THE SIMPLICIAL CONVEXITY OF CONVEX SURFACES

BY

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The aim of this note is to find the class of all (complete) convex surfaces which are simplicially convex. This problem will be completely solved here only for the 2-dimensional surfaces. However, one will probably find in this treatment some useful suggestions for attacking the n-dimensional case.

Let E_n be the n-dimensional Euclidean space. A set $A \subset E_n$ is said to be *l-simplicially convex* if there exists a subset $B \subset A$ such that

$$A = \mathcal{S}_l(B),$$

where $\mathcal{S}_l(B)$ denotes the union of all simplexes of dimension at most $l-1$, with vertices in B. A set which is l-simplicially convex for some $l \geqslant 2$ is called *simplicially convex* ([5]or [6]).

By Carathéodory's theorem ([2] or [3], p. 15), usual convexity and $(n+1)$-simplicial convexity are equivalent notions in E_n. (In general, usual convexity implies l-simplicial convexity for each l). Therefore we shall only speak about the 2- and the 3-simplicial convexity of convex surfaces, i.e. the frontiers of convex bodies *) in E_3 [1].

LEMMA 1. *If A is a simplicially convex set, and*

$$A = \mathcal{S}_l(B),$$

then

$$\text{ext conv } A \subset B^{**}).$$

We leave the (simple) proof of this lemma to the reader, remarking that a useful comment could eventually be found at the beginning of the proof of Theorem 4 in [5].

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*) By convex body we mean a convex set with interior points, different from the intersection of two half-spaces with parallel frontiers.

**) We use the (rather universal) notations of [3].

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LEMMA 2. *If $K \subset E_3$ is a bounded convex body and*

$$\mathcal{S}_2(\text{ext } K) \subset \text{bd } K,$$

then K is either a bounded cone (based on a planar convex body) or is combinatorially equivalent with a bounded triangular prism.

Proof. Let $x_1, \ldots, x_4 \in \text{ext } K$ be the vertices of a tetrahedron T not degenerate.

If

$$\text{ext } K = \{x_1, \ldots, x_4\},$$

then $K = T$ and nothing more needs a proof.

If

$$x_5 \in \text{ext } K - \{x_1, \ldots, x_4\}^{*}),$$

then

$$x_i \notin \text{conv } (\{x_1, \ldots, x_5\} - \{x_i\}) \ (i = 1, \ldots, 5).$$

By Radon's theorem ([4] or [3], p. 16), it follows that $\{x_1, \ldots, x_5\}$ can be decomposed in two disjoint subsets P and Q such that

$$\text{card } P = 2, \quad \text{card } Q = 3, \quad \text{and} \quad \text{conv } P \cap \text{conv } Q \neq \emptyset.$$

Now it is easily seen that either x_5 lies on the plane of a face of T, or

$$\text{conv } P \cap \text{int } T \neq \emptyset.$$

But the last relation is contradicted by

$$\text{bd } K \cap \text{int } T = \emptyset$$

together with

$$\text{conv } P \subset \mathcal{S}_2 (\text{ext } K)$$

and with the inclusion in the statement. Therefore there is a face F of T whose plane π contains x_5. Because $x_i \in \text{ext } K$ $(i = 1, \ldots, 5)$, $G = \text{conv } (F \cup \{x_5\})$ is a quadrilater.

If

$$\text{ext } K = \{x_1, \ldots, x_5\},$$

then

$$K = \text{conv } (T \cup G)$$

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*) Because no confusion can appear, we shall write "$-$" (instead of "$\sim$" [3]) for set-theoretic difference as well as for usual difference between numbers. Sums or differences of vectors will not be used.

is a bounded quadrilateral pyramid (therefore a special kind of bounded cone).

If

$$\text{ext } K - \{x_1, \ldots, x_5\} \neq \emptyset,$$

then either

$$\text{ext } K - T \subset \pi,$$

or there is a point

$$x_6 \in \text{ext } K - (T \cup \pi).$$

In the first case, K is a cone with apex ext $K - \pi$ and based on conv (ext $K \cap \pi$), and the proof does not need to be continued. In the second case, x_6 must belong to some face of each of the four tetrahedra not degenerate and with vertices in $\{x_1, \ldots, x_5\}$. This is possible only if the point x_6 lies on the segment s_1 joining the intersection of two opposite sides of G with ext $T-G$ (intersection at infinity is here permitted).

If

$$\text{ext } K = \{x_1, \ldots, x_6\},$$

then

$$K = \text{conv } (T \cup G \cup \{x_6\})$$

is combinatorially equivalent with a triangular prism, which verifies the lemma.

Suppose there would exist another point $x_7 \neq x_1$ with

$$x_7 \in \text{ext } K - (T \cup \pi).$$

By the preceding discussion, x_7 either belongs to s_1 or to an analoguous segment s_2 joining the intersection of the other two opposite sides of G with ext $T-G$. Since the extreme points ext $T-G$ and x_6 are already on s_1, it follows $x_7 \in s_2$. Let t be the triangular face of conv. $(T \cup G \cup \{x_6\})$ which contains x_6 and S be the tetrahedron having x_6, another vertex of t, and the two points of ext $G-t$ as vertices. Then x_7 does not belong to the plane of any face of S, and a contradiction is obtained.

The proof of Lemma 2 is complete.

LEMMA 3. *Let K be a bouded convex cone. Then*

a) bd K *is 2-simplicially convex,*

b) bd K *is 3-simplicially convex if and only if K is a tetrahedron.*

Proof. a) Let a be the apex of K, and

$$M_2 = \text{relbd conv (ext } K - \{a\}) \cup \{a\}.$$

One obviously obtains

$$\text{bd } K = \mathcal{S}_2 (M_2).$$

b) If K is a tetrahedron, then

$$\text{bd } K = \mathcal{S}_3(M_3)$$

for $M_3 = \text{ext } K$.

Let now bd K be 3-simplicially convex :

$$\text{bd } K = \mathcal{S}_3(M_3).$$

If K is a bounded cone (with apex a), but a tetrahedron, then its base B has a pair of extreme points b, c such that the segment joining them intersects int B. Then

$$\mathcal{S}_3(\{a, b, c\}) \cap \text{int } K \neq \emptyset,$$
$$\mathcal{S}_3(\{a, b, c\}) \subset \mathcal{S}_3(\text{ext } K) \subset \mathcal{S}_3(M_3),$$

whence

$$\mathcal{S}_3(M_3) \not\subset \text{bd } K,$$

which provides a contradiction.

LEMMA 4. *If K is combinatorially equivalent with a bounded triangular prism, then* bd K *is not simplicially convex.*

Proof. We have only to show that bd K is not 2- or 3-simplicially convex. Suppose this is not true ; then

$$\mathcal{S}_j(M_j) = \text{bd } K,$$

for some $j \in \{2, 3\}$. By Lemma 1,

$$\text{ext } K = \text{ext conv bd } K \subset M_j.$$

Clearly

$$M_j \not\subset \text{ext } K,$$

because

$$\mathcal{S}_2(\text{ext } K) \not\supset \text{bd } K\,;\ \mathcal{S}_3(\text{ext } K) \not\subset \text{bd } K.$$

Also $j = 2$, for

$$\mathcal{S}_3(\text{ext } K) \subset \mathcal{S}_3(M_3),$$

whence

$$\mathcal{S}_3(M_3) \not\subset \text{bd } K.$$

Let K have vertices $x_1, \ldots, x_6$, such that conv $\{x_1, x_2, x_3\}$, conv $\{x_4, x_5, x_6\}$ be faces and conv $\{x_1, x_4\}$, conv $\{x_2, x_5\}$ be edges of K. There exists a point $e \in M_2 - \text{ext } K$.

Suppose that

$$e \in \operatorname{conv}\{x_1, x_2\}.$$

Then

$$\operatorname{conv}\{e, x_6\} \cap \operatorname{int} K \neq \varnothing,$$

$$\operatorname{conv}\{e, x_6\} \subset \mathcal{S}_2(M_2),$$

whence

$$\mathcal{S}_2(M_2) \not\subset \operatorname{bd} K,$$

which is a contradiction.

Similarly,

$$e \notin \operatorname{conv}\{x_2, x_3\} \cup \operatorname{conv}\{x_3, x_1\}.$$

Also, if

$$e \in \operatorname{rel\,int}\operatorname{conv}\{x_1, x_2, x_3\},$$

then

$$\operatorname{conv}\{e, x_6\} \cap \operatorname{int} K \neq \varnothing,$$

and the same contradiction as before is obtained.

Therefore

$$M_2 \cap \operatorname{conv}\{x_1, x_2, x_3\} = \{x_1, x_2, x_3\},$$

whence

$$\mathcal{S}_2(M_2) \cap \operatorname{conv}\{x_1, x_2, x_3\} = \mathcal{S}_2(\{x_1, x_2, x_3\}) =$$

$$= \operatorname{relbd}\operatorname{conv}\{x_1, x_2, x_3\};$$

hence

$$\mathcal{S}_2(M_2) \not\supset \operatorname{bd} K,$$

which is again a contradiction, and Lemma 3 is completely proved.

We are now able to prove our first main result, regarding the closed convex surfaces (see [1]):

THEOREM 1. *a) The class of all closed convex surfaces in E_3 which are simplicially convex is that of frontiers of bounded convex cones.*

b) All bounded convex cones have 2-simplicially convex frontiers.

c) The class of all closed convex surfaces in E_3 which are 3-simplicially convex is that of frontiers of tetrahedra.

Proof. *a*) Let K be a bounded convex body in E_3 and bd K be simplicially convex. Then

$$\mathcal{S}_j(M_j) = \text{bd } K$$

for some $j \in \{2, 3\}$. Following Lemma 1, ext $K \subset M_j$. Therefore

$$\mathcal{S}_2(\text{ext } K) \subset \mathcal{S}_j(\text{ext } K) \subset \mathcal{S}_j(M_j) = \text{bd } K.$$

Now, by Lemma 2 combined with Lemma 4, K is a bounded convex cone.

b) This assertion of the Theorem 1 coincides with Lemma 3, a).
c) This is a consequence of point a) above and of Lemma 3, b).
Thus the proof of Theorem 1 is completed.

Let us pass now to the case of open and cylindrical surfaces.

LEMMA 5. *If the convex body K has precisely j extreme points ($j \geqslant 1$) and all of them are coplanar, then* bd K *is an open convex surface and K has at least $f(j)$ unbonded 1-faces* *), *where*

$$f(j) = \begin{cases} j+2 & \text{for } j = 1,\ 2 \\ j & \text{for } j \geqslant 3. \end{cases}$$

LEMMA 6. *If the convex body K has no extreme points, then* bd K *is a cylindrical convex surface and K has at least 3 (unbounded) 1-faces.*

The proof of these lemmas will also be left to the reader.

LEMMA 7. *If the convex body K has at least 4 unbounded 1-faces, then* bd K *is not simplicially convex.*

Proof. Let S_1, S_2, S_3, S_4 be unbounded 1-faces of K. If

$$\mathcal{S}_j(M_j) = \text{bd } K \quad (j = 2 \text{ or } 3),$$

then each of the preceding 1-faces must contain a sequence of points

$$\{s_{i,j,n}\}_{n=1}^{\infty} \ (i = 1, 2, 3, 4) \text{ of } M_j. \text{ Let}$$

$$s_{i,j,n_{ij}} \in \text{relint } S_i.$$

Then, clearly

$$\text{conv } \{s_{1,j,n_{1j}},\ s_{k,j,n_{kj}}\} \cap \text{int } K \neq \theta$$

for at least one index $k \in \{2, 3, 4\}$, which provides a contradiction, since

$$\text{conv } \{s_{1,j,n_{1j}}, s_{k,j,n_{kj}}\} \subset \mathcal{S}_j(\{s_{1,j,n_{1j}}, s_{k,j,n_{kj}}\}) \subset \mathcal{S}_j(M_j) = \text{bd } K.$$

*) By a d-face is meant a d-dimensional intersection of cl K with a supporting plane of K. In the case of Lemma 5, the considered 1-faces are semilines.

Consequently we obtain our second main result, concerning the open and the cylindrical convex surfaces (see [1]).

THEOREM 2. *a) The class of all open convex surfaces that are simplicially convex is that of frontiers of (unbounded) triangular pyramids.*

b) The class of all cylindrical convex surfaces that are simplicially convex is that of frontiers of (unbounded) triangular prisms.

c) The surface in both classes are 2-simplicially convex. No open or cylindrical convex surface is 3-simplicially convex.

Proof. a) and *c)* Let K be a convex body whose boundary is an open convex surface. By Lemma 6, K has at least one extreme point.

Suppose now that K has precisely one extreme point e; then, by Lemma 5, it has at least 3 unbounded 3-faces. If bd K is simplicially convex, then, by Lemma 7, K is an unbounded triangular pyramid. If, conversely, K is an unbounded triangular pyramid, then it has the (unbounded) 1-faces S_1, S_2, S_3. Since

$$\mathcal{S}_2(S_1 \cup S_2 \cup S_3) = \text{bd } K,$$

bd K is 2-simplicially convex. If

$$\mathcal{S}_3(M_3) = \text{bd } K,$$

then $e \in M_3$ and

$$M_3 \cap S_j - \{e\} \neq \emptyset \qquad (j = 1, 2, 3),$$

whence

$$\mathcal{S}_3(M_3) \cap \text{int } K \neq \emptyset$$

and a contradiction is obtained.

If K has precisely 2 extreme points, then, by Lemma 5, it has at least 4 unbounded 1-faces; therefore, by Lemma 7, bd K is not simplicially convex.

If K has precisely 3 extreme points e_1, e_2, e_3, then it has at least 3 semilines S_1, S_2, S_3, originating respectively in e_1, e_2, e_3, as 1-faces. Suppose bd K is simplicially convex. Then

$$\mathcal{S}_j(M_j) = \text{bd } K \qquad (j = 2 \text{ or } 3).$$

Since

$$M_j \cap S_k - \{e_k\} \neq \emptyset \qquad (j = 2, 3;\ k = 1, 2, 3),$$

we can find the points

$$x_{j,k} \in M_j \cap S_k - \{e_k\}.$$

But

$$\mathcal{S}_3(M_3) \supset \mathcal{S}_3(\{x_{3.1}, e_2, e_3\}); \quad \mathcal{S}_3(\{x_{3.1}, e_2, e_3\}) \cap \operatorname{int} K \neq \varnothing,$$

which provides a contradiction. Therefore $j = 2$. Now, like in the proof of Lemma 4, it can be shown that

$$M_2 \cap \operatorname{conv}\{e_1, e_2, e_3\} = \{e_1, e_2, e_3\}$$

and the same contradiction like there is obtained.

If there are 4 points in ext K, not all in one plane, then K also possesses at least 3 semilineas as 1-faces, one of which, say S, has the property that there exists a point $e \in \operatorname{ext} K$ such that

$$\operatorname{conv}\{x, e\} \cap \operatorname{int} K \neq \varnothing$$

for every $x \in \operatorname{relint} S$, and therefore bd K is not simplicially convex.

If ext K contains at least 4 points, but all lie on a plane, then, by Lemma 5, K has at least 4 unbounded 1-faces; following Lemma 7, bd K is not simplicially convex.

b) and *c*) Let now K be a convex body with cylindrical boundary. Then it has no extreme points and, by Lemma 6, possesses at least 3 (unbounded) 1-faces. If K has exactly 3 1-faces, then it is an (unbounded) triangular prism. All triangular prisms have obviously 2-simplicially convex boundaries since each such boundary is the 2-simplicially convex cover [5] of the union of the 3 1-faces. But these boundaries are not 3-simplicially convex. If K has at least 4 1-faces, bd K is not simplicially convex, following Lemma 7.

Theorem 1 together with Theorem 2 show:

The classes of all (complete, 2-dimensional) convex surfaces which are simplicially convex are those of frontiers of bounded cones, frontiers of unbounded triangular pyramids, and frontiers of unbounded triangular prisms. All of them are 2-simplicially convex. The class of all convex surfaces which are 3-simplicially convex is that of frontiers of tetrahedra.

We ask for characterizations of those convex which are l-simplicially convex, with (out) prescribed l, in more than 3 dimensions. A first step of special interest, and probably the most important, would be to find all the polyhedral simplicially convex surfaces, because our problem seems to look more natural when such surfaces are considered. Furthermore, the study of the simplicial convexity of boundary complexes of convex polytopes would be of a certain interest. Finally, let us remark that, even in the plane, the most general problem of charcterizing the arbitrary simplicially convex sets is not yet solved, and any significant combinatorial, topological, or geometrical properties (for some algebraic properties see [5] and [6]) have not yet been discovered.

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