Fix Point Theorems in Metric Spaces

By

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In this paper results similar to the well-known contraction theorems of BANACH are obtained*). These classical theorems, our mentioned similar results and theorems of KANNAN, EDELSTEIN and SINGH are all, in fact, corollaries of a few more general theorems, as we shall see.

Theorem 1. *Let M be a complete metric space, α, β, γ real numbers with $\alpha < 1$, $\beta < \frac{1}{2}$, $\gamma < \frac{1}{2}$, and $f: M \to M$ a function such that for each couple of different points $x, y \in M$, at least one of the following conditions is satisfied:*

1) $d(f(x), f(y)) \leqq \alpha d(x, y)$ **),
2) $d(f(x), f(y)) \leqq \beta(d(x, f(x)) + d(y, f(y)))$,
3) $d(f(x), f(y)) \leqq \gamma(d(x, f(y)) + d(y, f(x)))$.

Then f has a unique fixed point.

Proof. Consider the number

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}.$$

Obviously, $\delta < 1$.

Now, choose $x_0 \in M$ arbitrarily and fix an integer number $n \geqq 0$. Take $x = f^n(x_0)$ and $y = f^{n+1}(x_0)$. Suppose $x \neq y$; otherwise x is a fixed point of f. If for these two points condition 1) is satisfied, then

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq \delta d(f^n(x_0), f^{n+1}(x_0)).$$

If for x, y, condition 2) is verified, then

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq \beta(d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0))),$$

which implies

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq \frac{\beta}{1-\beta} d(f^n(x_0), f^{n+1}(x_0)) \leqq \delta d(f^n(x_0), f^{n+1}(x_0)).$$

In case condition 3) is satisfied,

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq \gamma d(f^n(x_0), f^{n+2}(x_0)) \leqq$$
$$\leqq \gamma(d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0))),$$

*) For example, Corollaries 3,4.

**) Throughout all the paper, d denotes the distance function.

which analogously implies

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq \delta d(f^n(x_0), f^{n+1}(x_0)) .$$

This inequality, true for every n, clearly implies that $\{f^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence and therefore converges to some point $z \in M$.

We now prove that z is a fixed point of f. Suppose $f(z) \neq z$ and consider the ball

$$B = \{x \in M : d(x, z) \leqq \tfrac{1}{4} d(z, f(z))\} .$$

Observe that $d(x, f(z)) \geqq \frac{3}{4} d(z, f(z))$ for every point $x \in B$. There exists a number N such that $f^n(x_0) \in B$ for each $n \geqq N$. Taking now $x = f^N(x_0)$ and $y = z$, we must have again one of the next three situations:

1) $$d(f^{N+1}(x_0), f(z)) \leqq \alpha d(f^N(x_0), z) ,$$

which, however, contradicts

$$\alpha d(f^N(x_0), z) \leqq d(f^N(x_0), z) \leqq \tfrac{1}{4} d(z, f(z)) < d(f^{N+1}(x_0), f(z)) ,$$

2) $$d(f^{N+1}(x_0), f(z)) \leqq \beta (d(f^N(x_0), f^{N+1}(x_0)) + d(z, f(z))) ,$$

contradicting

$$\begin{aligned} &\beta(d(f^N(x_0), f^{N+1}(x_0)) + d(z, f(z))) < \\ &< \tfrac{1}{2}(d(f^N(x_0), z) + d(z, f^{N+1}(x_0)) + d(z, f(z))) \leqq \\ &\leqq \tfrac{3}{4} d(z, f(z)) \leqq d(f^{N+1}(x_0), f(z)) , \end{aligned}$$

3) $$d(f^{N+1}(x_0), f(z)) \leqq \gamma (d(f^N(x_0), f(z)) + d(f^{N+1}(x_0), z)) ,$$

respectively contradicting

$$\begin{aligned} &\gamma(d(f^N(x_0), f(z)) + d(f^{N+1}(x_0), z)) < \\ &< \tfrac{1}{2}(d(f^N(x_0), z) + d(z, f(z)) + d(f^{N+1}(x_0), z)) \leqq \\ &\leqq \tfrac{3}{4} d(z, f(z)) \leqq d(f^{N+1}(x_0), f(z)) . \end{aligned}$$

Thus, $f(z) = z$.

Now we show that this fixed point z is unique. Suppose this is not true: $f(z') = z'$ for some point $z' \in M$ different from z. Then

$$\begin{aligned} &d(f(z), f(z')) = d(z, z') , \\ &d(f(z), f(z')) > d(z, f(z)) + d(z', f(z')) , \\ &d(f(z), f(z')) = \tfrac{1}{2}(d(z, f(z')) + d(z', f(z))) , \end{aligned}$$

so that none of the three conditions of the theorem is satisfied at the points z and z'.

Corollary 1 (Banach). *Let M be a complete metric space, $\alpha < 1$, and $f : M \to M$ a function such that for each couple of different points in M condition 1) of Theorem 1 is verified. Then f has a unique fixed point.*

Corollary 2 (Kannan [2]). *Let M be a complete metric space, $\beta < \frac{1}{2}$, and $f : M \to M$ a function such that for each couple of different points in M condition 2) of Theorem 1 is verified. Then f has a unique fixed point.*

We believe the following analogous result is new.

Corollary 3. *Let M be a complete metric space, $\gamma < \frac{1}{2}$, and $f: M \to M$ a function such that for each couple of different points in M condition* 3) *of Theorem* 1 *is verified. Then f has a unique fixed point.*

We give here an example of a space M and a function f such that Corollary 3 applies, but both Corollaries 1 and 2 fail to apply (even by taking suitable subsets of M and the corresponding restrictions of f).

Consider the complex plane $\mathbf{C}$, the half-plane $\mathbf{C}_+$ of all complex numbers with nonnegative coefficients of i, and the sequence $S = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$ on the real axis. To each couple of points $\left\{\frac{1}{2^{2m}}, \frac{1}{2^{2m+1}}\right\} \subset S$ (m integer), add two points $z_{2m}, z_{2m+1} \in \mathbf{C}_+$ with real parts $\frac{1}{2^{2m}}, \frac{1}{2^{2m+1}}$, respectively, such that all four points are the vertices of a square. Define

$$M = S \cup \{z_n\}_{n=0}^{\infty} \cup \{0\}$$

and $f: M \to M$ such that

$$f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}},$$
$$f(z_n) = z_{n+1},$$
$$f(0) = 0.$$

The space M is equipped with the distance induced from $\mathbf{C}$ and one easily verifies that it is complete and that f satisfies on all couples of different points in M the condition 3) of Theorem 1, but for each subset $M' \subset M$ with finite complement in M, at no couple of different points in M' condition 1) or condition 2) is satisfied.

Corollary 4. *Let M be a complete metric space, $\delta < 1$, and $f: M \to M$ a function such that for each couple of different points $x, y \in M$,*

$$d(f(x), f(y)) \leqq \delta g(x, y),$$

where $g(x, y)$ is the mean value of the first three, the last four, the first one and the last two or all five (the choice may depend on x, y) of the following numbers:

$$d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)).$$

Then f has a unique fixed point in M.

The proof is obvious: Let us say that for some $x, y \in M$, $g(x, y)$ is, for instance, the mean value of the first three numbers in the statement. Then $g(x, y) \leqq d(x, y)$ or $g(x, y) \leqq \frac{1}{2}(d(x, f(x)) + d(y, f(y)))$ (or both), so that at least one of the conditions 1) and 2) of Theorem 1 is satisfied at x, y.

The following statement slightly generalizes Theorem 1.

Theorem 2. *Let M be a metric space, α, β, γ real numbers with $\alpha < 1$, $\beta < \frac{1}{2}$, $\gamma < \frac{1}{2}$, and $f: M \to M$ a function such that for each couple of different points $x, y \in M$, at* *least one of the conditions* 1), 2), 3) *of Theorem* 1 *is satisfied. If for some* $x_0 \in M$, *the sequence* $\{f^n(x_0)\}_{n=0}^{\infty}$ *has a limit point* z *in* M, *then* z *is a unique fixed point of* f.

To prove Theorem 2, one shows like in the proof of Theorem 1 that $\{f^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence, whence $f^n(x_0) \to z$; further one follows again the proof of Theorem 1. That Theorem 2 generalizes Theorem 1 is clear since under the hypotheses of Theorem 1, $\{f^n(x_0)\}_{n=0}^{\infty}$ is convergent for every $x_0 \in M$.

Theorem 2 has, of course, corollaries analogous to Corollaries 1—4 of Theorem 1, but we omit writing them down explicitly. We just remark that the analogue of Corollary 2 is exactly Theorem 1 of KANNAN [3] if one drops from the last the condition (ii), which thus appears superfluous.

Theorem 3. *Let* M *be a set in a complete metric space* X, α, β, γ *real numbers with* $\alpha < 1$, $\beta < \frac{1}{2}$, $\gamma < \frac{1}{2}$ *and* $f: M \to M$ *a function such that for each couple of different points in* M, *at least one of the three conditions of Theorem* 1 *is satisfied. Then, for* $x_0 \in M$, *the sequence* $\{f^n(x_0)\}_{n=0}^{\infty}$ *converges to a point in* X, *independent on the choice of* x_0.

Proof. First, one finds like in the proof of Theorem 1 that there exists a point $z \in X$ such that

$$\lim_{n\to\infty} f^n(x_0) = z .$$

Choose now y_0 arbitrarily in M. Analogously, $f^n(y_0) \to z'$, where $z' \in X$, and we want to show $z = z'$. Suppose $d(z, z') > 0$ and choose the number ε such that

$$0 < \varepsilon < \frac{1}{2} d(z, z') \cdot \min\left\{\frac{1-\alpha}{1+\alpha}, \frac{1}{2}, \frac{1-2\gamma}{1+2\gamma}\right\}.$$

There is a natural number N such that $d(f^n(x_0), z) \leqq \varepsilon$ and $d(f^n(y_0), z') \leqq \varepsilon$ for every $n \geqq N$.

Suppose first that for $x = f^n(x_0)$, $y = f^n(y_0)$, where $n \geqq N$, condition 1) is verified (clearly, $x \neq y$). Then,

$$d(f^{n+1}(x_0), f^{n+1}(y_0)) \leqq \alpha\, d(f^n(x_0), f^n(y_0)) .$$

But

$$\alpha\, d(f^n(x_0), f^n(y_0)) \leqq \alpha(d(f^n(x_0), z) + d(z, z') + d(z', f^n(y_0))) \leqq$$
$$\leqq \alpha(d(z, z') + 2\varepsilon) .$$

Now, since

$$\varepsilon < \frac{1}{2} d(z, z') \cdot \frac{1-\alpha}{1+\alpha},$$

we have further

$$\alpha(d(z, z') + 2\varepsilon) < d(z, z') - 2\varepsilon \leqq$$
$$\leqq d(z, z') - d(f^{n+1}(x_0), z) - d(f^{n+1}(y_0), z') \leqq$$
$$\leqq d(f^{n+1}(x_0), f^{n+1}(y_0)),$$

which provides a contradiction.

Suppose now for these x, y condition 2) is satisfied. Then

$$d(f^{n+1}(x_0), f^{n+1}(y_0)) \leqq \beta(d(f^n(x_0), f^{n+1}(x_0)) + d(f^n(y_0), f^{n+1}(y_0))) .$$

But

$$\beta(d(f^n(x_0), f^{n+1}(x_0)) + d(f^n(y_0), f^{n+1}(y_0))) \leqq$$
$$\leqq \beta(d(f^n(x_0), z) + d(z, f^{n+1}(x_0)) + d(f^n(y_0), z') + d(z', f^{n+1}(y_0))) \leqq 4\beta\varepsilon$$

and since

$$\varepsilon < \frac{1}{2} d(z, z') \cdot \frac{1}{2} < \frac{1}{2} d(z, z') \cdot \frac{1}{1+2\beta},$$

we have further

$$4\beta\varepsilon < d(z, z') - 2\varepsilon \leqq d(f^{n+1}(x_0), f^{n+1}(y_0)),$$

which gives a contradiction.

If, finally, for this couple x, y, condition 3) is fulfilled, then analogously

$$d(f^{n+1}(x_0), f^{n+1}(y_0)) \leqq \gamma(d(f^n(x_0), f^{n+1}(y_0)) + d(f^n(y_0), f^{n+1}(x_0))),$$

but

$$\gamma(d(f^n(x_0), f^{n+1}(y_0)) + d(f^n(y_0), f^{n+1}(x_0))) \leqq$$
$$\leqq \gamma(d(f^n(x_0), z) + d(z, z') + d(z', f^{n+1}(y_0)) +$$
$$+ d(f^n(y_0), z') + d(z', z) + d(z, f^{n+1}(x_0))) \leqq$$
$$\leqq 2\gamma(d(z, z') + 2\varepsilon) < d(z, z') - 2\varepsilon \leqq d(f^{n+1}(x_0), f^{n+1}(y_0)),$$

and again a contradiction is obtained.

It follows $d(z, z') = 0$ and Theorem 3 is proved.

Surely, Theorem 3 has also corollaries (that can easily be reproduced by the reader) analogous to Corollaries 1—4. The one corresponding to Corollary 1 is the well-known Picard-Banach contraction principle.

Theorem 4. *Let M be a metric space and $f: M \to M$ a continuous function such that for each couple of different points x, $y \in M$, at least one of the following conditions is satisfied:*

1° $d(f(x), f(y)) < d(x, y)$,

2° $d(f(x), f(y)) < \frac{1}{2}(d(x, f(x)) + d(y, f(y)))$,

3° $d(f(x), f(y)) < \frac{1}{2}(d(x, f(y)) + d(y, f(x)))$.

If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M, then z is a unique fixed point of f.

Proof. Let $\{f^{n_i}(x_0)\}_{i=0}^{\infty}$ be a subsequence of $\{f^n(x_0)\}_{n=0}^{\infty}$ such that

$$\lim_{i\to\infty} f^{n_i}(x_0) = z.$$

Then

$$\lim_{i\to\infty} f^{n_i+1}(x_0) = f(z), \quad \lim_{i\to\infty} f^{n_i+2}(x_0) = f^2(z)$$

because f is continuous. We want to show now that for each nonnegative integer n either $f^n(x_0) = f^{n+1}(x_0)$ or

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) < d(f^n(x_0), f^{n+1}(x_0)).$$

Suppose, on the contrary, that for some $m \geqq 0$, $f^m(x_0)$ and $f^{m+1}(x_0)$ are distinct and

$$d(f^m(x_0), f^{m+1}(x_0)) \leqq d(f^{m+1}(x_0), f^{m+2}(x_0)).$$

Then, taking $x = f^m(x_0)$, $y = f^{m+1}(x_0)$, the preceding inequality contradicts each of the following three relations, respectively expressing the conditions 1°, 2°, 3°.

$$d(f^{m+1}(x_0), f^{m+2}(x_0)) < d(f^m(x_0), f^{m+1}(x_0)),$$

$$d(f^{m+1}(x_0), f^{m+2}(x_0)) < \tfrac{1}{2}(d(f^m(x_0), f^{m+1}(x_0)) + d(f^{m+1}(x_0), f^{m+2}(x_0))),$$

$$d(f^{m+1}(x_0), f^{m+2}(x_0)) < \tfrac{1}{2}(d(f^m(x_0), f^{m+2}(x_0)) \leqq$$
$$\leqq \tfrac{1}{2}(d(f^m(x_0), f^{m+1}(x_0)) + d(f^{m+1}(x_0), f^{m+2}(x_0))).$$

Thus $\{d(f^n(x_0), f^{n+1}(x_0))\}_{n=0}^{\infty}$ is convergent and

$$d(z, f(z)) = \lim_{i \to \infty} d(f^{n_i}(x_0), f^{n_i+1}(x_0)) = \lim_{n \to \infty} d(f^n(x_0), f^{n+1}(x_0)),$$
$$= \lim_{i \to \infty} d(f^{n_i+1}(x_0), f^{n_i+2}(x_0)) = d(f(z), f^2(z)).$$

Now, analogously, the above equality proves that $f(z) = z$. Finally, suppose $z' \neq z$ and $f(z') = z'$. Then, like in the proof of Theorem 1, none of the three conditions which are to be satisfied at $x = z$, $y = z'$ is fulfilled. The proof is finished.

In Theorem 4 and in its corollaries that will be mentioned, two of which explicitely, the existence of a sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ having some limit point in M is automatically guaranteed in case M is compact.

Corollary 5 (Edelstein [1]). *Let M be a metric space and $f: M \to M$ a function such that for each couple of different points in M, condition 1° of Theorem 4 is satisfied. If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M, then z is a unique fixed point of f.*

Corollary 6 (Singh [4]). *Let M be a metric space and $f: M \to M$ a continuous function such that for each couple of different points in M, condition 2° of Theorem 4 is satisfied. If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M, then z is a unique fixed point of f.*

In fact, we observe that this result of Singh (his Theorem 1 in [4]) is not exactly an extension of the proposition of Kannan cited here as Corollary 2, as he claims. It merely constitutes an improvement of Theorem 2 of Kannan [3], after remarking that the inequality condition imposed on some everywhere dense subspace may easily be extended to the whole space. (A remark on Theorem 3 of [4]: If the printing mistake of asking condition (b) to be satisfied by all $x, y \in X$ (as appeared, the condition (b) precisely implies the non-existence of any fixed point !) is eliminated, then the result immediately follows from Theorem 1 in [4] since condition (a) implies T is continuous!)

Before presenting the last three theorems of the paper, we only mention that two further corollaries to Theorem 4, analogous with Corollaries 3 and 4, may be formulated.

The proof of Theorem 4 suggests the following improvement.

Theorem 5. *Let M be a metric space and $f: M \to M$ a function such that for each couple of different points in M, at least one of the conditions $1°$, $2°$, $3°$ of Theorem 4 is satisfied. If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M and f is continuous at z and at $f(z)$, then z is a unique fixed point of f.*

The following theorems avoid in another way asking f to be continuous.

Theorem 6. *Let M be a metric space and $f: M \to M$ a function such that for each couple of different points in M, at least one of the conditions $1°$, $2°$, $3°$ of Theorem 4* is *satisfied. If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point z in M, then z is a unique fixed point of f.*

Proof. Since the uniqueness of the fixed point easily follows as in the proof of Theorem 1, the only thing to be done is to show $f(z) = z$. But this, again, may be almost copied from the proof of Theorem 1, because one doesn't use essentially there the existence of α, β, γ so that one of 1), 2), 3) is true, but only the validity of at least one of the conditions $1°$, $2°$, $3°$ of Theorem 4. It is moreover easy to see that in order to confirm $f(z) = z$ and the uniqueness of the fixed point, condition 2) may appear with β somewhat larger than $\frac{1}{2}$. Namely, we add the following result, which is less symmetric but obviously an improvement of Theorem 6.

Theorem 7. *Let M be a metric space, $\beta < 1$, and $f: M \to M$ a function such that for each couple of different points in M at least one of the conditions $1°$ and $3°$ of Theorem 4, and 2) of Theorem 1 (with $\beta < 1$) is satisfied. If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point z in M, then z is a unique fixed point of f.*

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Eingegangen am 26. 7. 1971

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