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A CHARACTERIZATION OF HAMILTONIAN GRAPHS

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In this Note we present a characterization of the hamiltonian graphs, using families of smaller circuits. The research is motivated by the almost perfect lack of theorems simultaneously providing necessary and sufficient conditions for a graph to be hamiltonian.

Let G be a graph without loops or multiple edges. We say that $\mathscr{F}$ is a covering family of circuits if $\mathscr{F}$ is a set of circuits the union of which spans G, i.e. $\mathscr{F}$ is so that the point-sets of G and $\cup \mathscr{F}$ are equal and the line-set of G includes that of $\cup \mathscr{F}$. Now, let E and F be two circuits in G. We shall write $E \,\&\, F$ if E and F have a line in common and their point-sets have just the end-points of that line in common. Now we can organize each covering family of circuits $\mathscr{F}$ as a graph $\mathscr{F}^*$ by establishing that the circuits E and F of $\mathscr{F}$ become points of $\mathscr{F}^*$ and determine a line in $\mathscr{F}^*$ if and only if $E \,\&\, F$. On the other hand, we can also organize $\mathscr{F}$ as a graph $\mathscr{F}^{**}$ by saying that the circuits E and F of $\mathscr{F}$ which become again points in $\mathscr{F}^{**}$, determine a line in $\mathscr{F}^{**}$ if and only if they (their point-sets) are not disjoint.

THEOREM. – *The graph G is hamiltonian if and only if it admits a covering family of circuits $\mathscr{F}$ such that $\mathscr{F}^*$ and $\mathscr{F}^{**}$ are trees* (**).

Proof. – If G is hamiltonian, then the hamiltonian circuit alone furnishes a covering family of circuits $\mathscr{F}$, $\mathscr{F}^*$ and $\mathscr{F}^{**}$ being degenerate trees.

Conversely, suppose $\mathscr{F}^*$ and $\mathscr{F}^{**}$ are trees for some covering family of circuits $\mathscr{F}$ of G. We have to prove that G is hamiltonian. We do this by induction on the number of circuits in $\mathscr{F}$ (or cardinality of the point-sets of $\mathscr{F}^*$ or $\mathscr{F}^{**}$). Let $\mathscr{F}$ have n circuits and suppose the theorem true if $\mathscr{F}$ would have $n-1$ circuits. Moreover suppose that if $\mathscr{F}$ has $n-1$ circuits then the hamiltonian circuit of G contains every line of $\cup \mathscr{F}$ which does not appear in the definition of the adjacency

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(*) Nella seduta del 25 novembre 1973.

(**) A point, a line (with its end-points), a path are all (degenerate) trees.

of points in $\mathscr{F}^*$. (Now, in order to use the induction correctly, we have to prove, in the case $\mathscr{F}$ has n circuits, not only that G is hamiltonian, but also this supplementary property of the hamiltonian circuit of G.) Let K be a circuit which, in $\mathscr{F}^*$, is a point k of degree one (*). Following the induction hypothesis, the graph $\cup (\mathscr{F} - \{K\})$ has a hamiltonian circuit C. Let K' be the circuit of $\mathscr{F}$ which, in $\mathscr{F}^*$, is the point k' adjacent with k. K and K' have exactly two points a, b and the line (a, b) in common. Following again the induction hypothesis, (a, b) belongs to C if there exists no circuit K'' in $\mathscr{F} - \{K\}$ such that K' and K'' have exactly the points a, b and the line (a, b) in common. Suppose (a, b) does not belong to C. Then there exists a circuit K'' in $\mathscr{F} - \{K\}$ which, in $\mathscr{F}^{**}$, is a point k'' adjacent to both k and k'. Now k and k' are themselves adjacent in $\mathscr{F}^{**}$ since they are so in $\mathscr{F}^*$. This implies $\mathscr{F}^{**}$ has the circuit $k\, k'\, k''\, k$, which contradicts the assumption that $\mathscr{F}^{**}$ is a tree. Therefore (a, b) belongs to C. Suppose K and $\cup (\mathscr{F} - \{K\})$ have, besides a and b, a third point c (at least) in common. Then there exists a circuit K''' in $\mathscr{F}$, distinct from K', containing the point c. This circuit is, in $\mathscr{F}^*$, a point k''' which is not adjacent with k, since k has degree one and is adjacent to k'. But, since $\mathscr{F}^*$ is a tree, there exists a path P (of length more than 1) in $\mathscr{F}^*$ joining k with k'''. The points k and k''' are joint by P in $\mathscr{F}^{**}$ too, because the line-set of $\mathscr{F}^{**}$ obviously includes that of $\mathscr{F}^*$. On the other hand, since c belongs to both K and K''', k and k''' are adjacent in $\mathscr{F}^{**}$. This line and P provide a circuit in $\mathscr{F}^{**}$, which contradicts the hypothesis. Therefore K and $\cup (\mathscr{F} - \{K\})$ have only a, b and (a, b) in common, and $C \cup K$ minus the line (a, b) is a hamiltonian circuit H in G. Moreover, H contains all lines which do not appear in the definition of the adjacency of points in $\mathscr{F}^*$ (since it contains the lines of K, except (a, b)). Thus, the theorem is proved.

I have the feeling that this theorem (wich admits several variations) may open the interest for a more general study of the graph-structure of families of subgraphs of a given graph G, in connection with properties of G itself.

(*) The circuits K, K', K'', K''' of $\mathscr{F}$ become the points k, k', k'', k''' (respectively) in both $\mathscr{F}^*$ and $\mathscr{F}^{**}$.