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ON SPANNING AND EXPANDING STARS

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1. – P. H. DOYLE [1] classified topologically the monotone unions of 1-cells. S. F. KAPOOR [2] extended the study to monotone unions of n-stars, called by him *expanding n-stars*, giving in particular a complete classification for $n = 3$. In his graph-theoretical terminology, vertices appear not only where the set is not locally euclidean, but also elsewhere, in order to avoid loops and multiple lines. Here, though we keep the graph-theoretical form, we shall prefer to consider points of the set as vertices of the graph if and only if the set is not locally euclidean there (as in an earlier version of [2]). Thus our graphs will not have vertices of degree 2, but possibly loops and multiple edges.

The graphs are understood here as sets in the plane or in the 3-dimensional euclidean space (**); hence we distinguish between *points* and *vertices* of a graph (a line of which has two vertices, but infinitely many points).

A point p of a graph G is *a vertex of degree* n ($n \neq 2$) if G is in p locally homeomorphic with an n-star of centre p. Let $\mathscr{V}(G) \subset G$ be the set of vertices of G. $\mathscr{V}(G)$ is supposed to be finite. The *line-degree of a vertex* p is the number of lines adjacent with p. Let $V(p)$ and $W(p)$ be the degree and the line-degree of p, respectively. Obviously $V(p) - W(p)$ equals the number of loops at p.

Let

$$W(G) = \max_{p \in \mathscr{V}(G)} W(p) .$$

As defined by KAPOOR [2], G is *an expanding n-star* if

$$G = \bigcup_{i=1}^{\infty} S_i(n) ,$$

(*) Nella seduta del 25 novembre 1973.

(**) The expanding n-stars are planar sets (Theorem 8 in [2]) and each graph is embadable in R^3.

where $S_i(n)$ are n-star graphs (with the same point p as vertex of degree n) and $S_i(n) \subset S_{i+1}(n)$ $(i = 1, 2, 3, ...)$. The point p is then called *n-source* of G. Since G may be written in more than one way as monotone union of n-star graphs, G possibly has more than one n-source. The set of all n-sources is the *n-centre* $C_n(G)$ of G.

The following two parts of this Note are independent.

2. – We present here a few results on the n-centres of expanding n-stars.

THEOREM 1. – *The point p is an n-source of the expanding n-star G if and only if either $V(p) \geqslant n$ and $W(p) = W(G)$, or $W(p) \geqslant n$.*

Proof. – First, suppose $p \in C_n(G)$. Then, obviously, $V(p) \geqslant n$. Suppose now $W(p) < n$. It follows that p has at least one loop. Let $q \in \mathscr{V}(G)$ be different from p. We have to show $W(q) \leqslant W(p)$.

We have

$$G = \bigcup_{i=1}^{\infty} S_i(n) ,$$

where $S_i(n)$ are n-star graphs with the same n-source p, and $S_i(n) \subset \subset S_{i+1}(n)$ $(i = 1, 2, 3, ...)$. Let $e_i^1, ..., e_i^n$ be the lines of $S_i(n)$, so that $e_i^k \subset e_{i+1}^k$ $(k = 1, ..., n)$. Evidently, q belongs to one line e_i^j of $S_i(n)$ for all $i \geqslant N$, where N is a natural number. Also, $q \in \overline{\cup_{i=1}^{\infty} e_i^k}$ is possible for each $k \in \{1, ..., n\}$. But because p has at least one loop, at most $W(p) - 1$ values are possible for k (i.e. at most $W(p) - 1$ lines from p are free to join q). Since $q \in e_i^j$ does not need to be an end-point of e_i^j, q may have one more adjacent line (possibly a loop), therefore $W(q) \leqslant W(p)$ (and $V(q)$ may attain $W(p) + 1$).

Conversely, suppose $W(p) \geqslant n$. Let $q \in C_n(G)$. If $q = p$, nothing more has to be proved. If $q \neq p$, then p lies in a union of n-star graphs with centre q, and again p belongs to one line e_i^j of the n-star graph $S_i^*(n)$ with q as n-source, for all i's greater than some natural number. Is interesting only the case when p lies in the interior of the e_i^j's, because otherwise G clearly reduces to the two points p and q, joined by exactly n lines, and $p \in C_n(G)$. Consider therefore this case. Let $e_k = \cup_{i=1}^{\infty} e_i^k$, where e_i^k are the lines of $S_i^*(n)$ $(k = 1, ..., n)$. Then e_j consists of a line m joining p with q and another line f_j. Since e_j covers two lines (one of which is possibly a loop) adjacent with p, and $W(p) \geqslant n$, it follows that for at most one index $l \in \{1, ..., n\}$, $p \notin \bar{e}_l$: Let $f_i^j = e_i^j \cap f_j$. Also, let $\{f_i^k\}_{i=1}^{\infty}$ be a sequence of arcs, all originating at p, so that $f_i^k \subset f_{i+1}^k$ and $\cup_{i=1}^{\infty} f_i^k = e_k$, for $k \in \{1, ..., n\} - \{l, j\}$. Finally, define $f_i = m \cup e_l$, where l is either the index precised above, or, if $q \in \bar{e}_k$ for each $k \in \{1, ..., n\}$, another arbitrary index different from j. Now, we can construct the n-star graphs

$$S_i(n) = \bigcup_{k=1}^{n} f_i^k ,$$

which have their n-sources at p and satisfy

$$\bigcup_{i=1}^{\infty} S_i(n) = G .$$

Hence $p \in C_n(G)$.

Finally, suppose $V(p) \geqslant n$ and $W(p) = W(G)$. The proof follows exactly the same way as above until the point where we explicitely motivated the existence of at most one index l such that $p \notin \bar{e}_l$, by recording $W(p) \geqslant n$. Suppose now $W(p) < n$. Since p obviously has at most one loop (contained in e_j), and $V(p) \geqslant n$, it follows

$$V(p) = n ; \qquad W(p) = n - 1 .$$

In this case e_j covers precisely one line and one loop adjacent with p, and since $W(q) \leqslant n - 1$, it follows that for one index $l \in \{1, ..., n\}$, $p \notin \bar{e}_l$ (more precisely e_l is a loop of q), and the proof continues as for the case $W(p) \geqslant n$ (or better ends immediately by observing the symmetry of G, which implies $p \in C_n(G)$ too).

THEOREM 2. – *If G is an expanding n-star, then*

$$W(G) \geqslant [(n+1)/2] .$$

If $C_n(G)$ has more than one point, then $W(G) \geqslant n - 1$.

Proof. – Since the minimal value of $W(p)$, where $p \in C_n(G)$, is taken when G consists only of the vertex p together with $n/2$ loops for even n and only of the vertices p and q, where possibly $p = q$, together with $(n-1)/2$ loops of p and a line joining p and q for odd n, it follows $W(G) \geqslant [(n+1)/2]$. From the second part of the proof of Theorem 1 it also results that if $C_n(G)$ has at least two points, then $W(G) \geqslant n - 1$.

THEOREM 3. – *If $W(p) \leqslant n - 1$ for some point p in the n-centre of an expanding n-star G, then G is also an expanding $(n-1)$-star.*

Proof. – Since $V(p) \geqslant n$, p necessarily has some loops $L_1, ..., L_m$.

Let $G = \cup_{i=1}^{\infty} S_i(n)$, where the $S_i(n)$'s are n-stars with rays $e_i^1, \ldots, e_i^n$ and with p as n-sources. Suppose

$$(\cup_{i=1}^{m} L_i) \cap S_i(n)$$

is an m-star; since

$$((G - \cup_{i=1}^{m} L_i) \cup \{p\}) \cap S_i(n)$$

is at most an $(n - m - 1)$-star, clearly $S_i(n)$ is at most an $(n-1)$-star, which is absurd. Therefore $(\cup_{i=1}^{m} L_i) \cap S_i(n)$ must be at least an $(m+1)$-star, which means that for at least one index $l \in \{1, \ldots, m\}$, there exist two indices $r, s \in \{1, \ldots, n\}$ so that

$$L_l \cap S_i(n) = e_i^r \cup e_i^s .$$

If $\{e_i^t\}_{i=1}^{\infty}$ is an increasing sequence of arcs originating in p, such that $\cup_{i=1}^{\infty} e_i^t = L_l$, then

$$S_i(n-1) = (S_i(n) - (e_i^r \cup e_i^s)) \cup e_i^t$$

is an $(n-1)$-star for each i, and

$$\bigcup_{i=1}^{\infty} S_i(n-1) = G .$$

THEOREM 4. – *If G is an expanding 3-star, then*

$$\text{card } C_3(G) \leqslant 4 .$$

If G is an expanding 4-star, then

$$\text{card } C_4(G) \leqslant 3 .$$

If G an expanding n-star, with $n \geqslant 5$, then

$$\text{card } C_n(G) \leqslant 2 .$$

Proof. – Since $p \in C_n(G)$ implies $V(p) \geqslant n$, it follows

$$\text{card } C_n(G) \leqslant D(n) ,$$

where $D(k)$ is the number of vertices of G, the degrees of which are greater than or equal to k (see [2]). Following Theorem 1 of [2], $D(n) \leqslant 1 + [n/(n-2)]$, which implies our theorem.

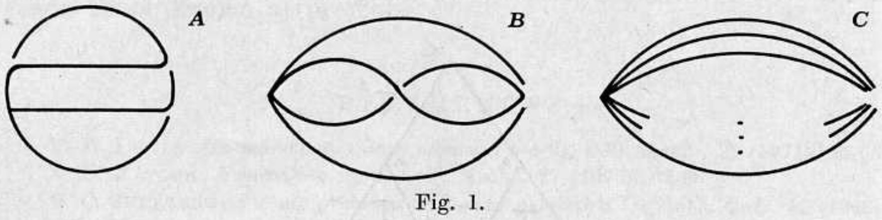

Fig. 1.

We verify now that the given bounds are attained. For $n = 3$, the graph A of fig. 1 appearing in the classification of [2] has all its vertices in its 3-centre. For $n = 4$, the graph B has its three vertices in $C_4(B)$. Finally, for $n \geqslant 5$, an example is provided by the graph C.

3. – Let G be a (finite) graph without loops or multiple edges and $n \leqslant \max_{p \in \mathscr{V}(G)} \deg p$ a natural number. Does there always exist an expanding n-star spanning G? Here, a graph H *spans* a graph G if $\mathscr{V}(G) \subset H \subset G$. It is easily seen that if a graph has no multiple edges and admits a spanning expanding n-star, then it also admits a spanning n-star, and conversely (*). For $n = 2$, the problem is: Does always G possess a hamiltonian path? The answer is known to be in the negative, even for planar G. We investigate now the case $n = 3$.

THEOREM 5. – *There exists a simple polytopal graph* (**) *without any spanning* 3-*star*.

Proof. – Let T be the graph of W. T. TUTTE [4] shown in fig. 2. Two copies of T are placed as indicated in fig. 3 to form a graph T'.

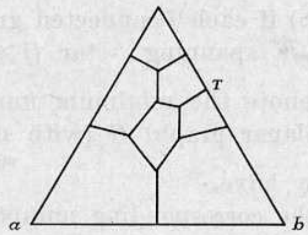

Fig. 2.

(*) In this section vertices of degree 2 will be tolerated.

(**) A *simple polytopal graph* means a planar 3-connected graph, each vertex of which has degree 3.

Now, five copies of the graph T' are placed as indicated in fig. 4 to form the graph G. Since T' is the well-known non-hamiltonian graph of Lederberg-Bosák-Barnette (the smallest known at present)

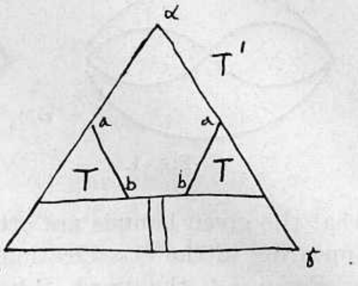

Fig. 3.

minus a vertex, it is obvious that no path joins any two of the points α, β, γ, passing through all the vertices of T'. The theorem now follows. G has 186 vertices.

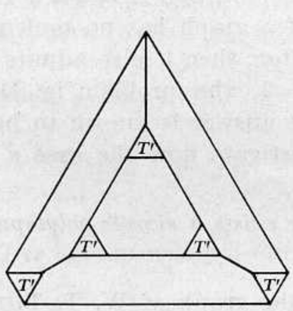

Fig. 4.

Let $S^i_j = \infty$ ($\bar{S}^i_j = \infty$) if each j-connected graph (planar graph) G with $\max_{p \in \mathcal{V}(G)} \deg p \geqslant i$ has a spanning i-star ($j \geqslant 1$, $i \geqslant 2$). If $S^i_j \neq \infty$ ($\bar{S}^i_j \neq \infty$), let S^i_j ($\bar{S}^i_j$) denote the minimum number of vertices that a j-connected graph (planar graph) G (with $\max_{p \in \mathcal{V}(G)} \deg p \geqslant i$) without any spanning i-star may have.

Let ${}_sS^i_j$ and ${}_s\bar{S}^i_j$ be the corresponding numbers if we restrict ourselves to the case $i \leqslant j$ and to graphs the degree of each vertex of which is j ($i \geqslant 2$, like before).

We ask for the determination of S^i_j, $\bar{S}^i_j$, ${}_sS^i_j$, ${}_s\bar{S}^i_j$. In particular, the question whether $\bar{S}^2_3 < 14$ or ${}_s\bar{S}^2_3 < 112$ is part of G. C. Shephard's Problem X in [3].

It is known that $\bar{S}_3^2 \leqslant 14$ [3] and ${}_s\bar{S}_3^2 \leqslant 88$; it is trivial that $S_1^n = \bar{S}_1^n = n + 2$ and $S_2^n = \bar{S}_2^n = n + 4$. The proof of Theorem 5 yields ${}_s\bar{S}_3^3 \leqslant 186$. Also, $S_3^2 = 8$ and ${}_sS_3^2 \leqslant 28$. Nothing more on these numbers seems to be known at present.

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