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SOME FIXED POINT THEOREMS IN METRIC SPACES

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The classical fixed point and contraction theorems have been generalized in various ways. We continue here a series of generalizations contained in the papers listed at the end of this note.

1. – Our purpose in this first chapter is to generalize Theorem 5 of [3]. Many theorems from [3] have been improved in [4] and among them also Theorem 5. Here we give a further improvement.

Theorem 1. Let M be a metric space, N a natural number, and $f\colon M \to M$ a function such that for each non-fixed point $x \in M$,

$$d(f(x), f^2(x)) < d(x, f(x)) ,$$

and for each couple of different points $x, y \in M$, at least one of the following conditions is satisfied:

$$d(f(x), f(y)) \neq d(x, y)$$

$$d(f(x), f(y)) < N(d(x, f(x)) + d(y, f(y)))$$

$$d(f(x), f(y)) \neq d(x, f(y))$$

$$d(f(x), f(y)) \neq d(y, f(x)) .$$

If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M and the functions f and f^2 are continuous at z then z is a unique fixed point of f.

(*) Nella seduta del 28 maggio 1972.

Proof. As in the argument used at the proof of Theorem 4 in [3], let $\{f^{n_i}(x_0)\}_{i=1}^{\infty}$ be a subsequence of $\{f^n(x_0)\}_{n=0}^{\infty}$ such that

$$\lim_{i\to\infty} f^{n_i}(x_0) = z .$$

Then

$$\lim_{i\to\infty} f^{n_i+1}(x_0) = f(z) , \qquad \lim_{i\to\infty} f^{n_i+2}(x_0) = f^2(z) .$$

Let m be an arbitrary nonnegative integer. Since either

$$f^m(x_0) = f^{m+1}(x_0)$$

or

$$d(f^{m+1}(x_0), f^{m+2}(x_0)) < d(f^m(x_0), f^{m+1}(x_0)) ,$$

the sequence $\{d(f^n(x_0), f^{n+1}(x_0))\}_{n=0}^{\infty}$ is convergent and

$$d(z, f(z)) = \lim_{i\to\infty} d(f^{n_i}(x_0), f^{n_i+1}(x_0)) = \lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0))$$

$$= \lim_{i\to\infty} d(f^{n_i+1}(x_0), f^{n_i+2}(x_0)) = d(f(z), f^2(z)) .$$

This equality shows that $f(z) = z$. Finally, suppose $z' \neq z$ and $f(z') = z'$. Then none of the (last) four conditions of the statement is satisfied at the couple of points $x = z$, $y = z'$. The proof is finished.

Theorem 1 implicitely generalizes results of Edelstein [1] and Singh [2] (see also [3] and [4]).

2. - Here we shall establish a result improving Theorem 7 of [3], which was the only one not already improved in [4], and in fact unimprovable by the means of the main idea of [4].

Theorem 2. Let M be a metric space, α, β real numbers with $\alpha > 1$, $0 < \beta < 1$, and $f : M \to M$ a function such that for each couple of different points $x, y \in M$, at least one of the following conditions is satisfied:

$$d(f(x), f(y)) < d(x, y)$$

$$d(x, y) < d(f(x), f(y)) < \alpha d(x, y)$$

$$d(f(x), f(y)) < \beta(d(x, f(x)) + d(y, f(y)))$$

$$d(f(x), f(y)) < \beta d(x, f(y)) + (1 - \beta)\, d(y, f(x))$$

$$d(f(x), f(y)) < \beta d(y, f(x)) + (1 - \beta)\, d(x, f(y)) .$$

If for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point z in M, then z is a unique fixed point of f.

Proof. First, we prove that z is a fixed point of f. Suppose $f(z) \neq z$ and consider the ball

$$B = \left\{ x \in M : d(x, z) \leqslant d((z, f(z)) \cdot \min \left\{ \frac{1}{\alpha + 1}, \frac{\beta}{2}, \frac{1-\beta}{1+2\beta} \right\} \right\}.$$

We observe that

$$d(x, f(z)) \geqslant d(z, f(z)) \cdot \max \left\{ \frac{\alpha}{2\alpha + 1}, \frac{2-\beta}{2}, \frac{3\beta}{1+2\beta} \right\}$$

for every point $x \in B$. Let N be such that $f^n(x_0) \in B$ for each $n \geqslant N$. Taking now $x = f^N(x_0)$ and $y = z$, the conditions in the statement imply that at least one of the following inequalities holds:

$$d(f^{N+1}(x_0), f(z)) < \alpha d(f^N(x_0), z)$$

$$d(f^{N+1}(x_0), f(z)) < \beta(d(f^N(x_0), f^{N+1}(x_0)) + d(z, f(z)))$$

$$d(f^{N+1}(x_0), f(z)) < \beta d(f^N(x_0), f(z)) + (1-\beta)\,(z, f^{N+1}(x_0))$$

$$d(f^{N+1}(x_0), f(z)) < \beta d(z, f^{N+1}(x_0) + (1-\beta)\, d(f^N(x_0), f(z)) .$$

On the other hand, the first one contradicts

$$\alpha d(f^N(x_0), z) \leqslant \alpha d(z, f(z)) \cdot \frac{1}{2\alpha + 1} \leqslant d(f^{N+1}(x_0), f(z)) ,$$

the second contradicts

$$\begin{aligned}
&\beta(d(f^N(x_0), f^{N+1}(x_0)) + d(z, f(z))) \\
&\qquad \leqslant \beta(d(f^N(x_0), z) + d(z, f^{N+1}(x_0)) + d(z, f(z))) \\
&\qquad \leqslant \beta d(z, f(z)) \left(\frac{1-\beta}{1+2\beta} + \frac{1-\beta}{1+2\beta} + 1 \right) \\
&\qquad \leqslant d(f^{N+1}(x_0), f(z)) ,
\end{aligned}$$

the third contradicts

$$\beta d(f^N(x_0), f(z)) + (1-\beta)\, d(z, f^{N+1}(x_0))$$

$$\leqslant \beta d(f^N(x_0), z) + \beta d(z, f(z)) + (1-\beta)\, d(z, f^{N+1}(x_0))$$

$$\leqslant d(z, f(z))\left(\min\left\{\frac{\beta}{2}, \frac{1-\beta}{1+2\beta}\right\} + \beta\right)$$

$$\leqslant d(z, f(z))\left(\frac{1-\beta}{2} + \beta\right)$$

$$\leqslant d(z, f(z)) \cdot \max\left\{\frac{2-\beta}{2}, \frac{3\beta}{1+2\beta}\right\}$$

$$\leqslant d(f^{N+1}(x_0), f(z)) ,$$

and the fourth contradicts

$$\beta d(z, f^{N+1}(x_0)) + (1-\beta)\, d(f^N(x_0), f(z))$$

$$\leqslant \beta(d(z, f^{N+1}(x_0)) + (1-\beta)\, d(f^N(x_0), z)) + (1-\beta)\, d(z, f(z)))$$

$$\leqslant d(z, f(z))\left(\frac{\beta}{2} + 1 - \beta\right)$$

$$\leqslant d(f^{N+1}(x_0), f(z)) .$$

Hence $f(z) = z$.

To show that the fixed point z is unique, suppose, on the contrary, that $f(z') = z'$ for some other point z' too. Then

$$d(f(z), f(z')) = d(z, z')$$

$$d(f(z), f(z')) > \beta(d(z, f(z)) + d(z', f(z')))$$

$$d(f(z), f(z')) = \beta d(z, f(z')) + (1-\beta)\, d(z', f(z))$$

$$d(f(z), f(z')) = \beta d(z', f(z)) + (1-\beta)\, d(z, f(z'))$$

and none of the five conditions in the statement is satisfied at the points z and z'. This completes the proof.

3. – In this and the next chapter we present two theorems of a different nature. Namely, the space will be divided into several parts having different special properties and the function will satisfy in each couple of points conditions depending on the part in which they lie.

Consider the metric space M, the set $L \subset M$, the function $f\colon L \to L$, the real number α, the real positive number $\beta < 1$, and the following conditions, each of which may be satisfied by f at a couple of points $x, y \in L$:

1) $d(f(x), f(y)) \leqslant \beta d(x, y)$

2) $d(f(x), f(y)) \leqslant \beta d(x, f(x))$

3) $d(f(x), f(y)) \leqslant \beta d(y, f(y))$

4) $d(f(x), f(y)) \leqslant \frac{\beta}{2}(d(x, f(y)) + d(y, f(x)))$

5) $d(f(x), f(y)) < d(x, y)$

6) $d(f(x), f(y)) < \beta d(x, f(x)) + (1-\beta)\, d(y, f(y))$

7) $d(f(x), f(y)) < \beta d(y, f(y)) + (1-\beta)\, d(x, f(x))$

8) $d(f(x), f(y)) < \frac{1}{2}(d(x, f(y)) + d(y, f(x)))$

9) $d(f(x), f(y)) \neq d(x, y)$

10) $d(f(x), f(y)) \leqslant \alpha(d(x, f(x)) + d(y, f(y)))$

11) $d(f(x), f(y)) \neq d(x, f(y))$

12) $d(f(x), f(y)) \neq d(y, f(x))$

13) $d(f(x), f^2(x)) < d(x, f(x))$.

Lemma 1. If M is complete and at each couple of different points in L at least one of the Conditions 1)-4) is satisfied, then for every point $x_0 \in L$ the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ converges in M.

This lemma doesn't need any proof, being just a weaker form of Theorem 4 from [4].

Lemma 2. If $L = M$, if at each couple of different points in M at least one of the Conditions 5)-8) is verified, and if for some $x_0 \in M$ the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M in which f and f^2 are continuous, then z is a unique fixed point of f.

Proof. Since each of the Conditions 6) and 7) implies

$$d(f(x), f(y)) < \max\{d(x, f(x)), d(y, f(y))\}\,,$$

at least one of the Conditions 1°-4° of the statement of Theorem 5 in [4] is satisfied. Whence the conclusion of Lemma 2.

Now, let L_1 and L_2 be subsets of M.

Theorem 3. If L_1 is complete, L_2 is compact, $M = L_1 \cup L_2$, at each couple of different points in M at least one of the Conditions 5)-8) is satisfied, at each couple of different points in $M - L_2$ at least one of the Conditions 1)-4) is satisfied, and the functions f and f^2 are continuous on L_2, then f has a unique fixed point in M.

Proof. Let $x_0 \in M$. Two cases are possible:

I. There exists an integer N, such that $f^n(x_0) \in M - L_2$ for all $n \geqslant N$.

II. There exists a subsequence $\{f^{n_i}(x_0)\}_{i=1}^{\infty}$ of $\{f^n(x_0)\}_{n=0}^{\infty}$, lying in L_2.

Case I. We consider the complete metric space L_1, the set $L_3 \subset L_1$ of all points of $\{f^n(x_0)\}_{n=N}^{\infty}$ and the function $f|L_3$. By Lemma 1, $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to some point z in L_1. Now, we can verify that at least one of the five conditions in Theorem 2 is fulfilled. Indeed, Conditions 5) coincides with the first condition of Theorem 2, Condition 6) implies the third one of Theorem 2 since

$$\beta d(x, f(x)) + (1-\beta)\, d(y, f(y)) \leqslant \gamma(d(x, f(x)) + d(y, f(y))) ,$$

where $\gamma = \max\{\beta, 1-\beta\}$, analogously Condition 7) implies the third one of Theorem 2, and finally Condition 8) implies one of the last two conditions of Theorem 2 because

$$\begin{aligned} &\tfrac{1}{2}(d(x, f(y)) + d(y, f(x))) \\ &\quad \leqslant \tfrac{1}{2}(d(x, f(y)) + d(y, f(x))) + (\gamma - \tfrac{1}{2})A - (\gamma - \tfrac{1}{2})B \\ &\quad \leqslant \gamma A + (1-\gamma)B , \end{aligned}$$

where

$$A = \max\{d(x, f(y)), d(y, f(x))\} ,$$

$$B = \min\{d(x, f(y)), d(y, f(x))\} .$$

Consequently, z is a unique fixed point of f.

Case II. Since L_2 is compact, there exists a subsequence of $\{f^{n_i}(x_0)\}_{i=1}^{\infty}$ convergent to some point $z \in L_2$. Because in z the functions f and f^2 are continuous, we may apply Lemma 2; it follows that z is a unique fixed point of f.

4. – Consider again the metric space M and the properties 1)-13) from the preceding paragraph, that may be verified by a function $f: M \to M$.

Theorem 4. Let M be covered by a family of sets $L_0, \ldots, L_s$, where L_0 is a subset of a complete subspace of M and $L_1, \ldots, L_s$ are compact subspaces of M. Suppose that f satisfies at each couple of different points $x, y \in L_0$ at least one of the Conditions 1)-4), at each couple of different points $x, y \in L_i$ $(i = 1, \ldots, s)$ at least one of the Conditions 9)-12), and at each non-fixed point $x \in M$ Condition 13). If f and f^2 are continuous on $\cup_{i=1}^{s} L_i$, then f has at least one and at most $s+1$ fixed points.

Proof. We first show that a fixed point must exist. Let $x_0 \in M$. Then, $f^n(x_0) \in L_0$ for all n's greater than a certain number N, or there is a subsequence $\{f^{n_j}(x_0)\}_{j=1}^{\infty}$ of $\{f^n(x_0)\}_{n=0}^{\infty}$, entirely lying in some L_i with $i \neq 0$.

In the first case, consider the complete metric space L with $L_0 \subset \subset L \subset M$ and the function $f|\{f^n(x_0)\}_{n=N+1}^{\infty}$, then apply Lemma 1. It follows that $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point $z \in L$. Suppose $f(z) \neq z$. Then, it is obvious that $f^n(x_0) \neq z$ for each n greater than some number N^*. Suppose $z \in L_0$. Since at least one of the Conditions 1)-4) must be valid at $x = f^n(x_0)$, $y = z$ $(n \geqslant \max\{N, N^*\} + 1)$, it results that at least one of the following inequalities is true:

$$d(f^{n+1}(x_0), f(z)) < d(f^n(x_0), z) ,$$

$$d(f^{n+1}(x_0), f(z)) < d(f^n(x_0), f^{n+1}(x_0)) ,$$

$$d(f^{n+1}(x_0), f(z)) < d(z, f(z)) ,$$

$$d(f^{n+1}(x_0), f(z)) < \tfrac{1}{2}(d(f^n(x_0), f(z)) + d(z, f^{n+1}(x_0))) .$$

Because $d(f^n(x_0), f(z)) \to d(z, f(z))$, $d(f^n(x_0), z) \to 0$ and $d(f^n(x_0), f^{n+1}(x_0)) \to 0$, it follows that, in any case,

$$d(z, f(z)) \leqslant \tfrac{1}{2}\, d(z, f(z)) ,$$

which contradicts $f(z) \neq z$. Hence z is a fixed point of f. Suppose now $z \in L_i$, where $i \neq 0$. Since f is continuous in z, $f^{n+1}(x_0) \to f(z)$, whence $f(z) = z$.

In the second case, i.e. if $f^{n_j}(x_0) \in L_i$ $(j = 1, 2, \ldots)$, there exists a subsequence $\{f^{m_j}(x_0)\}_{j=1}^{\infty}$ of $\{f^{n_j}(x_0)\}_{j=1}^{\infty}$, convergent to some point z of L_i, because L_i is compact $(1 \leqslant i \leqslant s)$ (*). Following Condition 13), either

(*) Compare the rest of the proof with that of Theorem 1 (which however cannot be applied!).

for some n_0, $f^{n_0}(x_0) = f^{n_0+1}(x_0)$, whence $z = f^{n_0}(x_0)$ and z is a fixed point of f, or for every n,

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) < d(f^n(x_0), f^{n+1}(x_0)) ,$$

which implies that $\{d(f^n(x_0), f^{n+1}(x_0))\}_{n=0}^{\infty}$ is convergent; in this case it follows that

$$\lim_{j\to\infty} d(f^{m_j}(x_0), f^{m_j+1}(x_0)) = \lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0))$$
$$= \lim_{j\to\infty} d(f^{m_j+1}(x_0), f^{m_j+2}(x_0)) .$$

On the other hand, f and f^2 are continuous at z, hence

$$f^{m_j+1}(x_0) \to f(z); \qquad f^{m_j+2}(x_0) \to f^2(z) .$$

Therefore $d(f, f(z)) = d(f(z), f^2(z))$, which by Condition 13) implies that z is a fixed point of f.

Now, suppose there are at least $s+2$ fixed points for f. Then at least one of the sets $L_0, \dots, L_s$ contains at least two (different) fixed points z, z'. It follows that at least one of the Conditions 2), 3), 9)-12) is satisfied at $x = z$, $y = z'$. These conditions are respectively:

$$d(f(z), f(z')) \leqslant \beta d(z, f(z))$$
$$d(f(z), f(z')) \leqslant \beta d(z', f(z'))$$
$$d(f(z), f(z')) \neq d(z, z')$$
$$d(f(z), f(z')) \leqslant \alpha(d(z, f(z)) + d(z', f(z')))$$
$$d(f(z), f(z')) \neq d(z, f(z'))$$
$$d(f(z), f(z')) \neq d(z', f(z)) .$$

each of which contradicts the fact that both z and z' are fixed points of f. Thus, the proof is achieved.

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