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Matematica. — *A theorem on fixed points.* Nota di TUDOR ZAMFIRESCU, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota vengono ampliati alcuni teoremi sulla presenza di un unico elemento unito in trasformazioni di spazi metrici generalizzati in se stessi.

Let (X, d) be a complete metric space or a generalized complete metric space (i.e. a complete metric space in which the distance-function d may attain ∞) and consider the function $f: X \to X$ and the number $\alpha \in (0, 1)$. We list the following properties f may possess at some couple of distinct points x, y:

1) $d(f(x), f(y)) < \alpha d(x, y)$,

2) $d(f(x), f(y)) < \alpha d(x, f(x))$,

3) $d(f(x), f(y)) < \alpha d(y, f(y))$,

4) $d(f(x), f(y)) < \frac{\alpha}{2}(d(x, f(y)) + d(y, f(x)))$.

The function f is called an *α-pseudocontraction* if at each couple $x, y \in X$ with $0 < d(x, y) < \infty$, at least one of the conditions 1)–4) is satisfied; f is called an *α-contraction* if at each couple $x, y \in X$ with $0 < d(x, y) < \infty$, condition 1) is satisfied.

The following proposition is a generalization of a well-known result of Banach and also of theorems of R. Kannan [1], S. Reich [3], I. A. Rus [4], R. M. Tiberio Bianchini [6].

PROPOSITION 1 [7]. *In a complete metric space, each α-pseudocontraction* $(\alpha \in (0, 1))$ *has a unique fixed point.*

The next result is due to S. P. Singh and C. W. Norris, and constitutes an improvement of a theorem of W. A. J. Luxemburg [2].

PROPOSITION 2 [5]. *Suppose* X *is a generalized complete metric space,* $f: X \to X$, $\alpha \in (0, 1)$ *and* $p \in N$ (1). *If*

a) for each point $x_0 \in X$, *there exists* $m \in N$ *such that* $d(f^{pm}(x_0), f^{pn}(x_0)) < \infty$ *for every* $n \in N$ *with* $n > m$,

b) f^p *is an α-contraction,*

c) the distance between every two fixed points of f^p *is finite,*

then f *has a unique fixed point in* X.

(*) Nella seduta del 16 giugno 1972.

(1) N denotes the set of natural numbers: $\{1, 2, 3, \cdots\}$.

Both Propositions 1 and 2 will be generalized here. Before presenting the main result, we record the following

PROPOSITION 3 [7]. *Suppose* M *is a set in a complete metric space* X, $f : M \to M$, $\alpha \in (0, 1)$, *and for each couple of different points* $x, y \in M$, *at least one of the conditions* 1)–4) *is satisfied. Then, for* $x_0 \in M$, *the sequence* $\{f^n(x_0)\}_{n=0}^{\infty}$ *converges to a point in* X *independent on the choice of* x_0.

THEOREM. *Suppose* X *is a generalized complete metric space*, $f : X \to X$, $\alpha \in (0, 1)$ *and* $p \in N$. *If*

a) *there exists a point* $x_0 \in X$ *and two distinct numbers* $m, n \in N$ *such that* $d(f^{pm}(x_0), f^{pn}(x_0)) < \infty$,

b) $f^{p|n-m|}$ *is an* α*–pseudocontraction*,

c) *the distance between every two fixed points of* $f^{p|n-m|}$ *is finite*,

then f *has a unique fixed point in* X.

Proof. Suppose $m < n$. Consider the set $S \subset X$ consisting of the elements of the sequence $\sigma = \{f^{pm+sp(n-m)}(x_0)\}_{s=0}^{\infty}$, and the restriction $f^{p(n-m)} | S$. If $f^{pm}(x_0) = f^{pn}(x_0)$, then $f^{pm}(x_0)$ is a fixed point of $f^{p(n-m)}$, and σ converges to $f^{pm}(x_0)$. If $f^{pm}(x_0) \neq f^{pn}(x_0)$, then

$$0 < d(f^{pm}(x_0), f^{pm+p(n-m)}(x_0)) < \infty .$$

and since $f^{p(n-m)}$ is an α–pseudocontraction, it follows that

$$d(f^{pm+p(n-m)}(x_0), f^{m+2p(n-m)}(x_0)) < \infty .$$

Analogously, for each s,

$$d(f^{pm+sp(n-m)}(x_0), f^{pm+(s+1)p(n-m)}(x_0)) < \infty .$$

Then, it easily follows that $d | S \times S$ does not attain ∞; moreover, $\{x \in X : d(x, f^{pm}(x_0)) < \infty\}$ is obviously a complete metric space (including the set S). Thus, we may use Proposition 3. It results that σ converges to some point $z \in X$. Because

$$\begin{aligned} d(f^{p(n-m)}(z), z) &\leq d(f^{p(n-m)}(z), f^{pm+(s+1)p(n-m)}(x_0)) \\ &\quad + d(f^{pm+(s+1)p(n-m)}(x_0), z) \\ &\leq \alpha d(z, f^{pm+sp(n-m)}(x_0)) + d(f^{pm+(s+1)p(n-m)}(x_0), z) \end{aligned}$$

for each $s \in N$, we have $d(f^{p(n-m)}(z), z) = 0$, and therefore z is a fixed point of $f^{p(n-m)}$.

Suppose now z' is a fixed point of $f^{p(n-m)}$. Following condition (*c*), $d(z, z') < \infty$; then $d | S^* \times S^*$, where $S^* = S \cup \{z, z'\}$, takes only finite values. Also, one easily verifies that S^* is a complete subspace of X. Consider $f^{p(n-m)} | S^*$ and apply Proposition 1: it follows $z = z'$.

Thus, $f^{p(n-m)}$ has a unique fixed point. Consequently, f has a unique fixed point too. The proof is achieved.

We finally observe that: if $d(x, y) < \infty$ for every couple $x, y \in X$ and if $p = |n - m| = 1$, one obtains Proposition 1; if f^p is an α-contraction and if $|n - m| = 1$, one obtains an improvement of Proposition 2.

References

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