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On a theorem of Chartrand, Kapoor and Kronk

Estratto

DIREZIONE E REDAZIONE:

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ON A THEOREM OF CHARTRAND, KAPOOR AND KRONK

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The following theorem of L. Pósa [3] can be regarded as a sufficient condition for the graph G (with p points $(2 \leqq p < \infty)$, undirected, and without loops or multiple lines) to be 2-connected, since every hamiltonian graph is obviously 2-connected. We denote by $M_m(G)$ the number of all points of degree at most m in G.

Pósa's theorem. *Let G satisfy the following conditions:*

(i) *for every k such that* $1 \leqq k < (p-1)/2$,

$$M_k(G) \leqq k - 1;$$

(ii)

$$M_{[(p-1)/2]}(G) \leqq [(p-1)/2].$$

Then G is hamiltonian.

Obviously, in Pósa's theorem, condition (ii) is implied by (i) in case p is even; therefore it is meaningfull only for odd p.

As a sufficient condition for G to be 2-connected, the above result can be improved replacing condition (ii) by the following:

(ii′)

$$M_{[(p-1)/2]}(G) \leqq p - 2,$$

which, of course, is also implied by (i) in case p is even. This fact follows from the more general

Theorem of Chartrand, Kapoor and Kronk. *Let* $1 \leqq n < p$ *and let* G *satisfy the following conditions:*

(i) *for every* k *such that* $n - 1 \leqq k < (p + n - 3)/2$,

$$M_k(G) \leqq k - n + 1;$$

(ii)

$$M_{[(p+n-3)/2]}(G) \leqq p - n.$$

Then G *is n-connected.*

This theorem, proved in [2], cannot be improved by changing the upper bounds $k - n + 1$ and $p - n$ appearing in conditions (i) and (ii), respectively, as shown at the end of [2]. However, the following theorem contains the one above as a particular case, by offering an alternative within condition (i):

Theorem. *Let* $1 \leqq n < p$ *and let* G *satisfy the following conditions:*

(i) *for every* k *such that* $n - 1 \leqq k < (p + n - 3)/2$,

$$M_k(G) \leqq k - n + 1$$

or

$$M_{p-k+n-3}(G) \leqq p - k - 2;$$

(ii)

$$M_{[(p+n-3)/2]}(G) \leqq p - n.$$

Then G *is n-connected.*

Proof. We shall follow a way similar to that used in [2]. The case $n = p - 1$ can be treated like there. Now, let G satisfy (i) and (ii), and suppose it is not n-connected ($n \leqq p - 2$). Then there are $n - 1$ points whose removal from G gives a disconnected graph G'. The graph G' contains at least one component G_1 having r points, where $r \leqq [p - n + 1]/2$. The removal of the points of G_1 from G' gives another graph G_2 (connected or not), with $p - n - r + 1$ points.

Three cases are to be considered:

Case 1. $r = (p - n + 1)/2$.

The proof in this case is identical with that of Case 1 of the proof of the Theorem in [2], the contradiction being obtained only from condition (ii).

Case 2. $r < (p - n + 1)/2$, *and suppose that the variant*

$$M_k(G) \leqq k - n + 1$$

is satisfied for $k = r + n - 2$. Then, again, the contradiction may be found as in [2].

Case 3. $r < (p - n + 1)/2$, *and suppose that the other variant*

$$M_{p-k+n-3}(G) \leqq p - k - 2$$

is satisfied for $k = r + n - 2$. Here we must give some proof.

Indeed, each of the $p - n - r + 1$ points of G_2 evidently has a degree not exceeding $(p - n - r + 1) + (n - 1) - 1 = p - r - 1$. Therefore,

$$M_{p-r-1}(G) > p - n - r,$$

i. e., expressing in terms of p, k, and n,

$$M_{p-k+n-3}(G) > p - k - 2,$$

which provides a contradiction.

That this theorem really improves that of [2], it may be seen from the example given by the graph Γ shown in the figure.

Γ is 2-connected. The sufficient condition of [2], demanding that

$$M_1(\Gamma) \leqq 0, \qquad M_2(\Gamma) \leqq 1$$

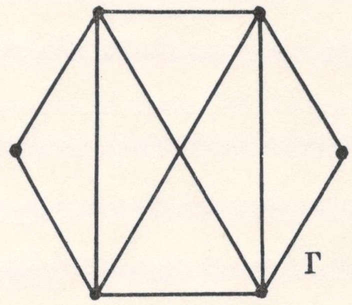
does not apply, since $M_2(\Gamma) = 2$. But the Theorem above, demanding only that

$$M_1(\Gamma) \leqq 0 \quad \text{or} \quad M_4(\Gamma) \leqq 3,$$

and

$$M_2(\Gamma) \leqq 1 \quad \text{or} \quad M_3(\Gamma) \leqq 2$$

does apply, because $M_1(\Gamma) = 0$ and $M_3(\Gamma) = 2$.



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Remarks. 1[0]) Like in the Theorem of Chartrand, Kapoor and Kronk, condition (ii) is relevant only when p and n are of opposite parity.

2[0]) Since the relation "improves" is transitive, the Theorem proved here also improves a result of G. Chartrand and F. Harary [1] (see [2], p. 51).

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