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Introduction

Let C be a convex bounded body in the n-dimensional Euclidean space E^n, and let B be its boundary. We use Busemann's notations: $E(a,b)$ for the segment $\{(1-\rho)a+\rho b\colon 0 \leqslant \rho \leqslant 1\}$ from a to b, and $S(z,\rho)$ for the sphere $\{x\colon \|z-x\| = \rho\}$. We now recall the definition of Hammer's associated bodies $C(r)$. Let $C_b(r)$ be the set obtained from C by a similitude transformation with a ratio r about centre b in B. The sets $C(r)$ are defined as follows:

$$C(r) = \begin{cases} \bigcap_{b \in B} C_b(r) & \text{if} \quad r \leqslant 1, \\ \bigcup_{b \in B} C_b(r) & \text{if} \quad r > 1. \end{cases}$$

Denote by $B(r)$ the boundary of $C(r)$.

For every chord $c = E(a,b)$ and point x belonging to the line through a and b, let

$$r(x,c) = \frac{\max\{\|a-x\|, \|b-x\|\}}{\|a-b\|},$$

$$r(x) = \max_c r(x,c), \quad \text{and} \quad r^* = \min_x r(x).$$

The number r^* is called (**4**) the critical ratio of C.

P. Hammer showed (**4**) that if $r > 1$ then $C = (C(r))(r/(2r-1))$, and that if $r < r^*$ then $C(r) = \emptyset$. He also proved that there exists a number r_i ($r^* \leqslant r_i \leqslant 1$) such that

$$C = (C(r))(r/(2r-1)) \quad \text{for} \quad r \geqslant r_i,$$

but

$$C \supset (C(r))(r/(2r-1))\dagger \quad \text{for} \quad r < r_i.$$

We call r_i the *reducibility number*, and $C(r_i)$ the *reducibility body* of C. If $r^* \leqslant r_i < 1$ then C is said to be *reducible* to $C(r_i)$; if $r_i = r^*$ then C is said to be *completely reducible*; if $r_i = 1$, then C is said to be *irreducible*.

Evidently, all the convex bodies $C(r)$ with $r \geqslant r_i$ have the same reducibility body, $C(r_i)$.

† We use $\subset$ and $\supset$ for strict inclusions, $\subseteq$ and $\supseteq$ for inclusions including the equality case, and $\setminus$ for relative complement.

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The reducibility number and the reducibility body are invariant under affine transformations (see ((**4**) Theorem 5, property 12)).

For E^2, P. Hammer proved that if $r_i = r^*$ then $r^* = \frac{1}{2}$; hence C is completely reducible if and only if B is a central curve.

By a *diameter* we mean a chord such that two parallel supporting planes pass through its end-points.

Associated hypersurfaces

We consider the set $\mathscr{D}$ of all the diameters of C together with the real number $r \geqslant \frac{1}{2}$. We put

$$\gamma(r) = \{x\colon\ r(x,d) = r,\ d \in \mathscr{D}\}.$$

We give now simple properties of the $\gamma(r)$, which we call the associated hypersurfaces of C.

THEOREM 1. *The associated hypersurface $\gamma(r)$ includes the boundary of $C(r)$.*

Proof. This assertion is trivial for $r \geqslant 1$ ($\gamma(r)$ bounds $C(r)$, see (**4**)), and is obviously true for $r < 1$ because in this case $B(r)$ is the set $\{x\colon r(x) = r\}$ (**4**) and if $r(x) = r$ then x lies on a diameter of C ((**4**) Theorem 4) and divides it in the ratio r, hence $x \in \gamma(r)$.

COROLLARY. *If $\gamma(r)$ contains no convex hypersurfaces (with more than one point) then $r \leqslant r^*$.*

THEOREM 2. *The associated hypersurface $\gamma(r)$ contains no interior points of $C(r)$.*

Proof. The theorem is again trivial for $r \geqslant 1$, since $\gamma(r) = B(r)$; let $r < 1$, let x, in $\gamma(r)$, be an interior point of $C(r)$, d the diameter (or one of the diameters) through x given by the definition of $\gamma(r)$, and $E(a,b) = d \cap C(r)$. It is clear that either $r(a,d)$ or $r(b,d)$ is greater than r. It follows that either

$$r(a) \geqslant r(a,d) > r$$

or

$$r(b) \geqslant r(b,d) > r.$$

Every inequality leads to a contradiction, because if $a,\ b \in B(r)$ then $r(a) = r(b) = r$.

Reducibility of projections

It is known that for $r > 1$, $C(r)$ is the set of all the points lying on diameters of C extended about their mid-points by the ratio $2r-1$. It follows that for $r < 1$, $(C(r))(r/(2r-1))$ can be obtained in the same manner from $C(r)$, the ratio being $1/(2r-1)$.

Let V be an $(n-s-1)$-flat in E^n, and C_V the convex body obtained by intersecting the supporting half-spaces of the n-dimensional convex body C which are bounded by hyperplanes parallel to V. The intersection of C_V with an $(s+1)$-flat orthogonal to V is $C_V{}^s$ $(s = 1, \ldots, n-2)$.

THEOREM 3. *If C is reducible then C_V† and $C_V{}^s$ are reducible for every $(x-s-1)$-flat V. The converse is false.*

Proof. Since C is reducible, there is a number $r < 1$ such that $C = (C(r))(r/(2r-1))$. It is very simple to prove that $C(r)_V = C_V(r)$. Indeed, if B_V is the boundary of C_V, and $B_V(r)$ that of $C_V(r)$, then

$$C(r)_V = \left(\bigcap_{b\in B} C_b(r)\right)_V = \bigcap_{b\in B} C_b(r)_V \stackrel{?}{=} \bigcap_{b\in B\cap B_V} C_b(r)_V = \bigcap_{b\in B_V} C_b(r)_V = C_V(r).$$

Since the points of B are the end-points of the extensions of the diameters of $C(r)$ about their mid-points in the ratio $1/(2r-1)$, the points of $B\cap B_V$ can be obtained from those diameters of $C(r)$ which have their extremities on $B(r)\cap B_V(r)$. Consequently, the diameters of C_V can also be obtained from those of $C_V(r)$, the ratio being $1/(2r-1)$ too. The reducibility of $C_V{}^s$ follows immediately.

The converse proposition is not true. For example, in the plane a triangle is not reducible, while its projection on every line is a segment, obviously completely reducible.

For further developments see (**9**).

COROLLARY. *If there exists an $(n-s-1)$-flat V such that C_V or $C_V{}^s$ is irreducible, then C is irreducible.*

Necessary conditions for reducibility

Let us consider the point p of B (the set $L\subset B$) and denote by $\nu(p)$ (respectively $\nu(L)$) the spherical image of p (resp. L) in Busemann's sense ((**2**) 25). Let also

$$\mu(q) = \{p\in B\colon\ q\in\nu(p)\},\quad \mu(M) = \bigcup_{q\in M}\mu(q),$$

$$\alpha(N) = \{p\in S(z,1)\colon\ p,\, p' \text{ antipods} \Rightarrow p'\in N\},$$

where $\{q\}$, $M, N \subset S(z,1)$.

Given a set $M\subset S(z,1)$ let $\mathcal{T}M$ be the interior of M relative to the topology of the smallest sphere containing M.

We define the following properties which C may have:

(1) If q_1 and q_2 are two arbitrary antipodal points on $S(z,1)$, then at least one of the sets $\mu(q_1)$ and $\mu(q_2)$ contains but one point.

† We observe that unbounded bodies whose frontiers are so-called cylindrical hypersurfaces ((**2**) 3) may be included in our considerations, using for them the same definition of reducibility.

(2) If $\mathscr{T}\nu(p_1)$ and $\mathscr{T}\nu(p_2)$ contain two antipodal points, then $\alpha(\nu(p_1)) = \nu(p_2)$, for every pair of points p_1, p_2 in B.

Denote by P_j the set of the convex bodies which have the property (j), $j = 1$ or 2.

THEOREM 4. *If C is reducible then $C \in P_2$.*

Proof. Suppose that, on the contrary, there exist points p_1, p_2 in B such that $\mathscr{T}\alpha(\nu(p_1)) \cap \mathscr{T}\nu(p_2) \neq \emptyset$ but $\alpha(\nu(p_1)) \neq \nu(p_2)$, say for definiteness $\alpha(\nu(p_1))\backslash\nu(p_2) \neq \emptyset$. First we investigate the plane case, $n = 2$.

Let $\beta = \mu(\mathscr{T}\alpha(\nu(p_1)))$ and $r_i < r < 1$. The only diameters of C with end-points p in the interior of β relative to the topology of B are $E(p, p_1)$; hence $\{x\colon r(x, E(p,p_1)) = r,\ p \in \beta\} \subset B(r)$, which is impossible if β is not a segment, because in this case the left-hand member is the union of two arcs which cannot belong to any convex curve. If β is a segment and p_3 one of its extremities, distinct from p_2, then B is necessarily non-differentiable in p_3. Since $\mathscr{T}\alpha(\nu(p_1)) \cap \mathscr{T}\nu(p_2) \neq \emptyset$, $p_2 \in \beta$. The only diameters with end-points on β are $E(p, p_1)$, where $p \in \beta$. Let q be an extremity of $\alpha(\nu(p_1))\backslash\nu(p_2)$ that does not belong to $\nu(p_2)$. The supporting line L through p_3 and orthogonal to $E(z, q)$ does not contain the segment β. The lines parallel to L through the points that divide $E(p_1, p_3)$ in the ratio r are supporting lines of $C(r)$, according to ((**4**) Theorem 5, property 7). We find the absurdity that one of these supporting lines separates points of the two segments $\{x\colon$ there exists p in $E(p_2, p_3)$ such that $r(x, E(p, p_1)) = r\} \subset B(r)$. Thus $\beta = \{p_2\}$, and the theorem for $n = 2$ is proved.

Now we pass to the case $n \geqslant 3$. Let p_0 be an interior point of both $\alpha(\nu(p_1))$ and $\nu(p_2)$, let V_1 and V_2 be neighbourhoods of p_0 in $\alpha(\nu(p_1))$ and $\nu(p_2)$, let $p \in \alpha(\nu(p_1))\backslash\nu(p_2)$, and let Γ be the great circle passing through p_0 and p. Using the notations of the preceding paragraph, we consider the convex planar body $C_V{}^1$, obtained for an $(n-2)$-flat V orthogonal to the plane of Γ. From Theorem 3, $C_V{}^1$ is reducible, together with C.

Consider, for every x in $B \cap C_V$, the projection $\varphi(x)$ of x on the plane of $C_V{}^1$. Let $\bar{\nu}(x)$ be the circular image on Γ of the point x belonging to the boundary of $C_V{}^1$. It is clear that $\bar{\nu}(y) = \Gamma \cap \nu(\varphi^{-1}(y))$. Also, $\bar{\nu}(\varphi(p_1)) = \Gamma \cap \nu(p_1)$ and $\bar{\nu}(\varphi(p_2)) = \Gamma \cap \nu(p_2)$; hence if $\bar{p}_1 = \varphi(p_1)$, $\bar{p}_2 = \varphi(p_2)$, $\overline{V}_1 = \Gamma \cap V_1$, and $\overline{V}_2 = \Gamma \cap V_2$, then $\overline{V}_1 \cap \overline{V}_2 = \Gamma \cap V_1 \cap V_2 \ni p_0$ and $\alpha(\bar{\nu}(\bar{p}_1))\backslash\bar{\nu}(\bar{p}_2) = (\Gamma \cap \alpha(\nu(p_1)))\backslash(\Gamma \cap \nu(p_2)) = \Gamma \cap (\alpha(\nu(p_1))\backslash\nu(p_2)) \ni p$, which is impossible because $C_V{}^1$ is reducible and we have found above that all planar reducible convex bodies belong to P_2. Thus the theorem is completely proved.

Let D be a convex set on B. Let $F_D \subset B$ be the maximal convex set such that B possesses two parallel different supporting planes containing respectively D and F_D. For a given D, we define

$$G_D = \{x\colon \text{ there exist } a \text{ in } D,\ b \text{ in } F_D \text{ such that } r(x, E(a,b)) = r\}.$$

If $G_D{}^1$ and $G_D{}^2$ are the connected components of G_D, let

$$\delta(r) = \bigcup_{\substack{i=1,2\\ D\subset B\\ G_D{}^i \cap B(r) \neq \emptyset}} G_D{}^i.$$

We have, obviously, $B(r) \subseteq \delta(r) \subseteq \gamma(r)$.

THEOREM 5. *If the inequality $r \geqslant r_i$ holds then $\gamma(r) = \delta(r)$.*

Proof. If $r = 1$, the result is obvious. Following ((**4**) Theorem 5, property 8), for $r > 1$, $\gamma(r) = B(r)$; hence $\gamma(r) = \delta(r)$. If $r < 1$ then $r_i < 1$, hence C is reducible. Now we consider this case.

Suppose that $\gamma(r)$ possesses a point x belonging neither to $B(r)$ nor to $\delta(r)\backslash B(r)$. Let $E(a,b)$ be a diameter through x given by the definition of $\gamma(r)$. According to Theorem 4, and to ((**7**) Theorem 7), three cases are possible:

1° $\nu(a) = \alpha(\nu(b))$ and $\mu(\nu(a)) = \{a\}$ or $\mu(\nu(b)) = \{b\}$;

2° $\mu(\nu(b)) \supseteq D \supset \{b\}$ and $F_D \supset \{a\}$;

3° $\mathscr{T}\nu(a) \cap \mathscr{T}\alpha(\nu(b)) = \emptyset$.

Case 1°. Suppose for instance that $\mu(\nu(b)) = \{b\}$. The diameter $E(a,b)$ through a is unique. On this diameter lie no diameters of $C(r)$ which, extended, give $E(a,b)$, because one of the two extremities of the segment which, extended about its mid-point in the ratio $1/(2r-1)$, gives $E(a,b)$ is x, and this point cannot be the end-point of any chord of $C(r)$. Thus we find that $C \supset (C(r))(r/(2r-1))$, hence $r < r_i$, which is absurd.

Case 2°. We have $x \in G_D{}^1 \cup G_D{}^2$; let, for instance, $x \in G_D{}^1$. By our assumption, $x \notin \delta(r)$; hence $G_D{}^1 \cap B(r) = \emptyset$. But there is a diameter of $C(r)$ which, extended about its mid-point, gives $E(a,b')$ ($b' \in F_D$), because C is reducible; then one of its extremities, say y, lies on $G_D{}^1$. We have obtained $y \in G_D{}^1 \cap B(r)$, which is absurd too.

Case 3°. Let $V\nu(b')$ intersect $\mathscr{T}\alpha(\nu(a))$. For a and b' only case 1° is possible. Then the point x' of $E(a,b')$ satisfying

$$\|x'-a\|/\|x'-b'\| = \|x-a\|/\|x-b\|$$

belongs to $B(r)$. Since $B(r)$ is closed, and $b' \to b$ implies $x' \to x$, $x \in B(r)$, which is absurd again. The theorem is proved.

Necessary and sufficient conditions for reducibility

A necessary and sufficient condition for the reducibility of C in the particular case that $C \in P_1$ can be derived from the following theorem.

THEOREM 6. *We have $C \in P_1$ and $r \geqslant r_i$ if and only if $\gamma(r) = B(r)$.*

Proof. Since $C \in P_1$, case 2° of the proof of Theorem 5 is not possible. From the examination of the other cases it follows $\gamma(r) \subseteq B(r)$, and since $B(r) \subseteq \gamma(r)$ obviously, we obtain $\gamma(r) = B(r)$.

Conversely, first we prove that if $\gamma(r) = B(r)$ then $C \in P_1$. Suppose that $C \notin P_1$; then there exists a convex set D, at least one-dimensional, such that F_D is at least one-dimensional too. Let Π_1 and Π_2 be the connected components of

$$\{x\colon \text{ there exist } p' \text{ in } D,\ p'' \text{ in } F_D \text{ such that } r(x, E(p', p'')) = r\}$$

(Π_1 between D and Π_2). Let d, in D, be a boundary point of $B\backslash D$, and let b, in F_D, be such that the supporting half-space of the convex closure $\tilde{D}$ of $D \cup \{b\}$, bounded by a hyperplane passing through b and d, does not contain a point c of F_D. It follows that $E(c,d) \cap \tilde{D} = \{d\}$. Since $\tilde{D} \cap \Pi_1$ is the part of the boundary of $C_b(r)$ which lies on Π_1, and

$$E(c,d) \cap \tilde{D} \cap \Pi_1 = \varnothing,$$

the point $\{p\} = E(c,d) \cap \Pi_1$ is exterior to $C_b(r)$ and, consequently, to $C(r)$. Hence $p \in \Pi_1 \subset \gamma(r)$ but $p \notin B(r)$, which is false.

Now we prove that if $\gamma(r) = B(r)$ then $r \geqslant r_i$. Suppose that $r < 1$. Let $E(a,b)$ be a diameter of C. Consider also x_1, x_2 in $E(a,b)$ such that $r(x_1, E(a,b)) = r(x_2, E(a,b)) = r$. Since $\gamma(r) = B(r)$, it follows that x_1, $x_2 \in B(r)$. The segment $E(x_1, x_2)$ is a diameter of $C(r)$ because two parallel supporting planes of $C(r)$ can be traced through x_1, x_2, both parallel to two parallel supporting planes of C through a, b, as we have seen above. The diameter $E(x_1, x_2)$, extended about $(x_1+x_2)/2$ in the ratio $1/(2r-1)$, gives $E(a,b)$. Thus $(C(r))(r/(2r-1)) = C$ and $r \geqslant r_i$.

Necessary and sufficient conditions for reducibility in the planar case

We now restrict our discussions to the plane case, $n = 2$.

We use the family $\mathscr{E}$ of all the essential diameters of C, which are defined in (**4**). Let

$$\varepsilon(r) = \{x\colon\ r(x,d) = r,\ d \in \mathscr{E}\}.$$

Evidently, $B(r) \subseteq \varepsilon(r) \subseteq \gamma(r)$.

THEOREM 7. *For a planar body, $r \geqslant r_i$ if and only if $\varepsilon(r) = B(r)$.*

Proof. If $r \geqslant r_i$ then $(C(r))(r/(2r-1)) = C$ and, according to ((**4**) Theorem 5, property 11), $C(r)$ is obtained from C by extending or shrinking the essential diameters of C about their mid-points in the ratio $2r-1$; hence $\varepsilon(r) = B(r)$.

Conversely, let $a \in B$, and let $E(a,b)$ be the essential diameter of C through a. The two points x_1, x_2 which divide $E(a,b)$ in the ratio $r < 1$ belong to $\varepsilon(r)$. From $x_1, x_2 \in B(r)$ and ((**4**) Theorem 5, property 7), it follows that $E(x_1,x_2)$ is a diameter of $B(r)$, which extended about $(x_1+x_2)/2$ by the ratio $1/(2r-1)$ gives $E(a,b)$; hence $a \in (C(r))(r/(2r-1))$. Therefore, for $r < 1$ and obviously for $r \geqslant 1$ too, the inequality $r \geqslant r_i$ holds.

LEMMA. *Let B be a bounded differentiable convex curve, let $E(a,b)$ be an essential diameter of the convex body C bounded by B, and let c be a point such that $r(c, E(a,b)) = r$. Then the tangents at a to B and at c to $\varepsilon(r)$ are parallel.*

Proof. Let x be a point such that the triangle abx remains similar to itself when a and b vary on B, and let y be the point of contact of $E(a,b)$ with its envelope. The normals of B at a and b intersect the line through y orthogonal to $E(a,b)$ in the points a' and b', respectively. The lines through a' and b' orthogonal to $E(a,x)$ and $E(b,x)$ respectively have z as a common point. It is known ((**1**) 37) that the normal at x to the curve described by this point passes through z. Since abx and $a'b'z$ are two similar triangles, $x \to c$ implies $z \to c'$, where

$$\|c'-a'\|/\|c'-b'\| = \|c-a\|/\|c-b\|.$$

Hence the normal at c on $\varepsilon(r)$ is the line through c and c', which is parallel to $E(a,a')$. Therefore we can conclude that the tangent at c to $\varepsilon(r)$ is parallel to that at a to B.

Remark. By an approximation argument, the result of the preceding lemma may be used for a convex curve differentiable at a and b and not necessarily everywhere differentiable.

The curve $\varepsilon(r)$ will be said to have a *direct sense* if for every two essential diameters $E(a,b)$ and d of C, with $a \notin d$, the point x of $E(a,b)$ satisfying $\|x-b\|/\|a-b\| = r$ either lies on the same side of the line containing d as a, or belongs to d.

THEOREM 8. *For a planar body, $r \geqslant r_i$ if and only if the curve $\varepsilon(r)$ has a direct sense.*

Proof. If $r \geqslant r_i$ then $\varepsilon(r) = B(r)$, by Theorem 7. Let d_1 and d_2 be two essential diameters of C. Since $C = (C(r))(r/(2r-1))$, the segments $\delta_1 = d_1 \cap C(r)$ and $\delta_2 = d_2 \cap C(r)$ are essential diameters of $C(r)$, following

((**4**) Theorem 5, property 11). According to ((**4**) Theorem 5, property 3), $\delta_1 \cap \delta_2 \neq \emptyset$, and then $d_1 \cap d_2 \in C(r)$. Therefore $\varepsilon(r)$ has a direct sense.

Conversely, let $\varepsilon(r)$ have a direct sense. If $r_i = r^*$ then $r \geqslant r_i = \frac{1}{2}$ trivially. If $r_i > r^*$, suppose that $r^* < r < r_i$; then $\varepsilon(r) \supseteq B(r)$. Let a, in $B(r)$, be a boundary point of $\varepsilon(r)\backslash B(r)$. According to ((**4**) Theorem 4), there exists an essential diameter d of C which is divided by a in the ratio $r(a) = r$. The diameter d is not a supporting line of $C(r)$ and intersects it, i.e. d cuts $C(r)$ ((**3**) 17). From the continuity argument, there exists a neighbourhood of a such that for every point of its intersection with $\varepsilon(r)\backslash B(r)$, the essential diameter given by the definition of $\varepsilon(r)$ cuts $C(r)$. Let b be such a point, and let g be the corresponding diameter. Let h, in $\mathscr{E}$, be the diameter passing through $c = g \cap B(r)$ and given by the definition of $\varepsilon(r)$. If h is not unique, let $\mathscr{H} \subseteq \mathscr{E}$ be the set of all such diameters. Since the point c does not coincide with b, $g \neq h$ (or $g \notin \mathscr{H}$). Let $E(c, c')$ be a chord of $C(r)$ which lies, entirely without the extremity c, on the same side of h (or, respectively, of every h in $\mathscr{H}$) as b. Let Y be a neighbourhood of an extremity of h (or of one of the two connected components of the set of the extremities of all h in $\mathscr{H}$) that does not intersect the line through c and c'. Again from the continuity argument, for every neighbourhood V of c there is another neighbourhood W_V of c such that through every point x of $W_V \cap B(r)$ can be traced an essential diameter $E(y,z)$ ($\|x-y\| < \|x-z\|$) of C such that $y \in Y$ and $E(y,z) \cap g \subseteq V$. Let us consider the neighbourhood U of c which does not contain b, and choose the point x_0 in $W_U \cap B(r)$ on the same side of both $E(c,c')$ and h (or every h in $\mathscr{H}$) as b. For this particular case we obtain $E(y_0, z_0)$. Hence it is clear that $E(y_0, z_0) \cap E(b, c) = \{v\} \neq \{c\}$. Therefore the points x_0 and y_0 are separated by g, which is impossible because $\varepsilon(r)$ has a direct sense. Now, supposing $r' \leqslant r^*$, let x_0' be the point of $E(y_0, z_0)$ such that $\|x_0' - y_0\|/\|x_0' - z_0\| = r'$. Since $r' < r$, it follows that x_0' and y_0 are also separated by g, which is impossible by the same argument. We have obtained $\varepsilon(r) = B(r)$, which concludes the proof.

Necessary conditions for reducibility in the planar case

In this section we shall give two properties of the reducible planar bodies relative to their essential diameters.

THEOREM 9. *For a reducible planar convex body, $\mu(p)$ and $\mu(\alpha(p))$ have the same cardinal, for every point p in $S(z, 1)$.*

Proof. It is sufficient to prove that if $\mu(p)$ is a point q_0, then $\mu(\alpha(p))$ is also a point. Suppose that $\mu(\alpha(p)) = E(q_1, q_2)$. Let $r < 1$, $r > r_i$, let x_1, x_2 be the points that divide $E(q_0, q_1)$ in the ratio r, and let U_1, U_2 be neighbourhoods of x_1, x_2 which do not intersect $E(q_0, q_2)$. There exists an essential diameter $E(q_0', q_1')$ ($q_0' \notin \{q_0\} \cup E(q_1, q_2)$) such that

$$M = \{x: \ r(x, E(q_0', q_1')) = r\} \subseteq U_1 \cup U_2.$$

Therefore M and one of the extremities of $E(q_0', q_1')$ are separated by $E(q_0, q_2)$, which is impossible by the inequality $r > r_i$ and Theorem 8.

This theorem is generalized and improved in (**7**) and (**9**).

THEOREM 10. *If m_a and m_b are the measures of the tangent angles (analogues in the plane of the tangent hypercones) in the extremities a and b of an essential diameter of a reducible convex body, then $m_a = m_b$. In particular, if the boundary is differentiable in a, it is in b too.*

Proof. First we shall prove, using the circular images, that the interiors of $\alpha(\nu(a))$ and $\nu(b)$ have common points. Suppose this is not true. Then, for instance, $\alpha(\nu(a))$ is not a point and $\alpha(\nu(a)) \cap \nu(b)$ is an extremity of $\alpha(\nu(a))$. The set $\beta = \mu(\alpha(\nu(a)))$ must be a segment, because if it is not there would be a point x of β such that $\nu(x)$ belongs to the interior of $\alpha(\nu(a))$, which is impossible by the reducibility of the given body, and Theorem 4. If β is a segment $E(b, b')$, from Theorem 9 it follows that $a \in E(a, a') \subseteq B$, the lines containing $E(a, a')$ and $E(b, b')$ are parallel, and $E(a, b) \cap E(a', b') = \emptyset$, which is impossible too because $E(a, b)$ is an essential diameter; hence the proof that $\mathscr{T}\alpha(\nu(a)) \cap \mathscr{T}\nu(b) \neq \emptyset$ is finished. Now, according to Theorem 4, $\alpha(\nu(a)) = \nu(b)$; hence $m_a = m_b$.

Remark. The theorem obtained from Theorem 10 by replacing the essential diameter by any diameter is not true. To see this, consider a rhomb which is not a square as the reducible body. The segment joining two non-opposite vertices is a diameter of the rhomb, whereas the angles at its extremities are not equal.

The reducibility number

In this section we estimate the reducibility number in the plane case, using the curvature of the boundary curve B.

Choose arbitrarily two distinct essential diameters d and δ.

THEOREM 11. *We have*

$$r_i = \sup_{d, \delta \in \mathscr{E}} r(d \cap \delta, d).$$

Proof. Let $\bar{r} = \sup_{d, \delta \in \mathscr{E}} r(d \cap \delta, d)$. We have $\bar{r} \geqslant r(d \cap \delta, d)$ for every pair d, $\delta \in \mathscr{E}$; hence, by Theorem 8, $\varepsilon(\bar{r})$ has a direct sense. Therefore $\bar{r} \geqslant r_i$. For every $\lambda > 0$, there is a pair of diameters d_λ, δ_λ, in $\mathscr{E}$, such that $\bar{r} - \lambda < r(d_\lambda \cap \delta_\lambda, d_\lambda)$, hence $\varepsilon(\bar{r} - \lambda)$ has not a direct sense. Consequently $\bar{r} - \lambda < r_i$, hence $\bar{r} \leqslant r_i$. Thus $\bar{r} = r_i$, and the theorem is proved.

Again, let us consider the pair of distinct diameters d, δ in $\mathscr{E}$.

THEOREM 12. *We have*

$$r_i = \sup_{d \in \mathscr{E}} \limsup_{\delta \to d} r(d \cap \delta, d).$$

Proof. Let $A_{d\delta}$ be the acute angle formed by d and δ. We have

$$r_i = \sup_{d,\delta \in \mathscr{E}} r(d \cap \delta, d) = \sup_{k \in (0,\pi/2]} \sup_{\substack{d,\delta \in \mathscr{E} \\ A_{d\delta}=k}} r(d \cap \delta, d)$$

and

$$\sup_{d \in \mathscr{E}} \limsup_{\delta \to d} r(d \cap \delta, d) = \sup_{d \in \mathscr{E}} \limsup_{A_{d\delta} \to 0} r(d \cap \delta, d).$$

One proves easily enough that

$$\sup_{d \in \mathscr{E}} \limsup_{A_{d\delta} \to 0} r(d \cap \delta, d) = \limsup_{k \to 0} \sup_{\substack{d,\delta \in \mathscr{E} \\ A_{d\delta}=k}} r(d \cap \delta, d).$$

Thus we must prove only that

$$\sup_{k \in (0,\pi/2]} r_k = \limsup_{k \to 0} r_k,$$

where $r_k = \sup_{\substack{d,\delta \in \mathscr{E} \\ A_{d\delta}=k}} r(d \cap \delta, d)$. Suppose this is not true; then there exists k_0 in $(0, \pi/2]$ such that $r_{k_0} = \sup_{k \in (0,\pi/2]} r_k$, which is a consequence of the continuity of r_k as a function of k. Consider d_0, δ_0 in $\mathscr{E}$ such that $\sup_{\substack{d,\delta \in \mathscr{E} \\ A_{d\delta}=k_0}} r(d \cap \delta, d) = r(d_0 \cap \delta_0, d_0)$, that is $r_i = r(d_0 \cap \delta_0, d_0)$. Suppose that $r_i \neq 1$. Let v, w be the points that divide δ_0 in the ratio r_i, and let a, in δ_0, and b be those that divide d_0 in the same ratio. If $a \in \{v, w\}$, for example if a and v coincide, then every chord c passing through v and intersecting $E(b, w)$ is an essential diameter. From the convexity of $\varepsilon(r_i)$ and the equality $r_i = \sup_{d,\delta \in \mathscr{E}} r(d \cap \delta, d)$ it follows that $r(v, c) = r_i$; hence $r_i = \lim_{c \to d_0} r(v, c)$, contrary to our supposition. Pass to the case $a \notin \{v, w\}$. All the essential diameters intersect $E(v, w)$. According to ((**4**) Theorem 5, property 7), for every essential diameter which intersects the interior of $E(v, w)$, unique supporting lines parallel to $E(v, w)$ can be traced through its extremities. It follows that there exists an essential diameter $d' \neq \delta_0$ such that $r(v, d') = r_i$. Now the proof can be continued as above. If $r_i = 1$, i.e. $\delta_0 = E(v, w)$, then there exist two parallel supporting lines at v and w. No pair of parallel boundary segments through v and w exists, because $\delta_0 \in \mathscr{E}$. It is clear that one can find an arc $\omega \subseteq B$ such that $E(v, w) \cap \omega = \{w\}$ and $E(v, v') \in \mathscr{E}$ for every v' in ω. Thus we obtain the same contradiction as above, and the proof is finished.

Let C be a reducible planar body. Consider a system of rectangular coordinates $(x; y)$, so that the supporting lines parallel to the y-axis are regular ((3) 31) and the projection of C on the x-axis is the segment $[0,1]$. Let f be a convex function and g a concave function such that $\{(x; y): x \in [0,1] \text{ and } y = f(x) \text{ or } y = g(x)\} = B$.

Let $x \in [0,1]$, $x+h \in [0,1]$, where $f'(x+h)$ exists, let m be the maximum (minimum) slope of a supporting line through $(x; f(x))$ if h is positive (negative), let $m_h = f'(x+h)$, $k = f(x+h)-f(x)$, and let $E((x; f(x)), (y_x; g(y_x)))$ be an essential diameter of B. The normal in $(x+h; f(x+h))$ to B either does not intersect the line through $(x; f(x))$ with slope $-m^{-1}$, or intersects it in a point whose distance from $(x; f(x))$ equals ((2) 8):

$$R_h{}^f(x) = (1+m^2)^{\frac{1}{2}} |(1-kh^{-1}m_h)h(m-m_h)^{-1}|.$$

We shall always choose h so that $f'(x+h)$ exists and, if possible, is not equal to $f'(x)$. In the same manner we define $R_{l_h}{}^g(y_x)$, where l_h is such that $g'(y_x+l_h)$ equals m_h in $(x+h; f(x+h))$.

Let I_x^f (I_x^g) be the maximal open interval containing x (possibly void) where f (respectively g) is linear, let

$$M = \{x: I_x^f = \emptyset\} \quad \text{and} \quad N = \{x: I_x^f \supset \{x\}\},$$

and let us denote by $\|I\|$ the length of an interval I.

THEOREM 13. *We have*

$$r_i = \max\left\{\sup_{x\in M} \limsup_{h\to 0} \frac{\max\{R_h{}^f(x), R_{l_h}{}^g(y_x)\}}{R_h{}^f(x)+R_{l_h}{}^g(y_x)}, \sup_{x\in N} \frac{\max\{\|I_x{}^f\|, \|I_{y_x}{}^g\|\}}{\|I_x{}^f\|+\|I_{y_x}{}^g\|}\right\}.$$

Proof. Let $a = (x; f(x))$, $b = (x+h; f(x+h))$, and let $E(a, a')$, $E(b, b') \in \mathscr{E}$. Let $c = E(a, a') \cap E(b, b')$, and let $p\,(p')$ be the intersection of the normals in $a\,(a')$ and $b\,(b')$ to the lines with slopes m and m_h. If a and b do not belong to the interiors of two segments of B, we have

$$\begin{aligned}
\limsup_{b\to a} \frac{\|a-c\|}{\|a'-c\|} &= \limsup_{b\to a} \frac{\|a-b\| \sin \angle cba}{\|a'-b'\| \sin \angle cb'a'} \\
&= \limsup_{b\to a} \frac{\|a-p\| \sin \angle p'b'a'.\sin \angle cba}{\|a'-p'\| \sin \angle pba.\sin \angle cb'a'} \\
&= \frac{\sin\left(\pi - \lim\limits_{b\to a} \angle cab\right)}{\sin\left(\pi - \lim\limits_{b'\to a'} \angle ca'b'\right)} . \limsup_{b\to a} \frac{\|a-p\|}{\|a'-p'\|} \\
&= \limsup_{b\to a} \frac{\|a-p\|}{\|a'-p'\|}.
\end{aligned}$$

If a and b are interior points of two maximal segments $E(a_1, a_2)$, $E(b_1, b_2)$

of B, then p and p' do not exist and

$$\limsup_{b\to a}\frac{\|a-c\|}{\|a'-c\|}=\frac{\|a-c\|}{\|a'-c\|}=\frac{\|a_1-a_2\|}{\|b_1-b_2\|},$$

because c is fixed.

In the first case, $x\in M$ and

$$\begin{aligned}\limsup_{\substack{\delta\to E(a,a')\\ \delta\in\mathscr{E}}} r(E(a,a')\cap\delta, E(a,a')) &= \limsup_{b\to a}\frac{\max\{\|a-c\|,\|a'-c\|\}}{\|a-a'\|}\\ &= \limsup_{b\to a}\frac{\max\{\|a-p\|,\|a'-p'\|\}}{\|a-p\|+\|a'-p'\|}\\ &= \limsup_{h\to 0}\frac{\max\{R_h{}^f(x), R_{t_h}{}^g(y_x)\}}{R_h{}^f(x)+R_{t_h}{}^g(y_x)}.\end{aligned}$$

In the second case, $x\in N$ and

$$\begin{aligned}\limsup_{\substack{\delta\to E(a,a')\\ \delta\in\mathscr{E}}} r(E(a,a')\cap\delta, E(a,a')) &= \frac{\max\{\|a_1-a_2\|,\|b_1-b_2\|\}}{\|a_1-a_2\|+\|b_1-b_2\|}\\ &= \frac{\max\{\|I_x{}^f\|,\|I_{y_x}{}^g\|\}}{\|I_x{}^f\|+\|I_{y_x}{}^g\|}.\end{aligned}$$

Using Theorem 12 and putting $r_x=\limsup_{\substack{\delta\to E(a,a')\\ \delta\in\mathscr{E}}} r(E(a,a')\cap\delta,\ E(a,a'))$, we have $r_i=\sup_{x\in[0,1]} r_x=\max\left\{\sup_{x\in M} r_x,\ \sup_{x\in N} r_x\right\}$, and the estimation of r_i given by the statement is found.

THEOREM 14. *If f and g possess finite non-vanishing second right and left de la Vallée Poussin derivatives (D_r'' and D_l'') on M, then*

$$\begin{aligned}r_i=\max\Bigg\{\sup_{x\in M}\max\Bigg\{&\frac{D_r''f(x)}{D_r''f(x)+D_l''g(y_x)}, \frac{D_l''g(y_x)}{D_r''f(x)+D_l''g(y_x)}, \frac{D_l''f(x)}{D_l''f(x)+D_r''g(y_x)},\\ &\frac{D_r''g(y_x)}{D_l''f(x)+D_r''g(y_x)}\Bigg\}, \sup_{x\in N}\max\left\{\frac{\|I_x{}^f\|}{\|I_x{}^f\|+\|I_{y_x}{}^g\|}, \frac{\|I_{y_x}{}^g\|}{\|I_x{}^f\|+\|I_{y_x}{}^g\|}\right\}\Bigg\}.\end{aligned}$$

Proof. We have used the notation

$$D_r''f(x)=\lim_{h\to 0+}\frac{2}{h}\left(\frac{f(x+h)-f(x)}{h}-f_r'(x)\right),$$

where f_r' is the right derivative of f, and another analogous one for $D_l''f(x)$. Considering the second right derivative $f_r''(x)=\lim_{h\to 0+}(f'(x+h)-f_r'(x))h^{-1}$ and the analogous second left derivative $f_l''(x)$, and using a theorem of Jessen (**5**), one can easily prove that the existence of $D_r''f$ and the existence of f_r'' are equivalent, and that these numbers are equal; the same thing is true for $D_l''f$ and f_l''. Using the value of $R_h{}^f(x)$ given above, one can deduce that the right and left limits of $R_h{}^f(x)$ at $h = 0$ exist, and one can calculate them (**2**); for instance,

$$\lim_{h\to 0+} R_h{}^f(x) = \frac{(1+f_r'^2(x))^{\frac{3}{2}}}{f_r''(x)} = \frac{(1+f_r'^2(x))^{\frac{3}{2}}}{D_r''f(x)}.$$

Continuing, we have

$$\lim_{h\to 0+} \frac{R_h{}^f(x)}{R_{l_h}{}^g(y_x)} = \frac{(1+f_r'^2(x))^{\frac{3}{2}}}{D_r''f(x)} \cdot \frac{D_l''g(y_x)}{(1+g_l'^2(y_x))^{\frac{3}{2}}} = \frac{D_l''g(y_x)}{D_r''f(x)}$$

and, similarly,

$$\lim_{h\to 0-} \frac{R_h{}^f(x)}{R_{l_h}{}^g(y_x)} = \frac{D_r''g(y_x)}{D_l''f(x)}.$$

Hence

$$\limsup_{h\to 0} \frac{\max\{R_h{}^f(x), R_{l_h}{}^g(y_x)\}}{R_h{}^f(x)+R_{l_h}{}^g(y_x)}$$

$$= \max\left\{\frac{\max\{D_r''f(x), D_l''g(y_x)\}}{D_r''f(x)+D_l''g(y_x)}, \frac{\max\{D_l''f(x), D_r''g(y_x)\}}{D_l''f(x)+D_r''g(y_x)}\right\}.$$

Now the formula of the statement can be easily established.

For the next discussion, let f and g possess continuous second derivatives f'' and g''. Also, let us define

$$f^*(x) = \begin{cases} f''(x) & \text{if } f''(x) > 0, \\ \|I_x{}^f\| & \text{if } x \in N, \\ 1 & \text{if } f''(x) = 0 \text{ and } x \in M, \text{ or } f''(x) = \infty, \end{cases}$$

and the analogous function $g^*(x)$.

THEOREM 15. *For a body satisfying the above regularity conditions,*

$$r_i = \sup_{x\in[0,1]} \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)}.$$

Proof. Let $0 < f''(x) < \infty$; then $f^*(x) = f''(x) = D_r''f(x) = D_l''f(x)$, and we shall prove that $g''(y_x)$ is also finite and strictly positive. If $I_{y_x}{}^g \supset \{y_x\}$ then $x \in N$, hence $f''(x) = 0$, which is absurd. If $I_{y_x}{}^g = \emptyset$ let $\{g''(y_{x_n})\}$ be a sequence of strictly positive numbers, where $x_n \to x$. From the continuity of f'' and g'' it follows that $f''(x_n) \to f''(x)$ and $g''(y_{x_n}) \to g''(y_x)$; hence if $g''(y_x)$ is 0 or ∞,

$$\lim_{n\to\infty} \frac{\max\{f''(x_n), g''(y_{x_n})\}}{f''(x_n)+g''(y_{x_n})} = 1.$$

From Theorem 14, we see that $r_i = 1$, i.e. our body is irreducible, which is absurd again. Hence $g''(y_x) > 0$, $g^*(y_x) = D_r''g(y_x) = D_l''g(y_x)$, and

$$\max\left\{\frac{D_r''f(x)}{D_r''f(x)+D''g(y_x)}, \frac{D_l''g(y_x)}{D_r''f(x)+D_l''g(y_x)}, \frac{D_l''f(x)}{D_l''f(x)+D_r''g(y_x)}, \frac{D_r''g(y_x)}{D_l''f(x)+D_r''g(y_x)}\right\}$$
$$= \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)}.$$

If $x \in N$ then

$$\max\left\{\frac{\|I_x^f\|}{\|I_x^f\|+\|I_{y_x}{}^g\|}, \frac{\|I_{y_x}{}^g\|}{\|I_x^f\|+\|I_{y_x}{}^g\|}\right\} = \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)},$$

from our notations.

Let $Q = \{x: f''(x) = 0 \text{ or } \infty\} \cap M$. If $x \in Q$ then $g''(y_x) = 0$ or ∞, because if $0 < g''(y_x) < \infty$ then $0 < f''(x) < \infty$. Also, $I_{y_x}{}^g = \emptyset$ because if $I_{y_x}{}^g \supset \{y_x\}$ then $x \in N$, which is false. Therefore $f^*(x) = g^*(y_x) = 1$ and

$$\frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)} = \tfrac{1}{2} \leqslant r_i.$$

One sees that

$$\sup_{x\in M\setminus Q} \frac{\max\{f''(x), g''(y_x)\}}{f''(x)+g''(y_x)} = \sup_{x\in M} \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)}$$

by the continuity of the used second derivatives and by the preceding inequality. Thus, by Theorem 14,

$$r_i = \sup_{x\in[0,1]} \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)}.$$

In the last case considered here, let the curve B have continuous curvature, which can attain 0 and ∞, let $E(a, b_a)$ be an essential diameter of B, and let $\Gamma(a)$ be (1) the curvature of B at a if a does not belong to the interior of a segment of B, or (2) the length of the maximal segment containing a and included in B in the other case.

THEOREM 16. *For the body described above,*

$$r_i = \sup_{a\in\tilde{B}} \frac{\Gamma(a)}{\Gamma(a)+\Gamma(b_a)},$$

$\tilde{B}$ being the subset of B where the quotient has a sense.

Proof. Since f'' and g'' are continuous at 0 and 1,

$$r_i = \sup_{x\in(0,1)} \frac{\max\{f^*(x), g^*(y_x)\}}{f^*(x)+g^*(y_x)},$$

by Theorem 15. Now

$$\sup_{x\in M}\frac{\max\{f^*(x),\, g^*(y_x)\}}{f^*(x)+g^*(yx)}$$

$$= \sup_{x\in M\setminus Q}\frac{\max\{f''(x),\, g''(y_x)\}}{f''(x)+g''(y_x)}$$

$$= \sup_{x\in M\setminus Q}\frac{\max\{(1-f'^2(x))^{\frac{3}{2}}\Gamma((x;f(x))),\, (1-g'^2(y_x))^{\frac{3}{2}}\Gamma((y;g(y_x)))\}}{(1-f'^2(x))^{\frac{3}{2}}\Gamma((x;f(x)))+(1-g'^2(y_x))^{\frac{3}{2}}\Gamma((y_x;g(y_x)))}$$

$$= \sup_{x\in M\setminus Q}\frac{\max\{\Gamma((x;f(x))),\, \Gamma((y_x;g(y_x)))\}}{\Gamma((x;f(x)))+\Gamma((y_x;g(y_x)))}$$

and

$$\sup_{x\in N}\frac{\max\{f^*(x),\, g^*(y_x)\}}{f^*(x)+g^*(y_x)} = \sup_{x\in N}\frac{\max\{\Gamma((x;f(x))),\, \Gamma((y_x;g(y_x)))\}}{\Gamma((x;f(x)))+\Gamma((y_x;g(y_x)))}.$$

Therefore

$$r_i = \sup_{x\in(0,1)}\frac{\max\{f^*(x),\, g^*(y_x)\}}{f^*(x)+g^*(y_x)}$$

$$= \sup_{x\in(0,1)\setminus Q}\frac{\max\{\Gamma((x;f(x))),\, \Gamma((y_x;g(y_x)))\}}{\Gamma((x;f(x)))+\Gamma((y_x;g(y_x)))}$$

$$= \sup_{a\in\tilde{B}_f}\frac{\max\{\Gamma(a),\Gamma(b_a)\}}{\Gamma(a)+\Gamma(b_a)} = \sup_{a\in\tilde{B}_{fg}}\frac{\Gamma(a)}{\Gamma(a)+\Gamma(b_a)},$$

where $\tilde{B}_f = \{(x;f(x)) : x\in(0,1)\setminus Q\}$ and $\tilde{B}_{fg} = \tilde{B}\setminus\{(0;f(0)),\, (1;f(1))\}$.

Again by the continuity of the curvature of B, we have

$$r_i = \sup_{x\in\tilde{B}_{fg}}\frac{\Gamma(a)}{\Gamma(a)+\Gamma(b_a)} = \sup_{x\in\tilde{B}}\frac{\Gamma(a)}{\Gamma(a)+\Gamma(b_a)},$$

which proves the theorem.

Final remarks

A natural partition of the family H of all Hammer's associated bodies can be established: $H = H_- \cup H_+$, where

$$H_- = \{h\in H;\, h\subset h_0\},\quad H_+ = \{h\in H;\, h\supseteq h_0\},\quad \text{and}\quad h_0 = C(r_i).$$

Then the body C is irreducible if C is the minimal element of H_+ and the upper bound of H_-, relative to the inclusion relation, and C is completely reducible if $H_- = \emptyset$.

We have established some properties showing qualitative differences between Hammer's bodies of H_+ and those of H_-. These results are useful to characterize the reducibility body.

Also, we notice that the principal properties established above, together with all the numerical estimations of r_i, are invariants of affine geometry.

Developments of some results of this paper and other related results can be found in (**6**), (**7**), (**8**), (**9**).

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