MATH. SCAND. 38 (1976), 211–239

ON LONGEST PATHS AND CIRCUITS IN GRAPHS

TUDOR ZAMFIRESCU

0. Introduction.

In this paper we are concerned with undirected connected graphs, without loops or multiple edges. Let $p(G)$ $(c(G))$ be the maximal number of vertices that a path (circuit) of the graph G may have. The *length* of a path or circuit will be its number of vertices (not edges!). A path (circuit) in G of length $p(G)(c(G))$ is also called *longest path* (*circuit*) of G. In [19] we introduced the numbers $P_k^j, C_k^j, \bar{P}_k^j, \bar{C}_k^j$ (j and k are natural numbers). They are defined as follows:

Let $P_k^j = \infty (C_k^j = \infty)$ if there exists no k-connected graph G such that for each j of its vertices there exists a longest path (circuit) in G avoiding them. If, on the contrary, there are such graphs G, then let $P_k^j(C_k^j)$ denote the minimal number of vertices G may have. Analogously are defined $\bar{P}_k^j$ and $\bar{C}_k^j$, for which the above definitions are restricted to planar graphs. (In the case of circuits (and for $k=1$), G is supposed not to be a tree.)

T. Gallai has asked (we reformulate his question by using one of the above numbers): "Is $P_1^1 < \infty$?" [5]. This surprisingly nontrivial question was answered by H. Walther in the affirmative (his example is a graph derived from part of Figure 2 of W. T. Tutte [15]). Thus, the desire of determining (or at least to know more about) all the above numbers appears as quite natural. Some results in this direction are given in this paper.

I am very much indebted to the referee, who read in great detail the manuscript and so found out several errors and made precious improvements. The present form of Lemma 13 belongs to him.

1. On the $\bar{P}_k^1$'s and $\bar{C}_k^1$'s.

1.1. First we remark that $\bar{P}_k^j$ and $\bar{C}_k^j$, involving k-connected planar graphs, have a sense only for $k \leqq 5$.

The graph used by Walther to answer Gallai's question is planar; it shows $\bar{P}_1^1 \leqq 25$ [17].

Received May 31, 1974; in revised form December 8, 1975.

The graph of Figure 1 provides $\bar{P}_1^1 \leqq 20$ and will be used later.

Now, we consider the graph A of Figure 2. A has 15 vertices. The proofs of the following lemmas are easy.

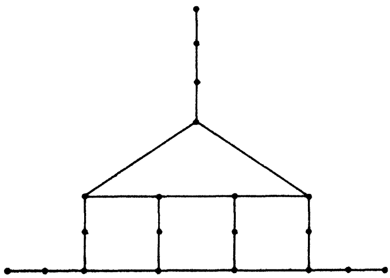

Fig. 1.

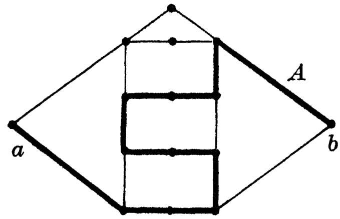

Fig. 2.

LEMMA 1. *The longest paths of A joining a with b have* 12 *vertices.*

LEMMA 2. *For each vertex v of A different from a and b, there exists a path of length* 12 *joining a with b and avoiding v.*

LEMMA 3. *There exists a path of length* 14 *in A, which has an endpoint in a and misses b (by symmetry, there also exists a path of length* 14, *with an endpoint in b and missing a).*

LEMMA 4. *The graph A has no hamiltonian path.*

THEOREM 1. $\bar{P}_1^1 \leqq 19$.

PROOF. Consider the graph B of Figure 3. B has 19 vertices.

First we prove that $p(B) = 16$. Obviously A is a subgraph of B. Following Lemma 1, the longest paths of B joining c with d have 16 vertices. Thus, $p(B) \geqq 16$. Let W be a path in B. If one of the endpoints of W is a, c, or the vertex between them, and the other endpoint b, d, or the vertex between them, then W has at most 4 vertices more than a subpath of

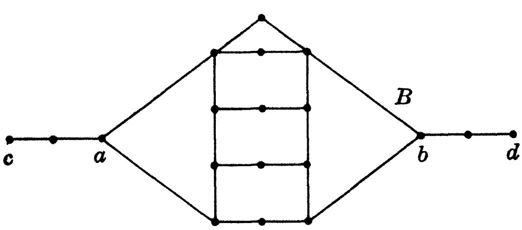

Fig. 3.

W joining a and b, which by Lemma 1 has at most 12 vertices. If one of the endpoints of W is a, c, or the vertex between them, and the other endpoint is different from b, d, and the vertex between them, W clearly does not contain the vertex d and that between b and d. Also, it cannot contain all the other vertices, because otherwise its restriction to A would be a hamiltonian path of A, which contradicts Lemma 4. Finally, omitting a case symmetric with the preceding one, if W has no endpoint outside A, then all 4 vertices of B not in A do not lie on W. Thus $p(B) = 16$.

Now we prove that for each vertex v of B, there exists a path of length 16 avoiding v. If v is in A, but is different from a and b, then such a path is provided by an obvious extension of the path of length 12 joining a with b in A and avoiding v, given by Lemma 2. If v is b, d, or the vertex between them (similarly if v is a, c, or the vertex between them), then a path of the demanded kind can be obtained by an obvious extension of the path of length 14 in A, which has an endpoint in a and misses b, given by Lemma 3.

1.2. The inequality $\overline{P}_2^1 \leqq 82$ (Zamfirescu [19]) which first demonstrated the finiteness of $\overline{P}_2^1$ can be drastically improved.

THEOREM 2. $\overline{P}_2^1 \leqq 32$.

PROOF. Consider the graph C of Figure 4, the subgraphs A and A' of which are isomorphic with the graph of Figure 2. C has 32 vertices.

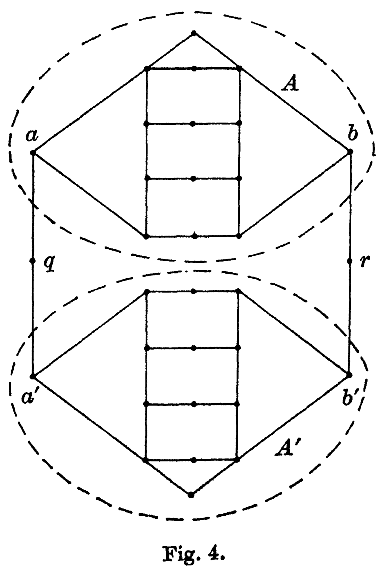

Fig. 4.

First we prove that $p(c) = 29$. Consider a path W_1 of length 12 joining in A' the vertices a' and b' (the existence of W_1 is guaranteed by Lemma 1) and a path W_0 with 14 vertices lying in A, having an endpoint in a, and missing b (the existence of which is guaranteed by Lemma 3). Then b, r, W_1, q, W_0 determine a path of length 29 in C. Thus $p(C) \geqq 29$.

Let W be a path in C.

CASE I. *No endpoint of W is in $A - \{a,b\}$.* If $W \cap A - \{a,b\} = \emptyset$, then W has at most 19 vertices. Otherwise, W contains a and b, and all the vertices of A belonging to W lie on the subpath W^* of W joining a with b. But Lemma 1 asserts that at least 3 vertices of A do not belong to W^*, whence W has at most 29 vertices.

CASE II. *No endpoint of W is in $A' - \{a',b'\}$.* (Analogous to Case I.)

CASE III. *One endpoint of W is in $A - \{a,b\}$, and the other endpoint of W is in $A' - \{a',b'\}$.* In this case it is clear that exactly one of the vertices q and r is in W. Also, we easily see that $W \cap A$ and $W \cap A'$ are both paths. But, since by Lemma 4 $W \cap A$ is not a hamiltonian path of A and $W \cap A'$ is not a hamiltonian path of A', there exist two more vertices not lying on W, whence W has at most 29 vertices.

Thus $p(C) = 29$.

Now we prove that for each vertex v of C there exists a path of length 29 avoiding v. Remember first that we already constructed a path of length 29 determined by b, r, W_1, q, W_0. Let v be a vertex of A' different from a' and b'. By Lemma 2, there exists a path W_v in A' joining a' with b', missing v, and having 12 vertices (as many as W_1. Then clearly b, r, W_v, q, W_0 determine a path of length 29 avoiding v. A symmetrical construction can be performed if v is in A, different from a and b. If v is b, r, or b', consider a path W_0' in A' of length 14 which has an endpoint in a' and misses b' (and existing by Lemma 3). Then W_0', q, W_0 determine a path of length 29 which misses v. (Analogously, if v is a, q, or a'.)

1.3. The first to prove the finiteness of $\bar{P}_3{}^1$ was B. Grünbaum, who established $\bar{P}_3{}^1 \leqq 484$ [9].

Let G and G^* be two graphs. The vertices of G^* have degree at most 3; three of the vertices of G are defined as *endpoints*. Let $\mathfrak{L}(G, G^*)$ be a graph (in general not unique), with the following properties:

1) $\mathfrak{L}(G, G^*)$ admits the family $\mathscr{G}$ of subgraphs, with $\cup \mathscr{G} = \mathfrak{L}(G, G^*)$,
2) each $G' \in \mathscr{G}$ is isomorphic to G,

3) each two distinct graphs of $\mathscr{G}$ are disjoint or have one vertex in common, which corresponds in both of them to an endpoint of G,

4) there exists a bijective function from $\mathscr{G}$ to the vertex-set of G^* such that two distinct graphs in $\mathscr{G}$ are not disjoint if and only if the corresponding vertices of G^* are adjacent.

5) no vertex of $\mathfrak{L}(G, G^*)$ belongs to more than two graphs of $\mathscr{G}$.

If now three vertices of degree two of G^* are *defined as* endpoints, then each graph G' of $\mathscr{G}$ which corresponds to an endpoint of G^* has exactly one endpoint which belongs to no graph in $\mathscr{G}$ except G'. These three vertices we define as endpoints of $\mathfrak{L}(G, G^*)$.

Throughout the paper, graphs of the type $\mathfrak{L}(G, G^*)$ will repeatedly play a central role.

In this subsection we consider the complete graph K_4 on 4 vertices and the graph K of Figure 5, which first appeared as a subgraph of a

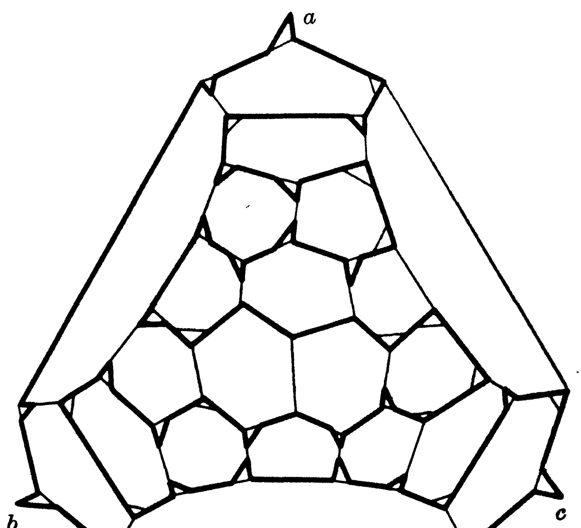

Fig. 5.

graph in [9]. The vertices a, b, c of K we call endpoints. The graph $\mathfrak{L}(K, K_4)$ (here one has unicity!) possesses 478 vertices and is 3-connected. It can be obtained by just contracting 6 of the edges of the graph used by Grünbaum in the proof of the second part of Theorem 1 from [9].

Theorems 4, 5, and 12 suggest that $\mathfrak{L}(K, K_4)$ could be used to prove $\bar{P}_3^1 \leq 478$. This is however wrong! All its longest paths (which have length 474) contain the six 4-valent vertices. Very much responsible for this is the fact that K is hamiltonian.

1.4. W. T. Tutte [16] proved that each 4-connected planar graph is hamiltonian; thus $\bar{P}_4^j = \infty$ for all j. Then, of course, $\bar{P}_5^j = \infty$ too (j arbitrary).

1.5. THEOREM 3. *$\bar{C}_1^1 = 6, \bar{C}_1^2 = 9$ and $\bar{C}_1^j \leqq 3j + 3$.*

PROOF. It is easy to see that there exists a connected graph with $3(j+1)$ vertices which contains $j+1$ disjoint triangles and no other circuits. Any such graph demonstrates the inequality $\bar{C}_1^j \leqq 3j+3$. That $\bar{C}_1^1 > 5$ and $\bar{C}_1^2 > 8$, it may be seen by investigating all connected planar graphs on up to 8 vertices.

1.6. C. Thomassen (private communication) found the graph of Figure 6, which shows $\bar{C}_2^1 \leqq 15$. This improves considerably the inequality $\bar{C}_2^1 \leqq 105$, which can be derived from an example of Walther [17], who first established the finiteness of $\bar{C}_2^1$. I believe $\bar{C}_2^1 = 15$.

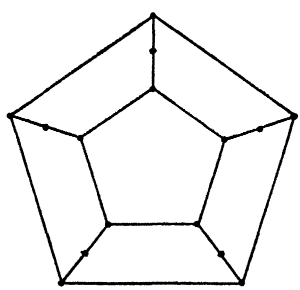

Fig. 6.

1.7. Grünbaum [9] was the first who proved the finiteness of $\bar{C}_3^1$. To do this he used the graph T of Figure 7. Thomassen [14] found recently planar hypohamiltonian and planar hypotraceable graphs. I am indebted to B. Toft and to the referee, who informed me about Thomassen's discoveries. The existence of a 3-connected, planar, hypohamiltonian graph with 105 vertices proves $\bar{C}_3^1 \leqq 105$.

1.8. Tutte's paper [16] yields $\bar{C}_4^j = \bar{C}_5^j = \infty$ for every j.

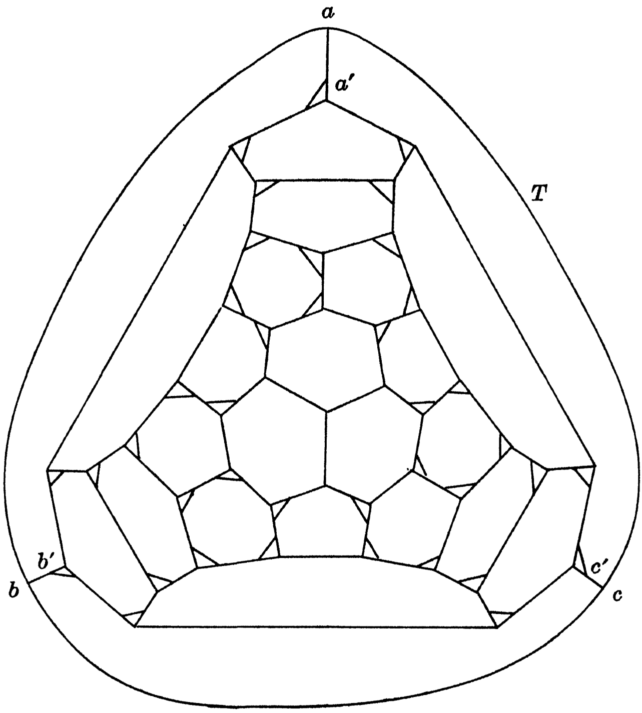

Fig. 7.

2. On the $\bar{P}_k^2$'s and $\bar{C}_k^2$'s.

2.1. We consider now the graph J of Figure 8a and again the graph K of Figure 5. The vertices l, m, n of J and a, b, c of K we shall call endpoints of J, respectively K. J has 14 vertices, K has 121 vertices. The following lemma is straightforward.

LEMMA 5. *The longest paths of J joining two of its endpoints have length* 12.

LEMMA 6. *For each vertex u in J, there exist two endpoints of J and a path of length* 12 *joining them without containing u.*

PROOF. Figures 8a-8e show for all essentially different vertices of J longest paths avoiding them.

The following two lemmas result from comments made by Grünbaum in [9] on the graph T (our Figure 7).

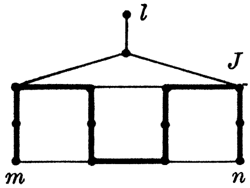

Fig. 8a.

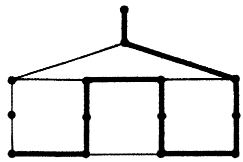

Fig. 8b.

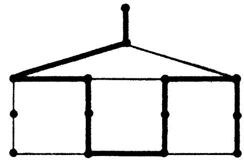

Fig. 8c.

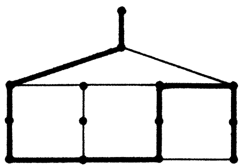

Fig. 8d.

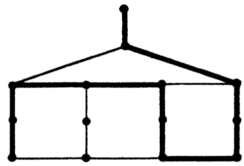

Fig. 8e.

LEMMA 7. *The longest paths of K joining two of its endpoints have length* 118.

LEMMA 8. *For each vertex v in K, there exist two endpoints of K and a path of length* 118 *joining them without containing v.*

The existence of the circuit indicated on Figure 5 proves

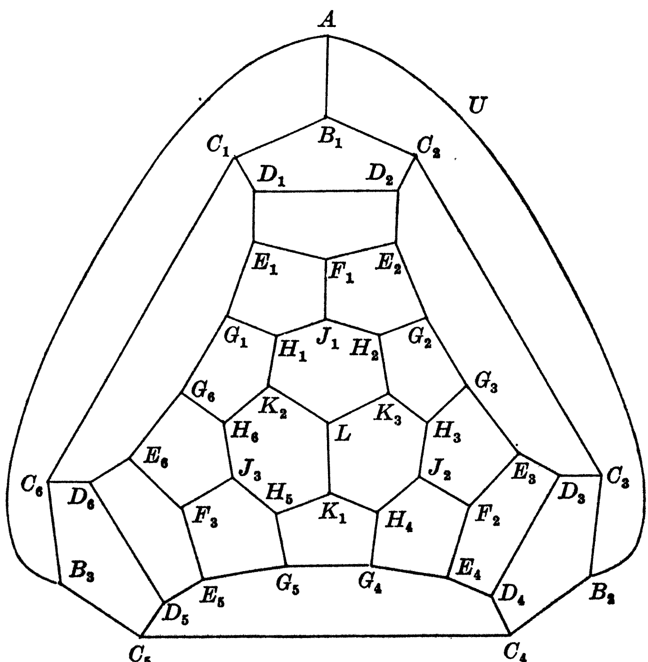

Fig. 9.

LEMMA 9. *The graph K is hamiltonian.*

Let U be the graph of Figure 9. This is a nonhamiltonian graph, discovered (independently) by E. J. Grinberg [7] and Tutte (see [8, p. 1145] and [6]). U has 44 vertices, while $c(U)=43$ [9].

LEMMA 10. *For each pair of nonadjacent edges in U, there exists a circuit of length* 43 *in U avoiding those edges.*

PROOF. Table 1 contains longest circuits in U. We shall now indicate for each pair of edges of U a longest circuit avoiding them. V&WX $=z$ means: the circuit numbered with z on Table 1 does not contain the vertex V and the edge WX. VW&XY $=z$ similarly means that VW and XY are not on the z-th circuit of Table 1. V&WX(V′&W′X′ $=z$) indicates that we replace V and WX through V′ and W′X′ by symmetry and V′&W′X′ $=z$. (Analogously for VW&XY(V′W′&X′Y′ $=z$).)

The symmetry of U allows us to reduce the number of considered cases. We have:

$A\&C_1D_1=1$; $A\&D_1D_2=2$; $A\&D_1E_1(A\&D_5E_5=1)$; $A\&E_1F_1(A\&E_2F_1=1)$; $A\&E_1G_1=1$; $A\&G_1H_1(A\&G_6H_6=1)$; $A\&H_1J_1(A\&H_2J_1=1)$; $A\&F_1J_1(A\&F_3J_3=3)$; $A\&G_1G_6(A\&G_2G_3=1)$; $A\&H_1K_2=1$; $A\&K_2L(A\&K_1L=1)$; $A\&C_1C_6(A\&C_4C_5=1)$. $AB_3\&B_1C_1=5$; $AB_3\&B_1C_2=12$.

$C_1\&C_2D_2=7$; $C_1\&D_2E_2=5$; $C_1\&E_2G_2=7$; $C_1\&G_2G_3=5$; $C_1\&G_3E_3=7$; $C_1\&E_3D_3=5$; $C_1\&D_3C_3=7$; $C_1\&C_3C_2=5$; $C_1\&E_2F_1=8$; $C_1\&F_1J_1=9$; $C_1\&J_1H_2=5$; $C_1\&H_2G_2=6$; $C_1\&H_2K_3=7$; $C_1\&K_3H_3=5$; $C_1\&H_3G_3=6$;

TABLE 1

Missed vertex	Nr	Circuit
A	1	$B_1C_1C_6B_3C_5D_5D_6E_6G_6G_1H_1J_1F_1E_1D_1D_2E_2G_2H_2K_3LK_2H_6J_3F_3E_5G_5H_5K_1H_4G_4E_4F_2J_2H_3G_3E_3D_3D_4C_4B_2C_3C_2$
A	2	$B_1C_1D_1E_1G_1H_1K_2H_6G_6E_6D_6C_6B_3C_5D_5E_5F_3J_3H_5G_5G_4E_4D_4C_4B_2C_3D_3E_3F_2J_2H_4K_1LK_3H_3G_3G_2H_2J_1F_1E_2D_2C_2$
A	3	$B_1C_1C_6B_3C_5D_5D_6E_6F_3E_5G_5H_5J_3H_6G_6G_1H_1K_2LK_1H_4G_4E_4F_2J_2H_3K_3H_2J_1F_1E_1D_1D_2E_2G_2G_3E_3D_3D_4C_4B_2C_3C_2$
B_1	4	$AB_2C_4D_4D_3C_3C_2D_2E_2G_2G_3E_3F_2E_4G_4H_4J_2H_3K_3H_2J_1F_1E_1D_1C_1C_6D_6E_6F_3J_3H_6G_6G_1H_1K_2LK_1H_5G_5E_5D_5C_5B_3$
C_1	5	$AB_1C_2D_2D_1E_1G_1H_1J_1F_1E_2G_2H_2K_3LK_2H_6G_6E_6F_3J_3H_5K_1H_4J_2H_3G_3E_3F_2E_4G_4G_5E_5D_5D_6C_6B_3C_5C_4D_4D_3C_3B_2$
C_1	6	$AB_1C_2D_2D_1E_1G_1H_1K_2LK_1H_5J_3H_6G_6E_6F_3E_5G_5G_4H_4J_2H_3K_3H_2J_1F_1E_2G_2G_3E_3F_2E_4D_4D_3C_3B_2C_4C_5D_5D_6C_6B_3$

TABLE 1 (*cont.*)

Missed vertex	Nr	Circuti
C_1	7	$AB_1C_2C_3B_2C_4C_5D_5E_5F_3J_3H_5G_5G_4E_4D_4D_3E_3F_2J_2H_4K_1L$ $K_3H_3G_3G_2H_2J_1F_1E_2D_2D_1E_1G_1H_1K_2H_6G_6E_6D_6C_6B_3$
C_1	8	$AB_1C_2C_3B_2C_4C_5D_5E_5G_5H_5K_1H_4G_4E_4D_4D_3E_3F_2J_2H_3G_3G_2$ $E_2D_2D_1E_1F_1J_1H_2K_3LK_2H_1G_1G_6H_6J_3F_3E_6D_6C_6B_3$
C_1	9	$AB_1C_2D_2D_1E_1F_1E_2G_2G_3E_3F_2E_4G_4G_5H_5J_3H_6K_2LK_1H_4J_2$ $H_3K_3H_2J_1H_1G_1G_6E_6F_3E_5D_5D_6C_6B_3C_5C_4D_4D_3C_3B_2$
C_1	10	$AB_1C_2C_3D_3E_3F_2J_2H_3G_3G_2H_2K_3LK_2H_1J_1F_1E_2D_2D_1E_1G_1$ $G_6H_6J_3F_3E_6D_6C_6B_3C_5D_5E_5G_5H_5K_1H_4G_4E_4D_4C_4B_2$
C_1	11	$AB_1C_2D_2D_1E_1G_1H_1J_1F_1E_2G_2H_2K_3LK_2H_6G_6E_6F_3J_3H_5K_1$ $H_4G_4G_5E_5D_5D_6C_6B_3C_5C_4D_4E_4F_2J_2H_3G_3E_3D_3C_3B_2$
D_1	12	$AB_1C_1C_6B_3C_5D_5D_6E_6G_6H_6J_3F_3E_5G_5H_5K_1LK_2H_1G_1E_1F_1$ $J_1H_2K_3H_3J_2H_4G_4E_4F_2E_3G_3G_2E_2D_2C_2C_3D_3D_4C_4B_2$
D_1	13	$AB_1C_1C_6B_3C_5D_5D_6E_6G_6H_6K_2LK_3H_3J_2H_4K_1H_5J_3F_3E_5G_5$ $G_4E_4F_2E_3G_3G_2H_2J_1H_1G_1E_1F_1E_2D_2C_2C_3D_3D_4C_4B_2$
D_1	14	$AB_2C_3D_3D_4C_4C_5D_5D_6E_6F_3E_5G_5G_4E_4F_2E_3G_3H_3J_2H_4K_1H_5$ $J_3H_6G_6G_1E_1F_1J_1H_1K_2LK_3H_2G_2E_2D_2C_2B_1C_1C_6B_3$
D_1	15	$AB_2C_3D_3E_3F_2J_2H_4K_1H_5J_3F_3E_6G_6H_6K_2LK_3H_3G_3G_2H_2J_1$ $H_1G_1E_1F_1E_2D_2C_2B_1C_1C_6D_6D_5E_5G_5G_4E_4D_4C_4C_5B_3$
D_1	16	$AB_1C_1C_6D_6E_6G_6H_6K_2H_1G_1E_1F_1J_1H_2G_2E_2D_2C_2C_3B_2C_4D_4$ $D_3E_3G_3H_3K_3LK_1H_4J_2F_2E_4G_4G_5H_5J_3F_3E_5D_5C_5B_3$
D_1	17	$AB_1C_1C_6D_6D_5E_5G_5G_4H_4K_1H_5J_3F_3E_6G_6H_6K_2LK_3H_3J_2F_2$ $E_4D_4D_3E_3G_3G_2H_2J_1H_1G_1E_1F_1E_2D_2C_2C_3B_2C_4C_5B_3$
E_1	18	$AB_1C_2C_3D_3D_4E_4F_2E_3G_3G_2H_2K_3H_3J_2H_4G_4G_5E_5D_5D_6E_6F_3$ $J_3H_5K_1LK_2H_6G_6G_1H_1J_1F_1E_2D_2D_1C_1C_6B_3C_5C_4B_2$
F_1	19	$AB_2C_3C_2B_1C_1C_6D_6D_5E_5G_5H_5K_1LK_3H_2J_1H_1K_2H_6J_3F_3E_6$ $G_6G_1E_1D_1D_2E_2G_2G_3H_3J_2H_4G_4E_4F_2E_3D_3D_4C_4C_5B_3$
F_1	20	$AB_1C_2D_2E_2G_2H_2J_1H_1K_2H_6G_6G_1E_1D_1C_1C_6B_3C_5C_4D_4E_4F_2$ $J_2H_4G_4G_5E_5D_5D_6E_6F_3J_3H_5K_1LK_3H_3G_3E_3D_3C_3B_2$
G_1	21	$AB_1C_1C_6D_6D_5E_5F_3E_6G_6H_6J_3H_5G_5G_4E_4D_4D_3E_3F_2J_2H_4K_1$ $LK_2H_1J_1H_2K_3H_3G_3G_2E_2F_1E_1D_1D_2C_2C_3B_2C_4C_5B_3$
G_1	22	$AB_1C_1D_1E_1F_1J_1H_1K_2H_6G_6E_6F_3J_3H_5K_1LK_3H_2G_2E_2D_2C_2$ $C_3D_3D_4E_4F_2E_3G_3H_3J_2H_4G_4G_5E_5D_5D_6C_6B_3C_5C_4B_2$
G_1	23	$AB_1C_2D_2E_2F_1E_1D_1C_1C_6B_3C_5C_4D_4E_4F_2J_2H_3K_3LK_1H_4G_4$ $G_5H_5J_3F_3E_5D_5D_6E_6G_6H_6K_2H_1J_1H_2G_2G_3E_3D_3C_3B_2$
G_1	24	$AB_1C_1C_6B_3C_5D_5D_6E_6G_6H_6J_3F_3E_5G_5H_5K_1LK_2H_1J_1H_2K_3$ $H_3J_2H_4G_4E_4F_2E_3G_3G_2E_2F_1E_1D_1D_2C_2C_3D_3D_4C_4B_2$
H_1	25	$AB_2C_3D_3E_3F_2J_2H_3G_3G_2E_2F_1J_1H_2K_3LK_2H_6J_3F_3E_5G_5H_5$ $K_1H_4G_4E_4D_4C_4C_5D_5D_6E_6G_6G_1E_1D_1D_2C_2B_1C_1C_6B_3$
J_1	26	$AB_2C_3D_3E_3G_3H_3J_2F_2E_4D_4C_4C_5D_5D_6E_6F_3E_5G_5G_4H_4K_1H_5$ $J_3H_6G_6G_1H_1K_2LK_3H_2G_2E_2F_1E_1D_1D_2C_2B_1C_1C_6B_3$

$C_1\&H_3J_2=7$; $C_1\&J_2F_2=5$; $C_1\&F_2E_3=11$; $C_1\&F_2E_4=7$; $C_1\&E_4D_4=5$; $C_1\&D_4D_3=10$; $C_1\&D_4C_4=6$; $C_1\&C_4B_2=5$; $C_1\&B_2C_3=10$; $C_1\&E_4G_4=6$; $C_1\&G_4G_5=8$; $C_1\&G_5E_5=7$; $C_1\&E_5D_5=6$; $C_1\&D_5C_5=5$; $C_1\&C_5C_4=10$; $C_1\&G_4H_4=5$; $C_1\&H_4J_2=8$; $C_1\&H_4K_1=6$; $C_1\&K_1H_5=7$; $C_1\&H_5G_5=5$; $C_1\&H_5J_3=8$; $C_1\&J_3F_3=6$; $C_1\&F_3E_5=5$; $C_1\&F_3E_6=7$; $C_1\&E_6D_6=5$; $C_1\&D_6D_5=7$; $C_1\&B_3C_5=6$; $C_1\&E_6G_6=8$; $C_1\&G_6G_1=5$; $C_1\&G_1E_1=8$; $C_1\&J_3H_6=5$; $C_1\&H_6G_6=9$; $C_1\&H_6K_2=6$; $C_1\&K_2H_1=5$; $C_1\&H_1G_1=10$; $C_1\&H_1J_1=6$; $C_1\&E_1F_1=5$; $C_1\&K_2L=7$; $C_1\&LK_3=6$; $C_1\&LK_1=5$. $C_1B_1\&C_6B_3=4$; $C_1B_1\&C_6D_6=18$; $C_1B_1\&D_1E_1=18$; $C_1B_1\&D_1D_2=4$; $C_1D_1\&C_6B_3=19$; $C_1D_1\&C_6D_6=12$; $C_1D_1\&B_1C_2=12$; $C_1C_6\&D_1E_1(C_2C_3\&D_2E_2=5)$; $C_1C_6\&D_1D_2=2$; $C_1C_6\&B_1C_2=22$.

$D_1\&E_2G_2=13$; $D_1\&G_2G_3=14$; $D_1\&G_3E_3=15$; $D_1\&E_3D_3=12$; $D_1\&E_2F_1=12$; $D_1\&F_1J_1=13$; $D_1\&J_1H_2=14$; $D_1\&H_2G_2=12$; $D_1\&H_2K_3=13$; $D_1\&K_3H_3=14$; $D_1\&H_3G_3=12$; $D_1\&H_3J_2=15$; $D_1\&J_2F_2=12$; $D_1\&F_2E_3=16$; $D_1\&F_2E_4=15$; $D_1\&E_4D_4=12$; $D_1\&D_4D_3=15$; $D_1\&E_4G_4=17$; $D_1\&G_4G_5=12$; $D_1\&G_5E_5=16$; $D_1\&E_5D_5=12$; $D_1\&G_4H_4=13$; $D_1\&H_4J_2=17$; $D_1\&H_4K_1=12$; $D_1\&K_1H_5=16$; $D_1\&H_5G_5=13$; $D_1\&H_5J_3=12$; $D_1\&J_3F_3=14$; $D_1\&F_3E_5=15$; $D_1\&F_3E_6=12$; $D_1\&E_6D_6=15$; $D_1\&D_6D_5=16$; $D_1\&E_6G_6=14$; $D_1\&G_6G_1=12$; $D_1\&J_3H_6=13$; $D_1D_2\&H_6G_6(D_3D_4\&H_2G_2=25)$; $D_1E_1\&H_6G_6(D_3E_3\&H_2G_2=6)$; $D_1\&H_6K_2=12$; $D_1\&K_2H_1=13$; $D_1\&H_1G_1=14$; $D_1\&H_1J_1=12$; $D_1\&K_2L=16$; $D_1\&LK_3=12$; $D_1\&LK_1=13$. $D_1D_2\&E_1C_1=4$; $D_1D_2\&E_1F_1=2$; $D_1E_1\&D_2E_2(D_3E_3\&D_4E_4=4)$.

$F_1\&G_2G_3=20$; $F_1\&G_3E_3=19$; $F_1\&G_2H_2=19$; $F_1\&H_2K_3=20$; $F_1\&K_3H_3=19$; $F_1\&H_3G_3(F_1\&H_6G_6=19)$; $F_1\&H_3J_2=20$; $F_1\&J_2F_2=19$; $F_1\&F_2E_3=20$; $F_1\&F_2E_4(F_1\&F_3E_5=19)$; $F_1\&E_4G_4=20$; $F_1\&G_4G_5=19$; $F_1\&G_4H_4(F_1\&G_5H_5=20)$; $F_1\&H_4J_2(F_1\&H_5J_3=19)$; $F_1\&H_4K_1=19$; $F_1\&LK_3(F_1\&LK_2=19)$; $F_1E_1\&LK_1=5$; $F_1J_1\&LK_1=26$. $F_1E_1\&E_2G_2=18$; $F_1E_1\&J_1H_1=25$; $F_1E_1\&J_1H_2=18$; $F_1J_1\&E_1G_1=21$.

$G_1\&E_2G_2=23$; $G_1\&G_3E_3=21$; $G_1\&E_4G_4=22$; $G_1\&G_2G_3=22$; $G_1\&J_1H_2=22$; $G_1\&H_2G_2=21$; $G_1\&H_2K_3=23$; $G_1\&K_3H_3=22$; $G_1\&H_3G_3=23$; $G_1\&H_3J_2=21$; $G_1\&G_4G_5=24$; $G_1\&G_4H_4=21$; $G_1\&H_4J_2=23$; $G_1\&H_4K_1=22$; $G_1\&K_1H_5=21$; $G_1\&H_5G_5=22$; $G_1\&H_5J_3=24$; $G_1\&J_3H_6=22$; $G_1\&H_6K_2=21$; $G_1\&K_2L=22$; $G_1\&LK_3=21$; $G_1E_1\&LK_1=1$; $G_1H_1\&LK_1=25$; $G_1G_6\&LK_1=5$; $G_1E_1\&H_1J_1=4$; $G_1E_1\&H_1K_2=1$; $G_1E_1\&G_6E_6=4$; $G_1E_1\&G_6H_6=1$; $G_1H_1\&G_6H_6=19$; $G_1G_6\&H_1J_1=12$; $G_1G_6\&H_1K_2=5$.

$J_1\&K_3H_3=26$; $J_1H_2\&H_3J_2(J_1H_1\&H_6J_3=7)$; $J_1H_1\&H_3J_2=16$; $J_1\&J_2H_4=26$; $J_1H_2\&H_4K_1=18$; $J_1H_1\&H_4K_1=4$; $J_1H_2\&LK_3=18$; $J_1H_1\&LK_3=4$; $J_1\&LK_1=26$. $J_1H_1\&H_2K_3(J_1H_2\&H_1K_2=1)$.

$K_1L\&K_3H_3=1$; $K_1H_4\&K_3H_3=19$; $K_1H_5\&K_3H_3(K_2H_6\&K_3H_2=6)$; $K_1L\&K_3H_2(K_2L\&K_3H_3=19)$; $K_1H_5\&K_3H_2(K_2H_6\&K_3H_3=26)$.

Thus, Lemma 4 is proved.

It is now an easy matter to derive from Lemma 10 the following

LEMMA 11. *For each pair of edges in T, there exists a circuit of length* 121 *avoiding those edges.*

The following two lemmas are consequences of Lemma 11.

LEMMA 12. *For each pair of edges of K, there exists a hamiltonian circuit or a path of length* 118 *joining two endpoints of K, which does not contain those edges.*

LEMMA 13. *For each edge and each endpoint of K, there exists a hamiltonian circuit missing that edge or a path of length* 118 *joining the other two endpoints of K and missing that edge.*

THEOREM 4. $\bar{P}_2^2 \leqq 6050$.

PROOF. We construct

$$M = \mathfrak{L}(\mathfrak{L}(J,K),K_4)\ ,$$

such that in each of the $\mathfrak{L}(J,K)$'s the endpoint l of each of the J's always corresponds to an edge of K which is not contained in a triangle or is an endpoint of that $\mathfrak{L}(J,K)$.

Let P be a longest path of M. We prove that P has 5263 vertices. There are four different cases with respect to the way in which any path can go through the four subgraphs L_1, L_2, L_3, L_4 of M isomorphic to $\mathfrak{L}(J,K)$. They are illustrated in Figure 10.

CASE I. P has an endpoint in the subgraph L_1 of M, isomorphic to $\mathfrak{L}(J,K)$, more precisely in a subgraph J_1 of L_1 isomorphic to J. By Lemma 9, P goes through the interior (i.e. not only the endpoints) of each of the 121 subgraphs of L_1 isomorphic to J. Suppose J_1 is once revisited (we mean its interior); then P contains all the vertices of J_1. By Lemma 5, P has in each of the other 120 J's of L_1 exactly 12 vertices. Thus P has 1333 vertices in L_1. If P does not revisit J_1 but contains all its vertices, then the same argument implies P has 1334 vertices in L_1. Such a path P can be obtained, for example, by developing a hamiltonian circuit of K reduced to a hamiltonian path beginning at an endpoint of K. (In this case J_1 corresponds to a vertex of K adjacent to an endpoint of K, and the position of J_1 in L_1 — see the construction of M — allows P to include a hamiltonian path of J_1.) Therefore, J_1 is not revisited. By Lemma 7, P avoids the interior of three of the 121 subgraphs of L_2 isomorphic to J. Following Lemma 5, P has in each of the other 118 exactly 12 vertices. Thus P has 1299 vertices in L_2. Analogously, P has 1299 vertices in L_4 and 1334 vertices in L_3. It follows that P has length 5263.

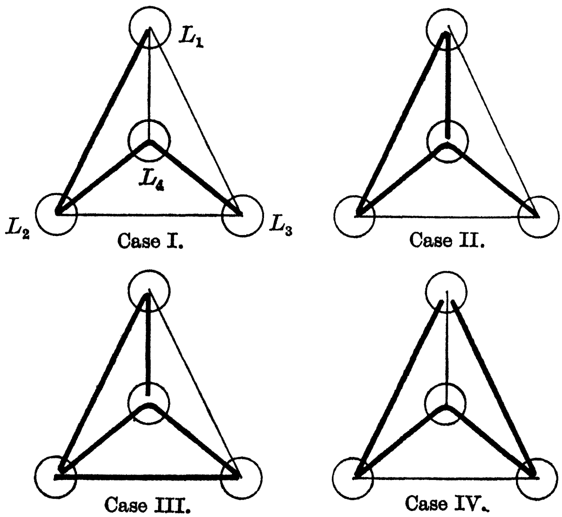

Fig. 10.

Case II. P begins now in L_4 and revisits L_4 after passing through L_1 and L_2. Let P_1, P_2 be the two subpaths of P lying in L_4. Let J_2 be the subgraph of L_4 isomorphic to J in which P (and P_1) begins (it is easily seen that P cannot begin in a vertex belonging to two different subgraphs of L_4 isomorphic to J). Like in Case I we see that J_2 is not revisited by P_1. Suppose J_2 is revisited by P_2. If α is the number of vertices P_1 has in J_2 and β the number of subgraphs of L_4 isomorphic to J through the interior of which P_1 passes, then P_1 has $\alpha+11(\beta-1)$ vertices in L_4 and P_2 has $11(121-\beta)+(14-\alpha)=1345-11\beta-\alpha$ vertices in L_4. Thus P has 1334 vertices in L_4. If P_2 does not revisit J_2 but P_1 contains all the vertices of J_2, then P_1 has $14+11(\beta-1)$ vertices in L_4 and P_2 has $11(121-\beta)+1$ vertices in L_4. Thus P has 1335 vertices in L_4. Therefore, J_2 is not revisited by P_2. Like in Case I one sees further that P has 1299 vertices in each of the subgraphs L_1 and L_2, and 1334 vertices in L_3. Since now four vertices have been counted twice, P has again length 5263.

Case III. Like in the Cases I and II, we show that P has 1335 vertices in each of the graphs L_4 and L_2, and 1299 vertices in each of the graphs L_1 and L_3. Therefore P has again the expected total length 5263.

CASE IV. P begins and ends in L_1. The subgraphs J_3 and J_4 of L_1 isomorphic to J in which P begins and ends are distinct. P has in each of the other 119 subgraphs of L_1 isomorphic to J 12 vertices and in each of the graphs J_3 and J_4 14 vertices. Thus P has 1337 vertices in L_1. Now, like in Case I, we prove that P has 1299 vertices in each of the graphs L_2, L_3, L_4. It follows that P has length 5230, which contradicts the fact that P is a longest path.

In conclusion we can have only Cases I, II, III, and $P(M)=5263$.

We prove now that for each pair of vertices v, w of M, there exists a longest path missing them.

CASE A. *v and w belong both to the same subgraph J_1 of L_1, isomorphic to J.* Let u be the vertex of K corresponding to J_1. By Lemma 8, there exists a path of length 118 joining two endpoints of K and missing u. This path can be easily developed to a path with 1299 vertices lying in L_1, joining two endpoints of L_1, and not containing vertices from J_1. Further this path can be extended to a path of length 5263 in M, like P in Case II above.

CASE B. *v and w belong to different subgraphs J_1 and J_2 of L_1, isomorphic to J.* By Lemma 6, there exists two paths π_1,π_2 of length 12 each, the first joining two endpoints of J_1 in J_1 and avoiding v, the second joining two endpoints of J_2 in J_2 and avoiding w. Let z_i be the endpoint of J_i which is not an endpoint of π_i $(i=1,2)$. If z_i is not an endpoint of L_1, then z_i belongs to two subgraphs, each of which is isomorphic to J. These subgraphs correspond to the endpoints of a certain edge ε_i in K.

Suppose z_1 and z_2 are not endpoints of L_1. If some hamiltonian circuit of K misses both ε_1 and ε_2, then there exists a hamiltonian path of K having exactly one of its endpoints in an endpoint of K belonging to a triangle which contains none of the vertices corresponding to J_1 and J_2, and avoiding both ε_1 and ε_2. This path can be easily developed to a path π_3 in L_1 of length 1334 (see Case I above) such that π_1, π_2 be subpaths of it, one of its endpoints be an endpoint of L_1, and z_i be not on $\pi_3-\pi_4$. Then clearly v and w are not on π_3. Further π_3 can be extended to a path of length 5263 in M, like P in Case I above. If no hamiltonian circuit of K misses both ε_1 and ε_2 then, following Lemma 12, there exists a path of length 118 in K joining two endpoints of K, such that ε_1 and ε_2 do not belong to it. This path can be easily developed to a path π_4 in L_1 joining two endpoints of L_1, being of length 1299, and admitting π_1 and π_2 as subpaths. Clearly again, v and w are not on π_4 and π_4 can be extended to a path of length 5263 in M, like P in Case II above.

Suppose z_1 is an endpoint of L_1, z_2 is not. By Lemma 13, there exists a hamiltonian path in K avoiding ε_2 and having one of its endpoints in an endpoint of K belonging to a triangle which contains none of the vertices corresponding to J_1 and J_2, or there exists a path of length 118 in K joining the endpoints of K which do not correspond to J_1 and avoiding ε_2. The proof follows like above.

If z_1 and z_2 are both endpoints of L_1, we take a hamiltonian path in K with en endpoint in the endpoint of K which does not correspond to J_1 or J_2, and proceed again as above.

CASE C. *v and w belong to different subgraphs L_1, L_2 of M.* Construct like in Case B the paths π_1 in L_1 and π_2 in L_2, each of which lying in a subgraph isomorphic with J, and consider also the vertices z_1, z_2 (introduced as in Case B). If z_i is not an endpoint of L_i, we define like in Case B the edge ε_i of K.

Suppose z_1 and z_2 are not endpoints of L_1, respectively L_2. Then there exists in K, by Lemma 13, a hamiltonian circuit missing ε_i or a path of length 118 joining two arbitrary endpoint and missing $\varepsilon_i (i=1,2)$. Consider the following property:

(P_i) *ε_i belongs to the same triangle T_i as one of the endpoints of K, without being adjacent to it.*

CASE 1°. (P_1) *and* (P_2) *hold.* Suppose z_1 is not the endpoint of L_1 belonging to L_3 or z_2 is not the endpoint of L_2 belonging to L_3. There exists a hamiltonian circuit in K missing ε_i. We can reduce in two ways this circuit to a hamiltonian path π_i' joining an endpoint of K with a vertex which is not adjacent to ε_i $(i=1,2)$. These paths π_1' and π_2' can be developed to two paths π_3 in L_1 and π_4 in L_2 including π_1 and π_2 respectively, and of length 1334 each. Further π_3 and π_4 can be extended to a path of length 5263 in M, symmetric to that shown in Figure 10 (Case I), with the endpoints in L_1 and L_2.

Suppose now that both z_1 and z_2 belong also to L_3. There exists a path π_i' of length 118 in K joining two endpoints of K and avoiding T_i $(i=1,2)$. The paths π_1' and π_2' can be developed to two paths π_3 in L_1 and π_4 in L_2, including π_1 and π_2 (respectively), and of length 1299 each. Further π_3 and π_4 can be extended to a path of length 5263 in M like in Figure 10 (Case II).

CASE 2°. (P_1) *holds,* (P_2) *not.* Again, there exists a path π_1' of length 118 in K joining two endpoints of K and avoiding T_1. Also, a) there is a hamiltonian circuit missing ε_2 or b) there exists a path of length 118 joining two arbitrary endpoints of K and missing ε_2. In the subcase a), we are able — since (P_2) doesn't hold — to reduce the hamiltonian circuit to a hamiltonian path π_2' beginning in an arbitrary endpoint of K, that we may choose conveniently. Thus, the two paths π_1' and π_2' can be developed to two paths π_3 in L_1 and π_4 in L_2, including π_1 and π_2 (respectively), the first of length 1299, the second of length 1334. Further π_3 and π_4 can be extended to a path of length 5263 in M with a shape like that shown in Figure 10 (Case I), but with one endpoint in L_2 and the other in L_3 or L_4. In the subcase b) we are led to a shape like in Figure 10 (Case I) ($L_4 - L_1 - L_2 - L_3$), Figure 10 (Case I) ($L_3 - L_1 - L_2 - L_4$), or Figure 10 (Case III) ($L_4 - L_1 - L_3 - L_2 - L_4 - L_3$).

The case that (P_1) doesn't hold and (P_2) holds is symmetric to Case 2°.

CASE 3°. *Neither* (P_1), *nor* (P_2) *holds.*

SUBCASE a): *There exists a hamiltonian circuit missing* ε_i ($i = 1, 2$). In this subcase we are able — since (P_i) don't hold — to reduce the hamiltonian circuit to hamiltonian paths beginning in conveniently chosen endpoints of K. The shape of the final path of length 5263 in M will be symmetric to that of Figure 10 (Case I), but with the endpoints in L_1 and L_2.

SUBCASE b): *There exists a path of length* 118 *joining two arbitrary endpoints of* K *and missing* ε_1 *and a hamiltonian circuit missing* ε_2. In this subcase we are again able to reduce the hamiltonian circuit to a hamiltonian path beginning in a conveniently chosen endpoint of K, and also to choose conveniently the endpoints of the path of length 118. Thus we finally obtain a longest path in M, the shape of which is for example like that indicated in Figure 10 (Case I), but with the endpoints in L_2 and L_3.

The subcase obtained from Subcase b) by replacing ε_1 with ε_2 and vice-versa is analogous to Subcase b).

SUBCASE c): *There exists a path of length* 118 *joining two arbitrary endpoints of* K *and missing* ε_i ($i = 1, 2$). Since the endpoints of these paths can be chosen arbitrarily (by Lemma 13), we can eventually obtain, for instance, a path like that of Figure 10 (Case II).

The case that one or both of the points z_1, z_2 are endpoints of L_1, L_2 doesn't need to be treated separately, because it suffices to immagine that property (P_i) holds when z_i is an endpoint of L_i, and then to proceed as above. Indeed, we used — each time (P_i) did hold — paths in K that missed not only ε_i but all the triangle T_i. Thus, the developed paths π_3, π_4 didn't contain the endpoints of L_1, L_2 belonging to J's which correspond to vertices (endpoints of K) of T_1, T_2.

M has 6050 vertices. It is easily seen that $\mathfrak{L}(J,K)$ is 2-connected. Since K_4 is also 2-connected, M is 2-connected too. Theorem 4 is completely proved.

2.2. Theorem 5. $\bar{P}_3^{\,2} \leqq 57838$.

Proof. The graph

$$M' = \mathfrak{L}(\mathfrak{L}(K,K); K_4)$$

has 57838 vertices. Since K is 3-connected, $\mathfrak{L}(K,K)$ is 3-connected too, and because K_4 is also 3-connected the same can be said about M'. The proof of the facts that $p(M') = 55933$ and that for each two vertices v, w of M' there exists a path of length 55933 which misses both of them is almost identical with that of Theorem 4. The only modifications consist in considering K instead of J and Lemmas 7 and 8 instead of Lemmas 5 and 6, respectively.

2.3. The graph T of Figure 7 has 124 vertices and 186 edges.

Lemma 14 (Grünbaum [9]). *We have* $c(T) = 121$ *and each vertex of* T *is missed by some circuit of* T *of length* 121.

Theorem 6. $\bar{C}_2^{\,2} \leqq 1550$.

Proof. The reader who followed the proof of Theorem 4 will have no diffeculty in understanding that of Theorem 6.

Let $S = \mathfrak{L}(J,T)$. If G is the graph obtained by replacing each vertex of T by a copy of J (two adjacent vertices in G belong to different copies of J only if each of them corresponds to one of the endpoints l,m,n), then S can be obtained from G by contracting each edge of T appearing in G (i.e. joining two distinct copies of J). Clearly, S has 1550 vertices. Label with $J_1, \ldots, J_{124}$ the (not pairwise disjoint!) copies of J contained by S.

By Lemma 14, a circuit W of length $c(S)$ in S avoids three copies of J (excepting perhaps their endpoints). By Lemma 5, W has in each other J_j exactly 12 vertices. Thus $c(S) = 1331$.

Let u,v be two vertices of S. We prove the existence of a circuit of length 1331 in S not containing u and v.

CASE 1. *u and v belong to the same* J_i. Use again l, m,n to denote the endpoints of J_i. These endpoints also belong to three other copies of J, say J_j, J_k, J_q respectively. By Lemma 14, there exists a circuit W_0 in S avoiding J_i (excepting perhaps the vertices l,m,n). We show that W_0 may be transformed, if necessary, so that it avoids also the point l (analogously for m and n). Suppose W_0 contains vertices from J_j which are not endpoints of J_j, otherwise of course W_0 does not contain l. Then it necessarily passes through the two endpoints of J_j different from l. Now, Lemma 6 asserts that $W_0 \cap J_j$ can be modified, if necessary, so that it does not contain l.

CASE 2. *u is in* J_i, *v is in* $J_j (i \neq j)$. By Lemma 6, there exist two paths Π_1 and Π_2 on 12 vertices each, the first joining two endpoints of J_i within J_i and missing u, the second joining two endpoints of J_j within J_j and avoiding v. Now, it is easily seen that Lemma 11 implies the existence of a circuit Γ of length 1331 in S, whose intersection with J_i is either Π_1, or $\Pi_1 \cup E(J_i)$, or is included in the set $E(J_i)$ of the endpoints of J_i, and whose intersection with J_j is either Π_2, or $\Pi_2 \cup E(J_j)$, or is included in $E(J_j)$. An argument similar to that of Case 1 shows further that Γ can be modified so that $\Gamma \cap J_i \subseteq \Pi_1$ and $\Gamma \cap J_j \subseteq \Pi_2$.

2.4. THEOREM 7. $\bar{C}_3^2 \leqq 14818$.

PROOF. Consider $S' = \mathfrak{L}(K,T)$. This graph has 14818 vertices. The proof of the fact that for each two vertices of S' there exists a longest circuit avoiding them is exactly like in Theorem 6; only, instead of Lemmas 5 and 6 we use here Lemmas 7 and 8 respectively. The planarity and 3-connectedness of S' are evident. $c(S') = 14157$.

Grünbaum [9] conjectured $\bar{C}_3^j, \bar{P}_3^j < \infty$ for all j; I conjecture the contrary.

3. On the P_k^1's and C_k^1's.

3.1. We mentioned that the first who proved the finiteness of P_1^1 was Walther [17]. His graph has 25 vertices.

THEOREM 8. $P_1^1 \leqq 12$.

PROOF. Consider the graph G of Figure 11. By identifying the vertices a,b,c one obtains Petersen's graph P. Since P is hypohamiltonian, it is easily seen that $p(G)=10$ and each vertex of G is missed by some longest path. G has 12 vertices and provides the smallest known graph answering to Gallai's original question.

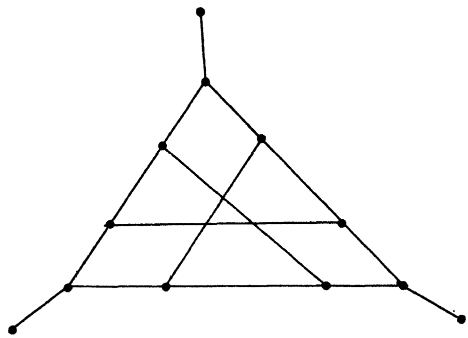

Fig. 11.

CONJECTURE 1. $P_1^1=12$.

3.2. The finiteness of P_2^1 was first proved by using a planar graph with 82 vertices (Zamfirescu [19]). J. D. Horton [12] then found a 3-connected hypotraceable graph on 40 vertices. Thomassen [13] has a 2-connected hypotraceable graph with only 34 vertices. Finally, the (planar!) graph of Figure 4 shows

THEOREM 9. $P_2^1 \leqq 32$.

Theorem 9 is a consequence of Theorem 2.

3.3. The first to prove the finiteness of P_3^1 was Grünbaum, who found, as we already saw, $\overline{P}_3^1 \leqq 484$ [9]. The mentioned graph of Horton [12] proves $P_3^1 \leqq 40$.

THEOREM 10. $P_3^1 \leqq 36$.

PROOF. Let F be the graph P of Petersen minus any vertex v of P (and its adjacent edges). By replacing each vertex of K_4 with a copy of F as in Figure 12 we obtain a graph H with 36 vertices and $p(H)=34$. Again, it is not hard to see that each point of H is missed by some longest path. In the terminology of section 5.2, $H=\mathfrak{L}^*(F,K_4)$.

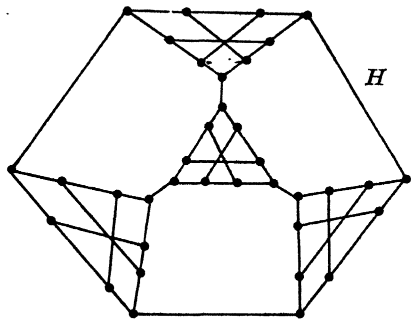

Fig. 12.

3.4. Theorem 11. *$C_1^1=6$, $C_1^2=9$ and $C_1^j \leqq 3j+3$.*

Proof. $C_1^j \leqq 3j+3$ follows from Theorem 3. That $C_1^1>5$ and $C_1^2>8$, it may be seen by investigating all connected graphs on at most 8 vertices (which are not trees).

Conjecture 2. $C_1^j=3j+3$.

3.5. We have $C_2^1=C_3^1=10$ (see C. Berge [1], [2] (pp. 213–217), R. G. Busacker and T. L. Saaty [3] (pp. 43–46), Grümbaum [9], J. C. Herz, J. J. Duby and F. Vigué [10], Herz, T. Gaudin and P. Rossi [10]).

4. On the P_k^2's and C_k^2's.

4.1. Grünbaum was the first to prove the finiteness of P_3^2. He established $P_3^2 \leqq 324$ [9].

Theorem 12. $P_3^2 \leqq 270$.

Proof. Let F be the graph used in the proof of Theorem 10 and let the three 2-valent vertices of F be considered endpoints. Construct

$$D = \mathfrak{L}(\mathfrak{L}(F,F),K_4).$$

The graph D has 270 vertices and is 3-connected. We consider as superflous to give here a proof of the fact that for each two vertices of D there exists a longest path avoiding them, since it contains no new ideas, compared with that of Theorem 4. It is very similar to and moreover simpler than that proof because it involves smaller graphs. $p(D)=241$.

4.2. Walther was the first who proved the finiteness of C_2^2; he established $C_2^2 \leqq 220$ [18]. Grünbaum improved this inequality and verified the finiteness of C_3^2 by establishing $C_3^2 \leqq 90$ [9].

THEOREM 13. $C_3^2 \leqq 75$.

PROOF. The proof is analogous to, but simpler than that of Theorem 6.

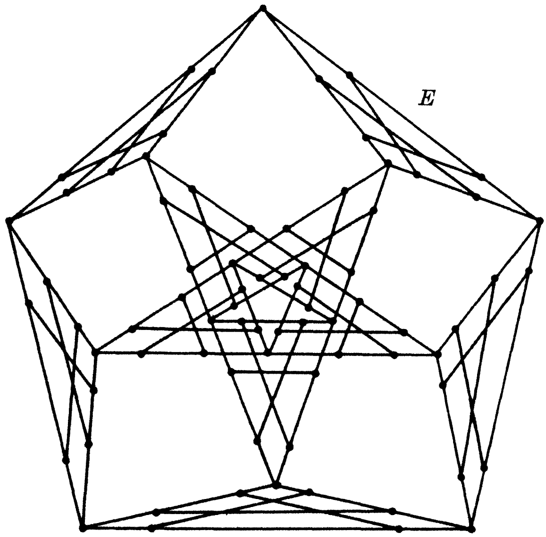

Fig. 13.

TABLE 2

P_k^j

k/j	1	2	3
1	$\leqq 12$	$\leqq 270$	?
2	$\leqq 32$	$\leqq 270$	?
3	$\leqq 36$	$\leqq 270$	?
4	?	?	?

C_k^j

k/j	1	2	3
1	6	9	$\leqq 12$
2	10	$\leqq 75$	?
3	10	$\leqq 75$	?
4	?	?	?

$\bar{P}_k^j$

k/j	1	2	3
1	$\leqq 19$	$\leqq 6050$	?
2	$\leqq 32$	$\leqq 6050$	?
3	$\leqq 484$	$\leqq 57838$	?
4	∞	∞	∞

$\bar{C}_k^j$

k/j	1	2	3
1	6	9	$\leqq 12$
2	$\leqq 15$	$\leqq 1550$	?
3	$\leqq 105$	$\leqq 14818$	?
4	∞	∞	∞

The graph we use is very much related to that used by Grünbaum for the first part of his Theorem 2 from [9]. It is

$$E = \mathfrak{L}(F,P)\,,$$

where P is Petersen's graph and F was introduced in the proof of Theorem 10. The graph E is illustrated in Figure 13. $c(E)=63$.

It is clear that, besides establishing better inequalities than the known ones, of great interest would be to decide about the finiteness of P_4^1, $C_4^1, P_1^3, C_2^3, \bar{P}_1^3$, and $\bar{C}_2^3$.

5. On the families $\Pi(j,m)$ and $\Gamma(j,m)$.

5.1. Grünbaum introduced in [9] the following families of graphs: $\Pi(j,m)$ — that of all graphs each of which 1) has m vertices more than its longest paths and 2) possesses, for each j vertices, a longest path missing them ($j \leqq m$), and $\Gamma(j,m)$ — the analogous family, in the definition of which circuits appear instead of paths, and the graphs are not trees.

Put $\mathfrak{P}_k^j = \infty$ if $\Pi(j,m)$ contains no k-connected graphs for every m, otherwise let us denote by $\mathfrak{P}_k^j$ the minimum number m such that $\Pi(j,m)$ contains k-connected graphs. Let $\mathfrak{C}_k^j$ be the analogous number defined with $\Gamma(j,m)$ instead of $\Pi(j,m)$ and $\overline{\mathfrak{P}}_k^j$, $\overline{\mathfrak{C}}_k^j$ the numbers obtained if the considered graphs are planar. Clearly, $j \leqq \mathfrak{P}_k^j \leqq \overline{\mathfrak{P}}_k^j$ and $j \leqq \mathfrak{C}_k^j \leqq \overline{\mathfrak{C}}_k^j$. Grünbaum [9] makes a number of interesting remarks and conjectures about $\Pi(j,m)$ and $\Gamma(j,m)$; some of them can be reformulated by using the numbers introduced above.

5.2. Walther [17] constructed, for each $m \geqq 4$, connected graphs belonging to $\Pi(1,m)$.

THEOREM 14. *For each $m \geqq 1$, there exists 2-connected planar graphs belonging to $\Pi(1,m)$.*

PROOF. Thomassen's planar hypotraceable graph belongs to $\Pi(1,1)$ and is 2-connected [14]. The graph V' of Figure 14, where V_1, V_2, V_3, V_4 are isomorphic to Thomassen's planar hypohamiltonian graph V opened in a vertex v of degree 3, belongs to $\Pi(1,2)$ and is 3-connected. By inserting k new vertices on each of the six edges connecting a V_i with a V_j ($i \neq j$), we obtain a 2-connected, planar graph belonging to $\Pi(1,2+k)$.

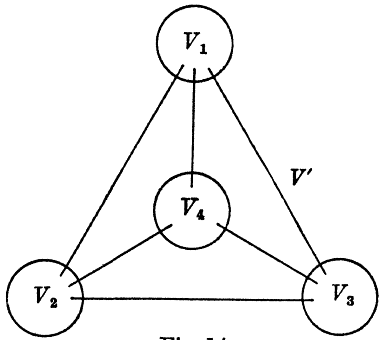

Fig. 14.

COROLLARY 1. $\overline{\mathfrak{P}}_2^1 = 1$.

THEOREM 15. *For each* $m \geqq 4\,639\,718$, *there exist* 3-*connected planar graphs belonging to* $\Pi(1,m)$; $\overline{\mathfrak{P}}_3^1 \leqq 2$.

PROOF. In section 1.3 we defined a certain composition law of graphs G and G^*, the resulting graph (not unique) being denoted by $\mathfrak{L}(G,G^*)$. We introduce now a variant of this composition in the following manner. Let $\mathfrak{L}^*(G,G^*)$ be a graph (in general not unique), satisfying the following conditions:

1) $\mathfrak{L}^*(G,G^*)$ admits a family $\mathscr{G}^*$ of subgraphs which cover all its vertices,

2) each $G' \in \mathscr{G}^*$ is isomorphic to G,

3) each two distinct graphs G', G'' of $\mathscr{G}^*$ are disjoint and either there is no edge connecting a vertex of G' with one of G'' (in which case we say that G' and G'' are *not adjacent*) or there is exactly one such edge, which then necessarily connects an endpoint of G' with an endpoint of G'' and is called *connecting edge* (in this case we say G' and G'' are *adjacent*),

4) there exists a bijective function from $\mathscr{G}^*$ to the vertex set of G^* such that two distinct graphs of $\mathscr{G}^*$ are adjacent if and only if the corresponding vertices of G^* are adjacent,

5) there is no vertex common to (at least) two connecting edges,

6) every edge of $\mathfrak{L}^*(G,G^*)$ either belongs to a graph of $\mathscr{G}^*$ or is a connecting edge.

It is easily seen that $\mathfrak{L}(G,G^*)$ can be obtained from $\mathfrak{L}^*(G,G^*)$ by simply contracting all the connecting edges.

It is in fact easier (and similar) to verify that

$$M'' = \mathfrak{L}^*(\mathfrak{L}^*(K,K),K_4)$$

belongs to some $\Pi(2,m)$, than that M' does (see the proof of Theorem 5). But M'' has more vertices than M', namely 58 564, therefore it was not used to provide an upper bound for $\overline{P}_3{}^2$. Instead, it is regular, of degree 3.

We choose now an arbitrary vertex α of M'' and denote by A the set of all other 58 563 vertices of M''. Let $M_{p,q}$ be the graph obtained by replacing each vertex in A through a copy of Y_p (see Figure 15) and α through a copy of $Y_q (0 \leqq q \leqq p)$.

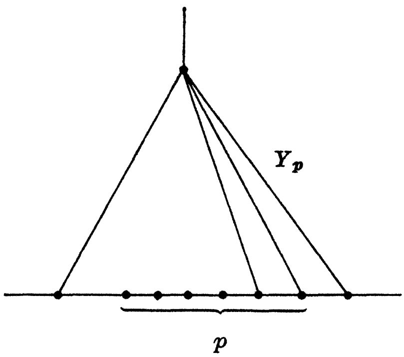

Fig. 15.

Each longest path of $M_{p,q}$ misses entirely the copy of Y_q which replaced α, if $q < p$. On the other hand, for each $\beta \in A$, there is a longest path in M'' that avoids α and β. Thus, for each vertex of $M_{p,q}$ there is a longest path avoiding it.

Since M'' and its longest paths have 58 564, respectively 56 410 vertices, the graph $M_{p,q}$ and its longest paths have 58 563 $(p+3)+(q+3)$, respectively 56 410 $(p+3)$ vertices. It follows that

$$M_{p,q} \in \Pi(1, 2153(p+3)+(q+3)) .$$

The equation

$$2153(p+3)+(q+3) = n$$

has for all $n \geqq 4\ 639\ 718$ positive integer solutions in p and q, such that $p \geqq q$. In other words, for all $n \geqq 4\ 639\ 718$,

$$M_{p,q} \in \Pi(1,n)$$

for some p, q.

The graph V' used in the proof of the preceding theorem is 3-connected and belongs to $\Pi(1,2)$, which proves $\overline{\mathfrak{P}}_3^1 \leqq 2$.

CONJECTURE 3. $\overline{\mathfrak{P}}_3^1 = 1$.

The graph T of Figure 7 found by Grünbaum [9] shows $\overline{\mathfrak{C}}_3^1 \leqq 3$. He (compare V. Chvátal [4]) conjectured in [9] that $\overline{\mathfrak{C}}_1^1 = 3$. However this is false, as the following theorem demonstrates.

THEOREM 16. *For each* $m \geqq 1$, *there exist* 3-*connected planar graphs belonging to* $\Gamma(1,m)$.

PROOF. The mentioned planar hypohamiltonian graph V of Thomassen is 3-connected. Moreover it can be used, as noted by the referee, to construct the graph of Figure 16, where each of the subgraphs $V_1', \ldots, V_k'$

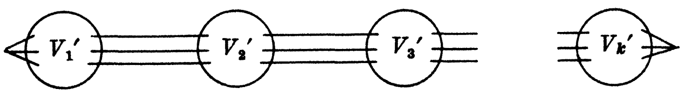

Fig. 16.

is a copy of V opened up in two nonadjacent vertices of degree 3, lying on the frontier of the same region. This graph belongs to $\Gamma(1,k)$ $(k \geqq 1)$ and is evidently also 3-connected.

COROLLARY 2. $\overline{\mathfrak{C}}_3^1 = 1$.

Grünbaum also conjectured that every $\Gamma(j,m)$ contains only finitely many planar graphs. In this general form, the conjecture can be disproved by considering the sequence of graphs $\{X_k\}_{k=1}^{\infty}$ illustrated in Figure 17, each of which belongs to $\Gamma(1,2)$. However, the question whet-

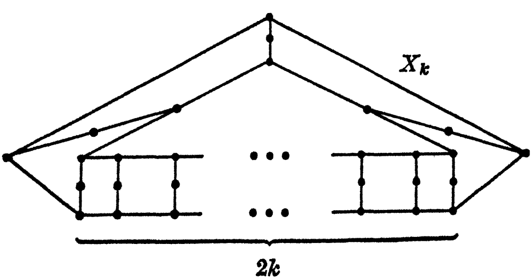

Fig. 17.

her there exist nonempty families $\Gamma(j,m)$ which contain only finitely many planar (3-connected) graphs remains open.

THEOREM 17. *The family* $\Pi(2,m)$ *contains* 3-*connected planar graphs for infinitely many m's*; $\overline{\mathfrak{P}}_2{}^2 \leqq 787$ *and* $\overline{\mathfrak{P}}_3{}^2 \leqq 1905$.

PROOF. Consider again the graph M'' constructed in the proof of Theorem 15. Its use in the present proof is again motivated by its regularity: each of its vertices has degree 3, and so we can replace all of them by copies of Y_p (see Figure 15). Since $M'' \in \Pi(1,2154)$, our graph belongs to $\Pi(1,2154(p+3))$ $(p \geqq 0)$.

The differences $m=787$ and $m=1905$ occur at the graphs M and M' from the proofs of Theorems 4 and 5.

THEOREM 18. *The family* $\Gamma(2,m)$ *contains* 3-*connected planar graphs for infinitely many m's*; $\overline{\mathfrak{C}}_1{}^2 \leqq 6$, $\overline{\mathfrak{C}}_2{}^2 \leqq 219$ *and* $\overline{\mathfrak{C}}_3{}^2 \leqq 661$.

PROOF. We replace each vertex of the graph $\mathfrak{L}^*(K,T)$, where K and T are the graphs of Figures 5 and 7, by a copy of the graph Y_p of Figure 15. The graph we obtain is 3-connected and planar. It belongs to $\Gamma(2, 726(p+3))$, because $\mathfrak{L}^*(K,T) \in \Gamma(2,726)$.

The differences $m=6$, $m=219$, and $m=661$ occur at one of the graphs used in the proof of Theorem 3, at the graph S from the proof of Theorem 6, and at the graph S' from that of Theorem 7.

Since $\overline{P}_4{}^j = \overline{C}_4{}^j = \infty$ for all j, we have

THEOREM 19. $\overline{\mathfrak{P}}_4{}^j = \overline{\mathfrak{C}}_4{}^j = \infty$.

5.3. THEOREM 20. *For each* $m \geqq 1$, *there exists* 3-*connected graphs belonging to* $\Pi(1,m)$.

PROOF. The 3-connected hypotraceable graph of Horton [12] belongs to $\Pi(1,1)$.

The 3-connected graph H of Figure 12 belongs to $\Pi(1,2)$ (see Theorem 10).

Further replace every vertex of Horton's mentioned graph by a copy of the graph Y_p of Figure 15. Obviously the obtained 3-connected graph belongs to $\Pi(1,p+3)$ for all $p \geqq 0$.

COROLLARY 3. $\mathfrak{P}_3{}^1 = 1$.

Grünbaum [9] proved $\mathfrak{P}_3^2 \leqq 50$ and $\mathfrak{C}_3^2 \leqq 18$. The next two theorems strengthen these inequalities.

THEOREM 21. $\mathfrak{P}_3^2 \leqq 29$.

PROOF. The graph D defined in the proof of Theorem 12 belongs to $\Pi(2,29)$.

THEOREM 22. $\mathfrak{C}_3^2 \leqq 12$.

PROOF. The graph E of Figure 13, used in the proof of Theorem 13, belongs to $\Gamma(2,12)$.

TABLE 3

		$\mathfrak{P}_k^j$				$\mathfrak{C}_k^j$	
k/j	1	2	3	k/j	1	2	3
1	1	$\leqq 29$	?	1	1	$\leqq 6$	$\leqq 9$
2	1	$\leqq 29$	?	2	1	$\leqq 12$	?
3	1	$\leqq 29$	?	3	1	$\leqq 12$	?
4	?	?	?	4	?	?	?

		$\overline{\mathfrak{P}}_k^j$				$\overline{\mathfrak{C}}_k^j$	
k/j	1	2	3	k/j	1	2	3
1	1	$\leqq 787$	?	1	1	$\leqq 6$	$\leqq 9$
2	1	$\leqq 787$	?	2	1	$\leqq 219$	?
3	$\leqq 2$	$\leqq 1905$	?	3	1	$\leqq 661$	?
4	∞	∞	∞	4	∞	∞	∞

5.4. Let $\mathscr{P}_k^j(\mathscr{C}_k^j)$ be the set of all families $\Pi(j,m)(\Gamma(j,m))$ which contain k-connected graphs. Let $\overline{\mathscr{P}}_k^j(\overline{\mathscr{C}}_k^j)$ be the subset of $\mathscr{P}_k^j(\mathscr{C}_k^j)$ of all families $\Pi(j,m)(\Gamma(j,m))$ which contain k-connected planar graphs. We shall say that $\mathscr{P}_k^j$ is *m_0-full* if it contains all families $\Pi(j,m)$ for $m \geqq m_0$. $\mathscr{P}_k^j$ will be called *full* if it is m_0-full for some m_0, and the minimum of the m_0's such that $\mathscr{P}_k^j$ be m_0-full will be denoted by $\mathfrak{p}_k^j$ (if $\mathscr{P}_k^j$ is not full we put $\mathfrak{p}_k^j = \infty$). If $\mathfrak{p}_k^j = j$, $\mathscr{P}_k^j$ will be said to be *completely full.* If $\mathfrak{p}_k^j = \mathfrak{P}_k^j$, $\mathscr{P}_k^j$ will be called *convex*. Analogously are defined the above notions and the numbers $\mathfrak{c}_k^j$, $\overline{\mathfrak{p}}_k^j$, $\overline{\mathfrak{c}}_k^j$ for the sets $\mathscr{C}_k^j$, $\overline{\mathscr{P}}_k^j$ and $\overline{\mathscr{C}}_k^j$ respectively.

By Theorems 14 and 16, $\overline{\mathscr{P}}_2^1$ and $\overline{\mathscr{C}}_3^1$ are completely full. By Theorem 15, $\overline{\mathscr{P}}_3^1$ is 4 639 718-full, but we do not know what $\overline{\mathfrak{p}}_3^1$ is.

By Theorem 20, $\mathscr{P}_3^1$ is completely full.

About $\overline{\mathscr{P}}_3{}^2$ and $\overline{\mathscr{C}}_3{}^2$ we know that they are infinite (see Theorems 17 and 18), but not whether they are full or not. It is not even known whether $\mathscr{P}_1{}^2$ or $\mathscr{C}_2{}^2$ is full. It is easily seen that $\overline{\mathscr{C}}_1{}^j$ is $3j$-full, but we do not know what $\mathfrak{c}_1{}^j$ and $\mathfrak{c}_1{}^j$ are ($j \geqq 2$).

TABLE 4

$\mathfrak{p}_k{}^j$

k/j	1	2
1	1	?
1	1	?
3	1	?
4	?	?

$\mathfrak{c}_k{}^j$

k/j	1	2	3
1	1	$\leqq 6$	$\leqq 9$
2	1	?	?
3	1	?	?
4	?	?	?

$\bar{\mathfrak{p}}_k{}^j$

k/j	1	2
1	1	?
2	1	?
3	$\leqq 4639718$	?
4	∞	∞

$\bar{\mathfrak{c}}_k{}^j$

k/j	1	2	3
1	1	$\leqq 6$	$\leqq 9$
2	1	?	?
3	1	?	?
4	∞	∞	∞

We obtained $\overline{\mathfrak{P}}_2{}^1 = \bar{\mathfrak{p}}_2{}^1 = 1$, $\overline{\mathfrak{C}}_3{}^1 = \bar{\mathfrak{c}}_3{}^1 = 1$, $\overline{\mathfrak{P}}_4{}^j = \bar{\mathfrak{p}}_4{}^j = \infty$, $\overline{\mathfrak{C}}_4{}^j = \bar{\mathfrak{c}}_4{}^j = \infty$, $\mathfrak{P}_3{}^1 = \mathfrak{p}_3{}^1 = 1$. These equalities suggest the question whether all considered sets of families are convex.

NOTE. W. Schmitz recently proved $\bar{P}_1{}^1 \leqq 17$, $P_1{}^2 \leqq 108$, and $\bar{C}_2{}^2 \leqq 170$ in *Über längste Wege und Kreise in Graphen*, to appear. Walther proved independently in his and Voss' book *Über Kreise in Graphen* (VEB-DVW, Berlin, 1974) that $P_1{}^1 \leqq 12$ and $P_1{}^2 \leqq 108$. Thomassen succeeded to prove Conjecture 2: $C_1{}^j = 3j + 3$ (private communication). I can prove now $P_1{}^2 \leqq 93$ and $\bar{C}_2{}^2 \leqq 135$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DORTMUND
46 DORTMUND - HOMBRUCH
WEST-GERMANY