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On k-Path Hamiltonian Graphs

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TUDOR ZAMFIRESCU (Dortmund)

Résumé. - *Dans cette Note on établit un résultat analogue à un théorème de C. Berge dans la théorie des graphes.*

An undirected graph G, without loops and multiple edges, is said to be:

— *k-path hamiltonian* if every path in G of length at most k can be extended to a hamiltonian circuit of G (1),

— *k-line hamiltonian* if every set of at most k lines constituting disjoint paths in G is included in a hamiltonian circuit of G.

Recently, C. BERGE proved the following theorem, which generalizes previous results of O. ORE [6], [7], [8], P. ERDÖS and T. GALLAI [3], L. PÓSA [9], J. A. BONDY [2], H. V. KRONK [4], [5]:

THEOREM 1 [1]. - *Let G be a (simple) graph on n points $x_1, \ldots, x_n$, such that*

$$d_G(x_1) \leqslant \ldots \leqslant d_G(x_n) .$$

Let k be an integer, $0 \leqslant k \leqslant n-2$. If

$$\left.\begin{array}{l} i<j \\ d_G(x_i) \leqslant i+k \\ d_G(x_j) \leqslant j+k-1 \end{array}\right\} \Rightarrow d_G(x_i)+d_G(x_j) \geqslant n+k ,$$

then G is k-line hamiltonian.

(1) The terminology and the notations used here are those of C. BERGE [1], except: for « chaine élémentaire » we use the word « path », $P(G)$ and $E(G)$ respectively denote the point- and the edge-set of G, and « circuit » is used for « cycle ».

From Theorem 1 one can immediately derive

COROLLARY 1. – *Under the hypotheses of Theorem* 1, *the graph* G *is* k*-path hamiltonian.*

We prove now the following wery simple

THEOREM 2. – *If for each connected subgraph* G' *of* G, *with at most* $k-1$ *vertices, the subgraph* G'' *of* G *with* $P(G'')=P(G)-P(G')$ *is hamiltonian-connected, then* G *is* k*-path hamiltonian.*

PROOF. – Let K be a path of length at most k in G. Consider the subgraph G' of G with $P(G')=P(K)-\{a, b\}$, where a and b are the end-points of K. Since G' is connected and has at most $k-1$ vertices, the subgraph G'' of G with $P(G'')=P(G)-P(G')$ is hamiltonian-connected, whence a and b are joined by a hamiltonian path Π in G''. Then $K \cup \Pi$ is a hamiltonian circuit of G.

COROLLARY 2 [10]. – *Let* G *be a graph on* n *points. If each subgraph of* G *on at least* $n-k+1$ *vertices is hamiltonian-connected, then* G *is* k*-path hamiltonian* $(1 \leqslant k \leqslant n-2)$.

The main aim of this Note is to establish the exact relation ship between Corollaries 1 and 2.

First we prove that Corollary 2 is not weaker than Corollary 1.

Suppose the hypotheses of Theorem 1 are satisfied for a graph G on n points and let G' be a subgraph of G on at least $n-k+1$ points. Choose $a, b \in P(G')$ arbitrarily. Consider the set

$$\{c_1, \ldots, c_l\} = P(G) - P(G') \qquad (l \leqslant k-1).$$

Construct the graph H such that $P(H)=P(G)$ and $E(H)=E(G) \cup V$, where

$$V=\{[a, c_1], [c_1, c_2], \ldots, [c_{l-1}, c_l], [c_l, b]\} .$$

It is easily seen that H also satisfies the hypotheses of Theorem 1. Then, following Corollary 1, the path Π with $E(\Pi)=V$ may be extended to a hamiltonian circuit Θ of H. Thus, one obtains the subgraph Π^* of Θ with $P(\Pi^*)=(P(\Theta)-P(\Pi)) \cup \{a, b\}$, which is a hamiltonian path in G', joining a with b.

Now, we show by an example that the domain of application of Corollary 2 is larger than that of Corollary 1, which proves that Corollary 2 is strictly stronger than Corollary 1 ([2]).

([2]) This fact has been stated (without proof) in a footnote of [10].

Let A, B, C, D be four pair-wise disjoint sets of points, each of cardinality $k+1$, and a, b, c, d four points not in $A \cup B \cup C \cup D$. Let G be a graph such that

$$P(G) = \{a, b, c, d\} \cup A \cup B \cup C \cup D ;$$

and

$$\begin{aligned} E(G) = \{[a, b],\ [b, c],\ [c, d]\} \cup \\ \cup \{[a, x]: x \in A\} \cup \{[b, x]: x \in B\} \cup \\ \cup \{[c, x]: x \in C\{ \cup \{[d, x]: x \in D\} \cup \\ \cup \{[x, y]: x, y \in A \cup B \cup C \cup D, x \neq y\} . \end{aligned}$$

We prove first that the hypothesis of Corollary 2 is fullfiled. Clearly, $n = 4k + 8$; $n - k + 1 = 3k + 9$. Let G' be a subgraph of G on at least $3k + 9$ vertices,

$$\begin{aligned} \{a_1, \ldots, a_p\} = P(G') \cap A , \\ \{b_1, \ldots, b_q\} = P(G') \cap B , \\ \{c_1, \ldots, c_r\} = P(G') \cap C , \\ \{d_1, \ldots, d_s\} = P(G') \cap D , \end{aligned}$$

and

$$E = P(G') - (A \cup B \cup C \cup D) .$$

We have to distinguish between 10 essentially different Cases: I: $E = \emptyset$, II: $E = \{a\}$, III: $E = \{b\}$, IV: $E = \{a, b\}$, V: $E = \{a, c\}$, VI: $E = \{a, d\}$, VII: $E = \{b, c\}$, VIII: $E = \{a, b, c\}$, IX: $E = \{a, b, d\}$, X: $E = \{a, b, c, d\}$. For all Cases I—IX, $p, q, r, s \geqslant 3$. (Suppose, on the contrary, $p \leqslant 2$. Then $q, r, s \leqslant k + 1$ and

$$\text{card}\, P(G') \leqslant p + q + r + s + 3 \leqslant 2 + 3(k + 1) + 3 = 3k + 8,$$

wich is absurd.) Analogously, for the Case X, $p, q, r, s \geqslant 2$. In Case I, G' is complete and therefore hamiltonian-connected. For Cases II—X, one proves that for each couple of vertices $x, y \in P(G')$, there is a hamiltonian path in G' with end-points x, y. The next table gives hamiltonian paths connecting essentially different pairs of vertices in G', for the Case II. Analogously, one may complete similar tables for Cases III—X.

x	y	Path
a	a_1	$[a, a_2, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r, d_1, \ldots, d_s, a_1]$
a	b_1	$[a, a_1, \ldots, a_p, b_2, \ldots, b_q, c_1, \ldots, c_r, d_1, \ldots, d_s, b_1]$
a	c_1	$[a, a_1, \ldots, a_p, b_1, \ldots, b_q, c_2, \ldots, c_r, d_1, \ldots, d_s, c_1]$
a	d_1	$[a, a_1, \ldots, a_p, b_1 \ldots, b_q, c_1, \ldots, c_r, d_2, \ldots, d_s, d_1]$
a_1	a_2	$[a_1, a, a_3, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r, d_1, \ldots, d_s, a_2]$
a_1	b_1	$[a_1, a, a_2, \ldots, a_p, b_2, \ldots, b_q, c_1, \ldots, c_r, d_1, \ldots, d_s, b_1]$
a_1	c_1	$[a_1, a, a_2, \ldots, a_p, b_1, \ldots, b_q, c_2, \ldots, c_r, d_1, \ldots, d_s, c_1]$
a_1	d_1	$[a_1, a, a_2, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r, d_2, \ldots, d_s, d_1]$
b_1	b_2	$[b_1, a_1, a, a_2, \ldots, a_p, b_3, \ldots, b_q, c_1, \ldots, c_r, d_1, \ldots, d_s, b_2]$
b_1	c_1	$[b_1, a_1, a, a_2, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r, d_1, \ldots, d_s, c_1]$

Now, let us show that the hypothesis of Corollary 1 is not satisfied. Indeed, if

$$\{y_1, \ldots, y_{4k+4}\} = A \cup B \cup C \cup D,$$

then, by putting $a = x_1$, $d = x_2$, $b = x_3$, $c = x_4$ and $y_i = x_{i+4}$ ($i = 1, \ldots, 4k+4$), we have

$$d_G(x_1) \leqslant \ldots \leqslant d_G(x_{4k+8})$$

and, contrarily to the hypothesis of Corollary 1, in this sequence there are indices i_0, j_0 such that $i_0 < j_0$, $d_G(x_{i_0}) \leqslant i_0 + k$ and $d_G(x_{j_0}) \leqslant j_0 + k - 1$, but $d_G(x_{i_0}) + d_G(x_{j_0}) < n + k$; take, for instance, $i_0 = 3$ and $j_0 = 4$: then

$$d_G(x_3) = d_G(b) = 3 + k,$$

$$d_G(x_4) = d_G(c) = 4 + k - 1,$$

but

$$d_G(x_3) + d_G(x_4) = 2k + 6 < 5k + 8.$$

D. Römer proved that for $n-4 \leqslant k \leqslant n-2$ Corollaries 1 and 2 are equivalent (private communication).

Thus, the relationship between Corollaries 1 and 2 is completely established. In other words, we proved the following strengthening of Corollary 1:

THEOREM 3. – *Under the hypoteses of Theorem 1, each subgraph of G on at least $n-k+1$ vertices is hamiltonian-connected.*

That Corollary 2 is strictly weaker than Theorem 2, it may be seen from the following example.

Consider the set

$$M = \{(x_1, \ldots, x_d) \in R^d : x_j \in N,\ x_j \leqslant m,\ j = 1, \ldots, d\}$$

and the graph G with $P(G) = M$ and

$$\begin{aligned} E(G) = \{[a, b] : \ & a = (x_1, \ldots, x_{c-1}, x_c, x_{c+1}, \ldots, x_d), \\ & b = (x_1, \ldots, x_{c-1}, x_c + 1, x_{c+1}, \ldots, x_d), \\ & a, b \in M,\ c \in \{1, \ldots, d\}\}\,. \end{aligned}$$

Then, for m large enough and

$$d-1 \leqslant k \leqslant 2d-8 \qquad (d \geqslant 7),$$

Theorem 2 applies, but Corollary 2 not.

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