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Two Characterizations of the Reducible Convex Bodies

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Introduction

In this note we give two algebraic characterizations of those convex bodies in $\mathbb{R}^n$ that are reducible in the sense of P. C. Hammer [3], and discuss some related types of convex bodies.

Let K and L be convex sets in $\mathbb{R}^n$, and $r \in \mathbb{R}$. We use the notations:

$$K + L = \{k + l : k \in K, l \in L\} \quad \text{(Minkowski addition)},$$
$$K - L = \{k - l : k \in K, l \in L\},$$
$$rK = \{rk : k \in K\}.$$

Let $\mathfrak{K}$ be the set of all convex bodies (i. e. n-dimensional compact convex sets) in $\mathbb{R}^n$.

Definition 1. Two convex sets $K, L \subset \mathbb{R}^n$ are called *homothetic* (respectively *I-homothetic*) if there exist $x \in \mathbb{R}^n$ (*centre*) and $r \in (0, \infty)$ (respectively $r \in (-\infty, 0)$) (*ratio*) such that*)

$$K = x + r(L - x).$$

Definition 2. The convex set $K \subset \mathbb{R}^n$ is said to be *symmetric* if there exists $x \in \mathbb{R}^n$ such that

$$K = x - K.$$

Definition 3. The convex body K is called a *summand* of $L \in \mathfrak{K}$ if there exists a set $K' \in \mathfrak{K}$ such that

$$L = K + K'.$$

Characterization Theorems

Lemmas. Let $C \in \mathfrak{K}$ and $r \geq \frac{1}{2}$. We define

$$C(r) = \begin{cases} \bigcap_{b \in \partial C} C_b(r) & \text{if } r \in [\frac{1}{2}, 1], \\ \bigcup_{b \in \partial C} C_b(r) & \text{if } r \in (1, \infty), \end{cases}$$

*) Throughout the paper, we identify $x \in \mathbb{R}^n$ with the convex set $\{x\}$.

where ∂C is the boundary of C and $C_b(r)$ is a convex set homothetic with C, centre and ratio being b and r, respectively.

Lemma 1. (P. C. Hammer [3]) *For* $r \geq 1$,

$$C = C(r)(r/(2r-1)).$$

Lemma 2. (D. Voiculescu [6]) *For* $r \geq 1$,

$$C(r) = rC + (1-r)C.$$

Definition 4. ([3], [7])

$$r_C = \inf\{r : C = C(r)(r/(2r-1))\}$$

is called the *reducibility number* of C.

Clearly, $r_C \in [\frac{1}{2}, 1]$.

Definition 5. ([3]) If $r_C < 1$, then C is called *reducible.* If $r_C = 1$, then C is called *irreducible.*

For example, all symmetric convex bodies are reducible, and have reducibility number $\frac{1}{2}$.

Lemma 3. *If* $K \in \mathfrak{K}$ *is the sum of two I-homothetic summands, then there exist a convex set* C *and a real number* $r > 1$, *so that*

$$K = rC + (1-r)C.$$

Proof. Let

$$K = L + x - rL,$$

with $L \in \mathfrak{K}$, $x \in \mathbb{R}^n$, and $r > 0$. If $r = 1$, then

$$K = 2 \cdot \tfrac{1}{3}(L-L) + x + (1-2) \cdot \tfrac{1}{3}(L-L).$$

If $r \neq 1$, then we may suppose $r < 1$, because if $r > 1$, then

$$K = M + x - (1/r)M,$$

where $M = -rL$ and $1/r < 1$.

Define $N = (1-r)L$ and $s = 1/(1-r)$. Then $L = sN$, $-rL = (1-s)N$, and

$$K = s(N+x) + (1-s)(N+x).$$

Theorems. This subsection contains two characterization theorems, which constitute the main results of the paper.

Theorem 1. *A convex body is reducible if and only if it is the sum of two I-homothetic summands.*

Proof. Let $C \in \mathfrak{K}$.

If C is reducible, then

$$C = C(r)(r/(2r-1))$$

for some $r < 1$. Following Lemma 2,

$$C = (r/(2r-1))C(r) + ((r-1)/(2r-1))C(r),$$

where the two summands are I-homothetic.

Now conversely, if C is the sum of two I-homothetic summands, then we may apply Lemma 3, and obtain

$$C = rC' + (1-r)C'$$

for some convex set $C' \subset \mathbb{R}^n$ and some real number $r > 1$. Since C is n-dimensional and compact, C' must also be n-dimensional and compact. Then it follows from Lemma 2 that $C = C'(r)$ and from Lemma 1 the second equality in

$$C(r/(2r-1)) = C'(r)(r/(2r-1)) = C'.$$

Further,

$$C(r/(2r-1))(r) = C'(r) = C;$$

hence $r_C \leq r/(2r-1) < 1$.

The proof is complete.

Theorem 2. *A convex body is reducible if and only if it possesses an I-homothetic summand.*

Proof. Let $K \in \mathfrak{K}$ be reducible; following Theorem 1,

$$K = L + x - rL$$

for some $L \in \mathfrak{K}$, $x \in \mathbb{R}^n$ and $r > 0$; as in the proof of Lemma 3 we may suppose $r \leq 1$. Then

$$K = (1-r^2)L + r^2L + x - rL = r(rL - L) + (1-r^2)L + x,$$

whence

$$K = -rK + (1-r^2)L + (1+r)x,$$

where K and $-rK$ are I-homothetic.

Conversely, let $K \in \mathfrak{K}$ equal $L - rK$, where $r > 0$. Then

$$\begin{aligned} K &= L - (r/2)K - (r/2)K \\ &= L - (r/2)K - (r/2)(L - rK) \\ &= L - (r/2)K - (r/2)(L - (r/2)K) + (r^2/4)K. \end{aligned}$$

Consequently,

$$(1 - (r^2/4))K = L - (r/2)K - (r/2)(L - (r/2)K).$$

Applying again Theorem 1, it follows now that K is reducible.

Remarks. Let $\mathfrak{C}$ ($\mathfrak{R}$) denote the set of all symmetric (reducible) convex bodies in $\mathbb{R}^n$, and $\mathfrak{I} = \mathfrak{K} \backslash \mathfrak{R}$. Then:

I. $\mathfrak{R}$ is a semigroup under Minkowski addition (this follows from Theorem 2).

II. $\mathfrak{I}$ doesn't form a semigroup.

III. $(\mathfrak{I} + \mathfrak{I}) \cap \mathfrak{I} \neq \emptyset$.

IV. $\mathfrak{R}$ is not an ideal of $\mathfrak{K}$ (see Figure 1).

V. $(\mathfrak{R} + \mathfrak{I}) \cap \mathfrak{R} \neq \emptyset$ (see Figure 2).

VI. Using Theorem 1 one can produce an immediate proof to Theorem 3 of [7].

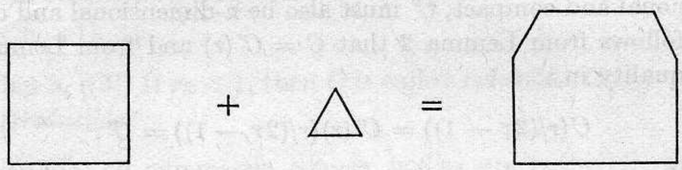


Figure 1

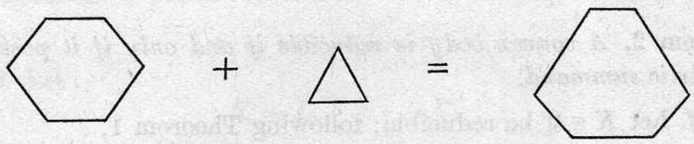


Figure 2

Related Types of Sets

S-reducible sets. In another subsection we shall be concerned with some special symmetric convex sets (the A-reducible sets), for which G. C. Shephard has given an equivalent definition in [5] (see further Theorem 7). By analogy to this definition, we introduce the *S-reducible* sets as those convex sets which possess symmetric summands. It is clear that the class $\mathfrak{S}$ of all S-reducible sets forms an ideal of $\mathfrak{K}$. Evidently, $\mathfrak{C} \subset \mathfrak{S}$.

Theorem 3. $\mathfrak{R} \subset \mathfrak{S}$.

Proof. Following Lemma 3 and Theorem 1, the reducible set K can be written as

$$K = rC + (1 - r)C \qquad (r > 1, C \text{ convex})$$

and rewritten as follows:

$$K = C + (r - 1)C + (1 - r)C = (r - 1)(C - C) + C.$$

Since $K \in \mathfrak{K}$ implies $C \in \mathfrak{K}$ and $C - C \in \mathfrak{C}$, we have $K \in \mathfrak{S}$.

Figure 1 illustrates that the inclusion $\mathfrak{R} \subset \mathfrak{S}$ is strict.

Theorem 4. $\mathfrak{S} = \mathfrak{R} + \mathfrak{K}$.

Proof. Since $\mathfrak{S} = \mathfrak{C} + \mathfrak{K}$ and $\mathfrak{C} \subset \mathfrak{R}$, we have $\mathfrak{S} \subset \mathfrak{R} + \mathfrak{K}$.

Conversely, Theorem 3 and the fact that $\mathfrak{S}$ is an ideal of $\mathfrak{K}$ yield

$$\mathfrak{R} + \mathfrak{K} \subset \mathfrak{S} + \mathfrak{K} \subset \mathfrak{S}$$

and the theorem is proved.

Decomposable sets. A convex set is said to be *decomposable* if it possesses a nonhomothetic summand [4]. We shall see that the decomposable sets generalize almost all the other types of convex bodies met throughout this paper. Let $\mathfrak{D}$ be the class of all decomposable sets in $\mathbb{R}^n$.

Theorem 5. *$\mathfrak{D}$ is an ideal of $\mathfrak{K}$.*

Proof. Let $D \in \mathfrak{D}$ and $K \in \mathfrak{K}$. Further, let C, $L \in \mathfrak{K}$ be such that $D = C + L$, where C and D are not homothetic. Then

$$D + K = C + L + K,$$

where either $C + L$ and $C + L + K$ are homothetic, and then C is a nonhomothetic summand of $D + K$, or $C + L$ itself is a nonhomothetic summand of $D + K$.

***A*-reducible sets and *AS*-reducible sets.** The term of "reducible set" is used in [2] for a symmetric set K which is the difference set of another set, nonhomothetic to K. In order to avoid confusion, we shall call it *A-reducible*. Using results of D. Gale [2], we note that parallelepipeds, octahedra, and other classes of sets are not A-reducible.

Theorem 6. *The set $\mathfrak{A}$ of all A-reducible convex bodies in $\mathbb{R}^n$ is an ideal of $\mathfrak{C}$.*

Proof. Following the theorem of [5], $K \in \mathfrak{C}$ is A-reducible if and only if it possesses an asymmetric summand. Further, the proof is obvious.

Let us define *AS-reducible* sets as S-reducible sets with A-reducible summands, and denote by $\mathfrak{A}\mathfrak{S}$ the class of all AS-reducible sets. The following theorem is immediate.

Theorem 7. *$\mathfrak{A}\mathfrak{S}$ is an ideal of $\mathfrak{K}$.*

Remarks.

I. Every A-reducible convex body K has the A-reducible summand $\frac{1}{2}K$, and is therefore AS-reducible.

II. Every reducible asymmetric set can be expressed in the form

$$K = (r - 1)(C - C) + C$$

(see the proof of Theorem 3). Here C cannot be symmetric, K itself being asymmetric; therefore $(r-1)(C - C) \in \mathfrak{A}$, whence K is AS-reducible.

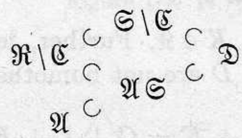
III. Every S-reducible asymmetric set is decomposable (because it is not homothetic with its symmetric summand).

IV. Every AS-reducible set K is of the form

$$K = H + L + M,$$

with $H \in \mathfrak{K}\backslash\mathfrak{C}$, $H + L \in \mathfrak{C}$, $M \in \mathfrak{K}$. Obviously, either H or $H + L$ is not homothetic with K, whence K is decomposable.

The four remarks above, and an inclusion derived from Theorem 3 are illustrated by the following diagram:



V. Let $\mathfrak{B}$ be one of the ideals $\mathfrak{S}$, $\mathfrak{A}\mathfrak{S}$, $\mathfrak{D}$ of $\mathfrak{K}$. If $L \in \mathfrak{B}$, and $K \in \mathfrak{K}$, then all the elements of the linear array determined by L and K, except perhaps K, belong to $\mathfrak{B}$.

VI. The family $\mathfrak{N}$ of all neighbourhoods of convex bodies in Eggleston's sense [1], i.e. of the form $K + B (K \in \mathfrak{K}$, B a ball), is another ideal of $\mathfrak{K}$; obviously $\mathfrak{N} \subset \mathfrak{S}$.

We habe observed that $\mathfrak{S}\backslash\mathfrak{R} \neq \emptyset$. It is also true that $\mathfrak{N}\backslash\mathfrak{R} \neq \emptyset$. However, $\mathfrak{R} + \mathfrak{C} \subset \mathfrak{R}$, which means that if $K \in \mathfrak{S}\backslash\mathfrak{R}$, then $K \in \mathfrak{J} + \mathfrak{C}$. The interesting fact that the inverse implication is not true, even for $K \in \mathfrak{N}$, has lead to the geometrical research contained in § 8 of [8] and in [9].

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