$.

1. Let $M(x)$ be a real continuous function of one real variable on the closed interval $[a, b]$. We can easily establish the following result:

If

$$\int_a^b M(x)\, \eta(x)\, \mathrm{d}x = 0$$

for every continuous function $\eta: [a, b] \to R$, then $M(x) \equiv 0$ on $[a, b]$.

A stronger proposition will be obtained if we restrict the set of the functions η to include only the functions satisfying several conditions especially required by the necessities of its applications.

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We give the more general

THEOREM 1. *Let the functions* $M_i(x): [a, b] \to R$ $(i = 1, \ldots, p)$ *be of class* D^0. If*)

$$\int_a^b M_i(x)\, \eta_i(x)\, \mathrm{d}\, x = 0 \qquad (i = 1, \ldots, p)$$

for all functions $\eta_i : [a, b] \to R$ *of class* C^∞ *vanishing respectively on the given non-dense sets* $S_i \subset [a, b]$ *together with all their derivatives, then* $M_i(x) \equiv 0$ *on* $[a, b]$.

Proof. Suppose on the contrary that $M_j(\xi) \neq 0$ for a $j \leqslant p$ and $\xi \in [a, b]$, say for definiteness $M_j(\xi) > 0$. Then we can, on account of the continuity of M_j on a subinterval**) of $[a, b]$ containing ξ, assign another subinterval I such that $M_j(x) > 0$ throughout I. We shall find, owing to the definition of S_i, the interval $(\alpha, \beta) \subset I$, such that $(\alpha, \beta) \cap S_j = 0$. Now, choose $\eta_i(x) \equiv 0$ for every $i \neq j$ and

$$\eta_j = \begin{cases} e^{\frac{1}{\alpha - x} - \frac{1}{\beta - x}} & \text{for} \quad x \in (\alpha, \beta) \\ 0 & \text{for} \quad x \in [a, \alpha] \cup [\beta, b], \end{cases}$$

which satisfy all the requirements of the theorem, as we can easily prove. But

$$\int_a^b M_i(x)\, \eta_i(x)\, \mathrm{d}\, x = \int_\alpha^\beta M_j(x) \cdot e^{\frac{1}{\alpha - x} - \frac{1}{\beta - x}} \mathrm{d} x,$$

where the improper integral exists and is strictly positive. This fact gives a contradiction, thus the hypothesis $M_j(\xi) \neq 0$ is untenable and the theorem is proved.

On the basis of the theorem 1 we can immediately prove the preceding result. The fundamental lemma used by Hadamard ***) is another simple consequence of this theorem. We now recall the following lemma due to Lagrange.

LAGRANGE'S LEMMA. *If* $M : [a, b] \to R$ *is a continuous function and if*

$$\int_a^b M(x)\, \eta(x)\, \mathrm{d}\, x = 0$$

for every function $\eta \in C^1$ *which vanishes at* a *and* b, *then* $M(x) \equiv 0$ *on* $[a, b]$.

But one easily sees that this lemma results from the theorem 1 taking $p = 1$ and $S_i = \{a, b\}$ and observing that $C^0 \subset D^0$ and $C^1 \supset C^\infty$ ****)

Remarking $C^m \supset C^\infty$ for $0 \leqslant m < \infty$, we can change Lagrange's lemma by a more useful proposition for the requirements of the Calculus of Variations, by taking $m = 2$.

*) A repeated subscript i indicates a summation with respect to i.

**) We understand by an interval (subinterval) a segment together with some of its extremities.

***) See Hadamard's work [3].

****) For other proofs see [2], [5].

2. We now intend to give an extension of the same problem, concerning the multiple integrals, which can consequently be also regarded as an extension of the theorem 1.

THEOREM 2. *Let the functions $M_i(x_1,\ldots,x_n): \mathcal{D} \to R$ $(i = 1,\ldots,p)$ be of class D^0 on the domain $\mathcal{D} \subset R^n$ containing the given non-dense sets $\mathcal{S}_i$. If*

$$\overbrace{\iint\cdots\int}^{n}_{\mathcal{D}} M_i(x_1,\ldots,x_n)\cdot\eta_i(x_1,\ldots,x_n)\,\mathrm{d}\omega = 0$$

for all functions $\eta_i: \mathcal{D} \to R$ of class C^∞, vanishing respectively on $\mathcal{S}_i$, together with all their partial derivatives, then $M_i(x_1,\ldots,x_n) \equiv 0$ on $\mathcal{D}$ for every $i \leqslant p$.

Proof. Suppose $M_j(\xi_1,\ldots,\xi_n) > 0$ for a $j \leqslant p$ in $(\xi_1,\ldots,\xi_n) \in \mathcal{D}$. From the continuity of M_j on subdomains of $\mathcal{D}$ it follows that $M_j(x_1,\ldots,x_n) > 0$ in a domain $\mathcal{E}$ containing $(\xi_1,\ldots,\xi_n)$. Let S_ε be an open sphere of centre $(x_1^0,\ldots,x_n^0)$ and radius ε, contained in $\mathcal{E}$ and satisfying: $S_\varepsilon \cap \mathcal{S}_j = 0$.

We choose $\eta_i(x_1,\ldots,x_n) \equiv 0$ for $i \neq j$ and

$$\eta_j = \begin{cases} e^{\frac{1}{\delta(x_1,\ldots,x_n)-\varepsilon}} & \text{for } (x_1,\ldots,x_n) \in S_\varepsilon \\ 0 & \text{for } (x_1,\ldots,x_n) \in \mathcal{D}-S_\varepsilon \end{cases}$$

where $\delta(x_1,\ldots,x_n) = \sqrt{(x_i - x_i^0)(x_i - x_i^0)}$. These functions satisfy all the requirements of the theorem. But

$$\overbrace{\iint\cdots\int}^{n}_{\mathcal{D}} M_i(x_1,\ldots,x_n)\,\eta_i(x_1,\ldots,x_n)\,\mathrm{d}\omega =$$

$$= \overbrace{\iint\cdots\int}^{n}_{S_\varepsilon} M_j(x_1,\ldots,x_n)\,e^{-\frac{1}{\varepsilon-\delta(x_1,\ldots,x_n)}}\,\mathrm{d}\omega > 0.$$

The contradiction obtained shows that the hypothesis $M_j(\xi_1,\ldots,\xi_n) > 0$ is untenable, as well as $M_j(\xi_1,\ldots,\xi_n) < 0$, consequently $M_j(x_1,\ldots,x_n) \equiv 0$

The usual fundamental lemma *) of the Calculus of Variations for multiple integrals is implied by the theorem 2, making the same remarks as above, especially that $C^2 \supset C^\infty$.

*) See, for instance, [7].

3. If we suppose $M(x) : [a, b] \to R$; $\eta(x) : [a, b] \to R$; $M, \eta \in C^1$ and $\eta(a) = \eta(b) = 0$, we shall obtain, integrating by parts,

$$\int_a^b M'(x)\,\eta(x)\,dx = -\int_a^b M(x)\,\eta'(x)\,dx.$$

Thus, if

$$\int_a^b M(x)\,\eta'(x)\,dx = 0$$

for all possible functions η, then $M(x)$ is constant on $[a, b]$*).

After giving an idea about this point, we prove

THEOREM 3. *Let the functions* $M_i(x) : [a, b] \to R$ $(i = 1, \ldots, p)$ *be of class* D^0. *If*

$$\int_a^b M_i(x)\,\eta_i(x)\,dx = 0$$

for all functions $\eta_i : [a, b] \to R$ *of class* C^∞, *satisfying*

$$\int_a^b \eta_i(x)\,dx = 0$$

and vanishing respectively on the given non-dense sets $S_i \subset [a, b]$, *then* $M_i(x)$ *are constant on* $[a, b]$.

Proof. Suppose M_j is not constant for a certain index j. Then we can find in $[a, b]$ the intervals I and J such that $M_j(\xi) > M_j(\zeta)$ for every $\xi \in I$ and $\zeta \in J$. We consider the intervals $(\alpha, \beta) \in I - S_j$ and $(\gamma, \delta) \in J - S_j$, of the same length. We choose the functions $\eta_i(x) \equiv 0$ for $i \neq j$ and

$$\eta_j(x) = \begin{cases} e^{\frac{1}{\alpha - x} - \frac{1}{\beta - x}} & \text{for } x \in (\alpha, \beta) \\ -e^{\frac{1}{\gamma - x} - \frac{1}{\delta - x}} & \text{for } x \in (\gamma, \delta) \\ 0 & \text{for } x \in [a, \alpha] \cup [\beta, \gamma] \cup [\delta, b]. \end{cases}$$

These functions obviously satisfy

$$\int_a^b \eta_i(x)\,dx = 0 \quad (i = 1, \ldots, p),$$

are of class C^∞ and vanish on S_i. We obtain

$$\int_a^b M_i(x)\,\eta_i(x)\,dx = \int_\alpha^\beta M_j(x)\,e^{\frac{1}{\alpha - x} - \frac{1}{\beta - x}}\,dx - \int_\gamma^\delta M_j(x)\,e^{\frac{1}{\gamma - x} - \frac{1}{\delta - x}}\,dx =$$

$$= \int_\alpha^\beta [M_j(x) - M_j(x - \alpha + \gamma)]\,e^{\frac{1}{\alpha - x} - \frac{1}{\beta - x}}\,dx > 0.$$

*) This fact results immediately from the preceding relation and from the Theorem 1.

Hence, M_i is necessarily constant on $[a, b]$ for every $i \leqslant p$.

This theorem directly generalizes Du Bois-Reymond's lemma, which can be obtained for $p = 1$, $S_i = 0$ and C^0 instead of both D^0 and C^∞. A more useful, but equivalent to Du Bois-Reymond's lemma is the following proposition :

If $M(x)$ is continuous on $[a, b]$, and if

$$\int_a^b M(x)\, \eta'(x)\, \mathrm{d}x = 0$$

for all functions $\eta \in C^1$ which vanish at a and b, then $M(x)$ is constant on $[a, b]$.

In proving the equivalency it is sufficient to consider the function $\nu(x) = \eta'(x)$, which satisfies

$$\eta(x) = \int_a^x \nu(t)\, \mathrm{d}t$$

and to use the theorem 3.

If we want to follow Du Bois-Reymond's manner of proof *), the assumption $M \in D^0$ will lead us only to the lemma used by Marston Morse [6] under Du Bois-Reymond's name, which can be obtained from the preceding proposition by taking D^0 instead of C^0 and D^1 instead of C^1, consequently enlarging the class of the functions η instead of restricting it.

THEOREM 4. *Let the functions* $M_i\,(x_1, \ldots, x_n) : \mathcal{D} \to R\ (i = 1, \ldots, p)$ *be of class* D^0 *on the domain* $\mathcal{D} \subset R^n$ *containing the given non-dense sets* S_i. *If*

$$\overbrace{\iint\cdots\int}^{n}_{\mathcal{D}} M_i\,(x_1, \ldots, x_n)\ \eta_i\,(x_1, \ldots, x_n)\, \mathrm{d}\omega = 0$$

for all functions $\eta_i : \mathcal{D} \to R$ *of class* C^∞, *which satisfy*

$$\overbrace{\iint\cdots\int}^{n}_{\mathcal{D}} \eta_i\,(x_1, \ldots, x_n)\, \mathrm{d}\omega = 0$$

and vanish respectively on S_i, *then* $M_i\,(x_1, \ldots, x_n)$ *are constant on* $\mathcal{D}$.

We let the proof be a reader's task.

Finally, we shall notice that the Euclidian spaces used above can be changed by other more general spaces, following the ideas of other articles written on this subject. On the other hand many other extensions can be reconsidered under our point of view.

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*) See Bolza's discussions, [1].

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