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Matematica. — *Generalizations of Banach's fixed point theorem.* Nota (*) di TUDOR ZAMFIRESCU, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Questa Nota contiene alcune generalizzazioni del Teorema di Banach sul punto unito, generalizzazioni relative al caso che per ogni m punti si sappia soltanto che vengono contratte almeno k delle loro coppie, m e k essendo numeri fissi, con $k < m(m-1)/2$.

Banach's classical theorem on contractions has been generalized in many ways. We intend to present here another direction in which generalizations are possible; we believe this direction to be new, but it is difficult to guarantee it!

Let M be a metric space and α a real number in $(0, 1)$. We say that a couple of points (x, y) *is contracted* through the function $f : M \to M$ if $x \neq y$ and

$$d(f(x), f(y)) \leq \alpha d(x, y),$$

where d is the distance function on M. We shall also call f *an* (m, k)*-contraction* if for every m points in M, at least k of the $\binom{m}{2}$ couples determined by them are contracted through f.

Each proposition of the type "each (m, k)-contraction in a complete metric space has a unique fixed point" is a generalization of Banach's theorem if $k < \binom{m}{2}$. However, we easily see that almost always unessential additional assumptions have to be made.

(*) Pervenuta all'Accademia il 29 settembre 1972.

We ask for an (as complete as possible) enumeration of all valid generalizations of this type (under as weak additional assumptions as possible).

Clearly, not all such propositions are true (at least in the absence of any additional assumptions): Take $n = 3$, $k = 1$, $M = \{0, 1, \alpha, \alpha^2, \alpha^3, \cdots\}$, and $f : M \to M$ defined by

$$f(0) = 0 \quad , \quad f(1) = 1 \quad , \quad f(\alpha^n) = \alpha^{n+1} \qquad (n = 1, \cdots, \infty).$$

We shall present here just a few partial results.

LEMMA 1. *Each $(m, (m^2 - 3m + 4)/2)$-contraction is continuous.*

Proof. Let $f : M \to M$ be an $(m, (m^2 - 3m + 4)/2)$-contraction, $a \in M$ and $\varepsilon > 0$. If in $S(a, \varepsilon/2m)$ there are less than m points (1), then a is isolated and f is trivially continuous in a. In the other case, choose $m - 1$ (distinct) points $x_1, \cdots, x_{m-1}$ in $S(a, \varepsilon/2m) - \{a\}$. At least

$$\frac{m^2 - 3m + 4}{2} = \binom{m}{2} - (m - 2)$$

couples from all those determined by $x_1, \cdots, x_{m-1}$, a are contracted through f, i.e. at most $m - 2$ couples (of distinct points) are not contracted. Now, two cases are possible:

Case I. *(x_1, a) is contracted.*

Then, clearly, $f(x_1) \in S(f(a), \varepsilon/2m)$.

Case II. *(x_1, a) is not contracted.*

Since each graph on m points having at least $\binom{m}{2} - (m - 2)$ edges is connected, there exists a path (of length less than m) joining x_1 and a in the graph with point-set $x_1, \cdots, x_{m-1}$, a and edge-set $\{e : e \text{ is contracted}\}$. Let $x_1, y_1, \cdots, y_p$, a be this path ($p \leq m - 2$). We have

$$\begin{aligned} d(f(x_1), f(a)) &\leq d(f(x_1), f(y_1)) + d(f(y_1), f(y_2)) \\ &\quad + \cdots + d(f(y_{p-1}), f(y_p)) + d(f(y_p), f(a)) \\ &\leq d(x_1, y_1) + d(y_1, y_2) + \cdots + d(y_{p-1}, y_p) + d(y_p, a) \\ &\leq 2p \frac{\varepsilon}{2m} + \frac{\varepsilon}{2m} < \frac{p+1}{m} \varepsilon < \varepsilon . \end{aligned}$$

Thus, in both cases, $f(x_1) \in S(f(a), \varepsilon)$, which proves the lemma.

By $f^n(x)$ we shall mean $f(f^{n-1}(x))$; $f^0(x) = x$.

LEMMA 2. *Suppose* M *is a metric space and* $f : M \to M$ *is an* $(m, (m^2 - 3m + 4)/2)$*-contraction in* M. *If, for some* $x_0 \in M$, $\{f^n(x_0)\}_{n=0}^{\infty}$

(1) $S(a, \beta) = \{x \in M : d(x, a) \leq \beta\}$.

has an infinity of distinct points and converges to a point z in M, *then z is a unique fixed point of f.*

Proof. From the continuity of f, assured by Lemma 1, it follows that z is a fixed point for f. Now, suppose z' is another fixed point of f. Then one can find a number p' such that $(z', f^n(x_0))$ is contracted for no $n \geq p'$. Choose $m - 1$ distinct points $f^{p_1}(x_0), f^{p_2}(x_0), \cdots, f^{p_{m-1}}(x_0)$, with $p_i \geq p'$ $(i = 1, \cdots, m-1)$. They plus the point z' admit $m - 1$ non-contracted couples, namely $(z', f^{p_1}(x_0)), \cdots, (z', f^{p_{m-1}}(x_0))$, which contradicts the hypothesis. Hence f has exactly one fixed point.

THEOREM 1. *If the complete metric space* M *has at most finitely many isolated points, f is an $(m, (m^2 - 3m + 4)/2)$-contraction in* M *and, for some $x_0 \in$ M, $\{f^n(x_0)\}_{n=0}^{\infty}$ has an infinity of distinct points, then f has a unique fixed point.*

Proof. There exists a number r such that $\{f^n(x_0)\}_{n=r}^{\infty}$ contains no isolated points. Then, suppose $(f^q(x_0), f^{q+1}(x_0))$ is not contracted through f $(q \geq r)$. Consider the sequence of distinct points $\{s^n\}_{n=1}^{\infty}$ converging to $f^{q+1}(x_0)$. By Lemma 1, $f(s^n) \to f^{q+2}(x_0)$. Obviously, one can find again a number p such that $(f^q(x_0), s^n)$ is contracted for no $n \geq p$. Then the m points $f^q(x_0), f^{q+1}(x_0), s^p, s^{p+1}, \cdots, s^{p+m-3}$ admit $m - 1$ non-contracted couples, namely $(f^q(x_0), f^{q+1}(x_0)), (f^q(x_0), s^p), \cdots, (f^q(x_0), s^{p+m-3})$, which contradicts the hypothesis. Hence, A_r denoting the set of all points in $\{f^n(x_0)\}_{n=r}^{\infty}$, $f | A_r$ is a usual contraction and therefore, by the Picard–Banach contraction principle, $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point $z \in$ M.

Now, from Lemma 2, it follows that z is a unique fixed point of f.

THEOREM 2. *Suppose the metric space* M *has at most finitely many isolated points and f is an $(m, (m^2 - 3m + 4)/2)$-contraction in* M. *If, for some $x_0 \in$ M, $\{f^n(x_0)\}_{n=0}^{\infty}$ has an infinity of distinct points and also a limit point z in* M, *then z is a unique fixed point of f.*

Proof. Like in the proof of Theorem 1, we find a number r such that $f | A_r$ is a usual contraction, where A_r denotes the set of all points in $\{f^n(x_0)\}_{n=r}^{\infty}$. Then, like in the standard proof of the Picard–Banach contraction principle, we can show that $\{f^n(x_0)\}_{n=r}^{\infty}$ is a Cauchy sequence. Hence, this sequence must converge to its limit point z. Now, following Lemma 2, z is a unique fixed point of f.

THEOREM 3. *Suppose* M *is a set in a complete metric space* X *and has at most finitely many isolated points. Let f be an $(m, (m^2 - 3m + 4)/2)$-contraction in* M *and denote $s_x = \{f^n(x)\}_{n=0}^{\infty}$, for $x \in$ M. If s_{x_0} has an infinity of distinct points for some $x_0 \in$ M, then there exists a point in* X, *to which all s_x with $x \in$ M converge.*

Proof. Like in the proof of Theorem 1, we show that s_{x_0} converges to some point $z \in X$. Take now $y \in M$. Suppose s_y does not converge to z. Two cases are possible:

Case I. s_y *contains finitely many distinct points*. Then s_y has a limit point $w \in M$, different from z. We can find a number $t > 0$ with the property that $f^t(w) = w$. Hence there exists another number $u < t$ such that $d(f^u(w), z) \leq d(f^{u+1}(w), z)$. Obviously, $f^{u+1}(w) \neq z$, otherwise also $f^u(w) = z$ and s_y would converge to z. We choose p', like in the proof of Lemma 2, such that $(f^u(w), f^n(x_0))$ is contracted for no $n \geq p'$. Following further the same argument as in the mentioned proof (of Lemma 2), we find a contradiction.

Case II. s_y *contains infinitely many distinct points*. Then s_y converges to a point in X different from z and there exist two numbers p' and q such that $(f^n(x_0), f^q(y))$ is contracted for no $n \geq p'$. Now, again like in the proof of Lemma 2, we can find a contradiction.

Therefore s_y converges to z and the proof is finished.

Similar are the proofs of the next three theorems; therefore the last two of them we only state.

THEOREM 4. *If* M *is a complete metric space, f is an* $(m, [(m^2 - 2m + 3)/2])$–*contraction in* M *and, for some* $x_0 \in M$, $\{f^n(x_0)\}_{n=0}^{\infty}$ *has an infinity of distinct points, then f has a unique fixed point.*

Proof. Let

$$B = \{ q \geq 0 : (f^q(x_0), f^{q+1}(x_0)) \text{ is not contracted} \}$$

and suppose

$$\text{card } B \geq [m/2] .$$

Take a subset C of B with card $C = [m/2]$ and put

$$D = \{ q \geq 1 : q - 1 \in C \} .$$

We have obviously card C = card D and therefore

$$\text{card } (C \cup D) \leq 2 [m/2] \leq m .$$

Now, if E is a set of positive integers including $C \cup D$ and with card $E = m$, and $F = \{f^q(x_0) : q \in E\}$, then evidently among all couples of distinct points in $F \times F$ there are at least $[m/2]$ which are not contracted. On the other hand, since f is an $(m, [(m^2 - 2m + 3)/2])$–contraction, there are in $F \times F$ at least $[(m^2 - 2m + 3)/2]$ contracted couples, which would mean that $F \times F$ possesses at least

$$[m/2] + [(m^2 - 2m + 3)/2] = \binom{m}{2} + 1$$

couples of distinct points, which is absurd.

Hence card $B < [m/2]$ and we may choose a number r such that $f \mid A_r$ is a usual contraction (A_r being the set of all points in $\{f^n(x_0)\}_{n=r}^{\infty}$). Now, as in the proof of Theorem 1, we conclude that f has a unique fixed point in M.

THEOREM 5. *Suppose* M *is a metric space and* f *is an* $(m, [(m^2 - 2m + 3)/2)$–*contraction in* M. *If, for some* $x_0 \in M$, $\{f^n(x_0)\}_{n=0}^{\infty}$ *has an infinity of distinct points and also a limit point* z *in* M, *then* z *is a unique fixed point of* f.

THEOREM 6. *Let* M *be a set in a complete metric space* X, f *an* $(m, [(m^2 - 2m + 3)/2])$–*contraction in* M, *and* $s_x = \{f^n(x)\}_{n=0}^{\infty}$, *for* $x \in M$. *If* s_{x_0} *has an infinity of distinct points for some* $x_0 \in M$, *then there exists a point in* X, *to which all* s_x *with* $x \in M$ *converge.*

ROMA, 1973 — Dott. G. Bardi, Tipografo dell'Accademia Nazionale dei Lincei