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METRIC SPACES CONSISTING OF CLASSES OF CONVEX BODIES (*)

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SOMMARIO. - *In questa Nota si mostra come in due spazi di classi di corpi convessi si può introdurre una metrica derivata dalla « difference body metric » di G. C. Shephard. Inoltre la famiglia di tutti i corpi convessi riducibili viene considerata come semigruppo topologico.*

SUMMARY. - *In this Note it is shown how spaces of classes of convex bodies can be equipped with a certain metric, derived from the « difference body metric » of G. C. Shephard. Also, the family of all reducible convex bodies is organized as topological semigroup.*

1. Introduction.

Let E^n denote the Euclidean n-dimensional space and $\mathcal{K}^n$ the family of all convex bodies (n-dimensional compact convex sets) in E^n. The aim of this Note is to show how two natural factorizations of $\mathcal{K}^n$ introduced by using Hammer associated bodies, displacements or affine transformations of E^n, may be equipped with a metric structure, derived each time from the difference body metric of $\mathcal{K}^n$.

Let us recall the definition of the *Hammer associated bodies* [2] of the convex body $C \in \mathcal{K}^n$:

$$C(r) = \begin{cases} \bigcap\limits_{b \in \partial C} (b + r(C-b)) & \text{for } r \in (0, 1) \\ \bigcup\limits_{b \in \partial C} (b + r(C-b)) & \text{for } r \in [1, \infty) \end{cases}$$

(∂C means the boundary of C).

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Following D. Voiculescu [4], if $r \geq 1$, then the Hammer associated body $C(r)$ can be written in the form:

$$C(r) = rC + (1-r)C = (r-1)C + C + (1-r)C$$

i. e.

$$C(r) = C + (r-1)D,$$

where $D = C - C$ (D is called difference body or vector domain of C).

It is easily seen that the *difference body metric* ρ^D introduced by G. C. Shephard in [3] can be expressed as follows:

$$\rho^D(C_1, C_2) = ln(2q-1),$$

where

$$q = \min\{r\colon C_1 \subset C_2(r) \text{ and } C_2 \subset C_1(r)\}.$$

We observe that, if $\mathscr{A}$ is the family of all affine transformations in E^n,

$$\rho^D(C_1, C_2) = \rho^D(aC_1, aC_2) = \rho^D(C_1(r), C_2(r))$$

for all $a \in \mathscr{A}$ and $r \geqq 1$.

P. C. Hammer has proved in [2] that for every convex body $C \in \mathcal{K}^n$, there exists a number $r_C \leqq 1$, called reducibility number of C, so that

$$C(r)(r/(2r-1)) = C$$

for each $r > r_C$. If $r_C < 1$, then C is said to be *reducible* [2].

2. First factorization.

Now, let us define an equivalence relation in the space $\mathcal{K}^n$ by

$$C_1 \sim C_2 \Leftrightarrow C_1 = aC_2(r),$$

with $r \geq r_{O_2}$, and $a \in \mathcal{B}$, where $\mathcal{B} \subset \mathscr{A}$ is the family of displacements of E^n. By the continuity argument it is clear that

$$C(r)(r/(2r-1)) = C$$

for each $r \geq r_C$, if r_C does not equal the critical ratio of C, i. e. if C is not a centrally symmetric convex body [5]. But in the case of a centrally symmetric body C_2,

$$r > r_{O_2} = 1/2$$

because C_1 is n-dimensional. Therefore the equivalence relation introduced above can also be written as follows:

$$C_1 \sim C_2 \Leftrightarrow C_1 = aC_2\,(r) \text{ or } C_2 = aC_1\,(r),$$

where $r \geqq 1$ and $a \in \mathcal{B}$.

If $[C]$ denotes the family of all reducible Hammer associated bodies of C, then an element $[C]_{\mathcal{B}}$ of $\mathcal{K}^n/\sim$ has the form

$$[C]_{\mathcal{B}} = \{a\,(K) \colon a \in \mathcal{B},\ K \in [C] \cap \mathcal{K}^n\}.$$

Let us introduce a metric structure in $\mathcal{K}^n/\sim$. Choose the elements C_1', C_2' in the classes $[C_1]_{\mathcal{B}}$, $[C_2]_{\mathcal{B}}$ so that

$$C_1' \subset C_2' \subset aC_1'\,(r),$$

with $a \in \mathcal{B}$, and define

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}},\ [C_2]_{\mathcal{B}}) = \ln\,(2q-1),$$

where q is the infimum (necessarily attained by Blaschke's theorem) of the set of r's which can appear in the proceding relations.

THEOREM 1.

$$\rho_{\mathcal{B}} \colon \mathcal{K}^n/\sim \times\, \mathcal{K}^n/\sim\ \rightarrow [0, \infty)$$

is a metric in $\mathcal{K}^n/\sim$.

PROOF. If

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}},\ [C_2]_{\mathcal{B}}) = 0,$$

then $q = 1$ and

$$C_1' \subset C_2' \subset a\,(C_1'),$$

for a certain displacement a and two bodies $C_1' \in [C_1]_{\mathcal{B}}$ and $C_2' \in [C_2]_{\mathcal{B}}$; therefore $C_1' = C_2'$; conversely, $[C_1]_{\mathcal{B}} = [C_2]_{\mathcal{B}}$ obviously implies

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}},\ [C_2]_{\mathcal{B}}) = 0.$$

If

$$\ln\,(2r-1) = \rho_{\mathcal{B}}([C_1]_{\mathcal{B}},\ [C_2]_{\mathcal{B}}),$$

then we can find $C_1' \in [C_1]_{\mathcal{B}}$, $C_2' \in [C_2]_{\mathcal{B}}$, and $a \in \mathcal{B}$ so that

$$C_1' \subset C_2' \subset aC_1'\,(r).$$

It follows that
$$C_2' \subset aC_1'(r) \subset aC_2'(r),$$
hence
$$\rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_1]_{\mathcal{B}}) \leqq ln\,(2r-1) = \rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}).$$

Similarly one finds the converse inequality; therefore
$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = \rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_1]_{\mathcal{B}}).$$

Let
$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}) = ln\,(2r-1)$$
and
$$\rho_{\mathcal{B}}([C_2]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) = ln\,(2s-1);$$
show that
$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) \leqq ln\,(2r-1) + ln\,(2s-1).$$

One has
$$C_1' \subset C_2' \subset aC_1'(r)$$
and
$$C_2'' \subset C_3' \subset bC_2''(s),$$
for some displacements $a, b \in \mathcal{B}$ and bodies $C_1' \in [C_1]_{\mathcal{B}}$, $C_2', C_2'' \in [C_2]_{\mathcal{B}}$ and $C_3' \in [C_3]_{\mathcal{B}}$. Either
$$C_2' = cC_2''(t),$$
or
$$C_2'' = cC_2'(t),$$
with $c \in \mathcal{B}$ and $t \geqq 1$. Consider, for instance, the first case; the proof of the other being similar, will be omitted. We can write
$$C_1' \subset C_2' = cC_2''(t) \subset cC_3'(t)$$
and
$$cC_3'(t) \subset cbC_2''(s)(t) = cbc^{-1}cC_2''(t)(s) =$$
$$= cbc^{-1}C_2'(s) \subset cbc^{-1}aC_1'(r)(s)$$
Since
$$C_1'(r)(s) = C_1'(1-r-s+2rs),$$
we have
$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}}, [C_3]_{\mathcal{B}}) \leq ln\,(1-2r-2s+4rs) = ln\,((2r-1)(2s-1)).$$

3. Second factorization.

Now we shall consider another equivalence relation in $\mathcal{K}^n$, namely

$$C_1 \approx C_2 \Leftrightarrow C_1' = aC_2',$$

with $C_1' \in [C_1]_{\mathcal{B}}$, $C_2' \in [C_2]_{\mathcal{B}}$ and $a \in \mathcal{A}$. In other words, an element of the space $\mathcal{K}^n/\approx$ can be written as follows:

$$[C]_{\mathcal{A}} = \{a(K):\ a \in \mathcal{A},\ K \in [C]\} \cap \mathcal{K}^n.$$

A metric structure of $\mathcal{K}^n/\approx$ will be obtained by introducing the distance

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}},\ [C_2]_{\mathcal{A}}) = ln\,(2q-1),$$

where q is the infimum (necessarily attained) of the set of r's which satisfy simultaneously

$$\begin{cases} C_1' \subset aC_2'\,(r) \\ aC_2' \subset fC_1'\,(r), \end{cases}$$

for certain $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{T}$, where $\mathcal{T} \subset \mathcal{A}$ is the family of translations in E^n.

THEOREM 2.

$$\rho_{\mathcal{A}}:\ \mathcal{K}^n/\approx \times\ \mathcal{K}^n/\approx\ \to [0, \infty)$$

is a metric in $\mathcal{K}^n/\approx$.

PROOF. If

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}},\ [C_2]_{\mathcal{A}}) = 0,$$

then $q=1$ and we can find $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{T}$ so that

$$C_1' \subset aC_2' \subset fC_1',$$

hence $C_1' = aC_2'$ and $[C_1]_{\mathcal{A}} = [C_2]_{\mathcal{A}}$. The converse is obvious.

If

$$ln\,(2r-1) = \rho_{\mathcal{A}}([C_1]_{\mathcal{A}},\ [C_2]_{\mathcal{A}}),$$

then, for certain $C_1' \in [C_1]_{\mathcal{A}}$, $C_2' \in [C_2]_{\mathcal{A}}$, $a \in \mathcal{A}$ and $f \in \mathcal{T}$,

$$\begin{cases} C_1' \subset aC_2'(r) \\ aC_2' \subset fC_1'(r), \end{cases}$$

whence

$$fC_1' \subset faC_2'(r).$$

From the second and the third relation, it follows:

$$\rho_{\mathcal{A}}([C_2]_{\mathcal{A}}, [C_1]_{\mathcal{A}}) = \rho_{\mathcal{A}}([aC_2']_{\mathcal{A}}, [C_1']_{\mathcal{A}}) \leqq$$

$$\leqq \ln(2r-1) = \rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}).$$

The converse inequality can be similarly obtained, therefore equality holds. Show that

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}}) + \rho_{\mathcal{A}}([C_2]_{\mathcal{A}}, [C_3]_{\mathcal{A}}) \geqq \rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_3]_{\mathcal{A}}).$$

If

$$r = \frac{1}{2} e^{\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_2]_{\mathcal{A}})} + \frac{1}{2},$$

$$s = \frac{1}{2} e^{\rho_{\mathcal{A}}([C_2]_{\mathcal{A}}, [C_3]_{\mathcal{A}})} + \frac{1}{2},$$

then, for some $C_1' \in [C_1]_{\mathcal{A}}$, C_2', $C_2'' \in [C_2]_{\mathcal{A}}$, $C_3' \in [C_3]_{\mathcal{A}}$, a, $b \in \mathcal{A}$, f, $g \in \mathcal{T}$,

$$\begin{cases} C_1' \subset aC_2'(r) \\ aC_2' \subset fC_1'(r) \end{cases}$$

and

$$\begin{cases} C_2'' \subset bC_3'(s) \\ bC_3' \subset gC_2''(s). \end{cases}$$

Either $C_2' = cC_2''(t)$, or $C_2'' = cC_2'(t)$, with $c \in \mathcal{A}$ and $t \geqq 1$. In the first case,

$$C_1' \subset aC_2'(r) = acC_2''(t)(r) \subset acbC_3'(t)(r)(s)$$

and

$$acbC_3'(t) \subset acgC_2''(s)(t) = acgc^{-1}C_2'(s) \subset acgc^{-1}a^{-1}fC_1'(r)(s)$$

Since $\mathcal{T}$ is a normal subgroup of $\mathcal{A}$, $acgc^{-1}a^{-1}f$ is a translation; therefore

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}}, [C_3]_{\mathcal{A}}) \leqq \ln(1-2r-2s+4rs),$$

which proves our inequality. In the second case,

$$C_1'(t)\subset aC_2'(r)(t)=ac^{-1}C_2''(r)\subset ac^{-1}bC_3'(r)(s)$$

and

$$ac^{-1}bC_3'\subset ac^{-1}gC_2''(s)=ac^{-1}gcC_2'(t)(s)\subset ac^{-1}gca^{-1}fC_1'(t)(r)(s)$$

and our inequality is again obtained.

4. Continuity of canonical maps.

THEOREM 3. *The canonical maps*

$$(\mathcal{K}^n,\rho^D)\to(\mathcal{K}^n/\sim,\rho_{\mathcal{B}})\quad and\quad(\mathcal{K}^n,\rho^D)\to(\mathcal{K}^n/\approx,\rho_{\mathcal{A}})$$

are continuous.

PROOF. Let

$$\rho^D(C_1,C_2)=ln(2r-1).$$

Then

$$C_1\subset C_2(r);\ C_2\subset C_1(r),$$

whence, on one hand,

$$C_1\subset C_2(r)\subset C_1(r)(r)=C_1(1-2r+2r^2),$$

whence

$$\rho_{\mathcal{B}}([C_1]_{\mathcal{B}},\ [C_2]_{\mathcal{B}})\leqq ln(1-4r+4r^2)=2\rho^D(C_1,C_2),$$

which proves the continuity of the first canonical map, and on the other hand,

$$\rho_{\mathcal{A}}([C_1]_{\mathcal{A}},\ [C_2]_{\mathcal{A}})\leqq ln(2r-1)=\rho^D(C_1,C_2),$$

which proves that the second canonical map is continuous.

5. Structure of the space of reducible convex bodies.

THEOREM 4. *The space $\mathcal{K}^n$ and its subspace $\mathcal{K}_r^n$ of all n-dimensional reducible convex bodies in E^n are topological abelian semigroups, with Minkowskian addition and distance ρ^D.*

PROOF. If $C_1, C_2\in\mathcal{K}_r^n$, then r_{σ_1}, $r_{\sigma_2}<1$. Let us choose r such that

$$\max\{r_{\sigma_1}, r_{\sigma_2}\}<r<1.$$

Put

$$K_i = C_i\,(r) \qquad (i=1,2);$$

we have $K_i \supset C_i\,(r\alpha_i)$, $K_i \in \mathcal{K}_r^n$ and $C_i = K_i\,(s)$ with $s = r/(2r-1) > 1$. Hence

$$C_1 = sK_1 + (1-s)\,K_1,$$

$$C_2 = sK_2 + (1-s)\,K_2,$$

whence

$$C_1 + C_2 = s\,(K_1+K_2) + (1-s)(K_1+K_2) = (K_1+K_2)\,(s),$$

therefore

$$C_1 + C_2 \in \mathcal{K}_r^n.$$

If $C_1, C_2, C_3 \in \mathcal{K}^n$ and

$$\rho^D\,(C_1, C_2) = ln\,(2r-1),$$

then

$$C_1 \subset C_2\,(r),$$

whence

$$C_1 + C_3 \subset C_2\,(r) + C_3 \subset C_2\,(r) + C_3\,(r) = (C_2+C_3)\,(r)$$

and similarly

$$C_2 + C_3 \subset (C_1+C_3)\,(r);$$

hence

$$\rho^D\,(C_1+C_3,\; C_2+C_3) \leqq \rho^D\,(C_1, C_2),$$

which proves that the abelian semigroup $\mathcal{K}^n$ is topological; $\mathcal{K}_r^n$ being closed for Minkowskian addition is itself a topological abelian semigroup.

It would be very interesting if, using the described spaces $\mathcal{K}^n/\sim$ and $\mathcal{K}^n/\approx$, an increased algebraic structure will be obtained (though this doesn't seem easy), as G. Ewald and G. C. Shephard did in [1], using other factorizations of $\mathcal{K}^n$.

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