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# GENERALIZED CONTRACTIONS AND FIXED POINTS IN METRIC SPACES

RIASSUNTO - *Il teorema del punto fisso di Banach e diverse sue generalizzazioni sono corollari di risultati dimostrati in questo lavoro per una classe di funzioni che è più ampia di quella costituita dalle α-contrazioni* ($\alpha < 1$).

The main purpose of this paper is to present in the context of a metric, sometimes complete metric space new kinds of functions, which on one hand admit as special cases several previously known generalized contractions and on the other hand enjoy like these the same strong fixed point properties.

## 1. NOTATIONS AND DEFINITIONS.

Let $(M, d)$ be a metric space and $\delta : 2^M \times 2^M \to 2^{[0, \infty]}$ be a map defined by

$$\delta(A,B) = \begin{cases} \{\infty\} & \text{if } A = \emptyset \text{ or } B = \emptyset \\ \{d(x, y) : (x, y) \in A \times B\} & \text{if } A \neq \emptyset \text{ and } B \neq \emptyset \end{cases}.$$

For $m \in M$ and $A \subset M$ we put $\delta(m, A) = \delta(A, m) = \delta(A, \{m\})$; for $m, m' \in M$, $\delta(m, m') = \{d(m, m')\}$.

For $x \in [-\infty, \infty]$ and $U, V \subset [0, \infty]$, $x < U$ means $x < u$ for all $u \in U$, $xU$ means $\{xu : u \in U\}$, and $U + V$ means $\{u + v : (u, v) \in U \times V\}$. $\mathbb{N}$ denotes the set $\{1, 2, 3, \dots\}$ of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

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*Classificazione per soggetto* AMS (MOS) 1970: 47H10.

Our main attention will be concentrated on a function $f : M \to M$. Consider also the bounded functions

$$r, r^* : M \times M \to \mathbb{N}$$

and the functions

$$s, s^* : M \times M \to \mathbb{Z}_+ ;$$

let $\alpha \in (0, 1]$, $\gamma \in (0, 1)$, $\tau \in (0, \infty)$ and $t \in \mathbb{N}$. Whenever these functions and numbers will later appear, their meaning will be as described here.

Let $f^t$ denote $f \circ \ldots \circ f$ ($t$ times), $f^0(x) = x$ and $f^{-t}(x) = \{ y \in M : f^t(y) = x \}$, where $x \in M$. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ be defined by [1]

$$\boldsymbol{a}\ (\tau, r, s)\,(x, y) = \tau\delta\,(f^{r\,(x, y)}\,(x), f^{s\,(x, y)}\,(y))$$

$$\boldsymbol{b}\ (\tau, r, s)\,(x, y) = \frac{-\tau}{(r-s)\,(x, y)}\ \delta\,(f^{r\,(x, y)}\,(x), f^{s\,(x, y)}\,(x))$$

$$\boldsymbol{c}\ (\tau, r, s)\,(x, y) = \frac{-\tau}{(r-s)\,(x, y)}\ \delta\,(f^{r\,(x, y)}\,(y), f^{s\,(x, y)}\,(y))$$

$$\boldsymbol{d}\ (\tau, r, r^*, s\ s^*)\,(x\ y) = \frac{-\tau}{(r + r^* - s - s^*)\,(x, y)}\ (\delta(f^{r(x, y)}(x), f^{s(x, y)}(y)) + \\ + \delta\,(f^{s^*(x, y)}\,(x), f^{r^*(x, y)}\,(y)))\quad (r + r^* \neq s + s^*).$$

We say that $f$ is an *$\alpha$-precontraction* if for some $r > s$, $r^* > s^*$, $\gamma \leqq \alpha$ and $t$, at each pair $(x, y)$ of distinct points from $f^t(M)$, $d$ is smaller than at least one of the folowing eight set-valued functions:

$$\boldsymbol{a}\ (\gamma, -r, -r), \quad \boldsymbol{b}\ (\gamma, -r, -s), \quad \boldsymbol{c}\ (\gamma, -r, -s), \quad \boldsymbol{d}\ (\gamma, -r, -r^*, -s, -s^*)$$

$$\boldsymbol{a}\ (\alpha, -1, -1), \quad \boldsymbol{b}\ (\alpha, -r, 0), \quad \boldsymbol{c}\ (\alpha, -r, 0), \quad \boldsymbol{d}\ (\alpha, -r, -r^*, -s, 0)$$

(the choice depends on choice of $x$ and $y$). It is clear that if $\alpha < 1$, the comparison with the last four functions becomes superflous.

We say that the $\alpha$-precontraction $f$ is an *$\alpha$-contraction* if there exists numbers $\tau$ and $t$ and functions $r$ and $s > 0$ (not necessarily

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(1) Exceptionally, in these definitions $r$, $r^*$, $s$, $s^*$ are considered to be integer-valued.

the same as those in the definition of an $\alpha$-precontraction), such that for each pair of distinct points $x \in M$, $y \in f^t(M)$, the value of $d$ taken at $(f^{r(x,y)}(x), f^{s(x,y)}(y))$ is smaller than that of at least one of the following three functions:

$$\boldsymbol{a}\ (\tau, 0, 0), \qquad r\,\boldsymbol{b}\ (\alpha, 0, r) + s\,\boldsymbol{c}\ (\alpha, 0, s), \qquad (s+r)\ \boldsymbol{d}\ (\alpha, 0, 0, s, r),$$

taken at $(x, y)$.

The set $\{f^n(x) : n \in \mathbb{Z}_+\}$ and the sequence $\{f^n(x)\}_{n=0}^{\infty}$ will be called the *orbit* and the *orbital sequence* of $x$.

## 2. A LEMMA.

LEMMA. *Suppose $M$ is a metric space and $f$ an $\alpha$-precontraction, with $\alpha < 1$. Then each orbital sequence is Cauchy and for every couple of points $x, y \in M$, $d(f^n(x), f^n(y)) \to 0$.*

*Proof.* Let $x_0 \in M$, $x_n = f^n(x_0)$ $(n \in \mathbb{N})$ and $a_n = d(x_n, x_{n+1})$ $(n \in \mathbb{Z}_+)$. We want to show that $\{x_n\}_{n=0}$ is a Cauchy sequence. Let

$$a_m = \max\{a_0, \ldots, a_{k-1}\},$$

where $k \geqq t$ is a common upper bound for $r$ and $r^*$. First, we intend to show that $a_l \leqq a_m \alpha^{\left[\frac{l}{k}\right]}$ for all $l \in \mathbb{Z}_+$. This is clear for $l < k$. We shall prove it for arbitrary $l \geqq k$ supposing it works for each index smaller than $l$.

If for some $n_0 \in \mathbb{Z}_+$, $a_{n_0} = 0$, then $\{x_n\}_{n=0}^{\infty}$ converges to $x_{n_0}$. Suppose now that no $a_n$ is zero.

The conditions defining an $\alpha$-precontraction, put on the couple $(x_l, x_{l+1})$, say that at least one of the following inequalities holds:

$$d(x_l, x_{l+1}) < \gamma d(x_{l-r}, x_{l+1-r})$$

$$d(x_l, x_{l+1}) < \frac{\gamma}{r-s}\, d(x_{l-r}, x_{l-s})$$

$$d(x_l, x_{l+1}) < \frac{\gamma}{r-s}\, d(x_{l-r+1}, x_{l-s+1})$$

$$d(x_l, x_{l+1}) < \frac{\gamma}{r+r^*-s-s^*}\, (d(x_{l-r}, x_{l-s+1}) + d(x_{l-s^*}, x_{l-r^*+1})),$$

where $r(x_l, x_{l+1})$, $s(x_l, x_{l+1})$, $r^*(x_l, x_{l+1})$, $s^*(x_l, x_{l+1})$ were abbreviated by $r$, $s$, $r^*$, $s^*$. The above inequalities successively imply

$$a_l < \alpha a_{l-r}$$

$$a_l < \frac{\alpha}{r-s}\ (a_{l-1}+\dots+a_{l-s-1})$$

$$a_l < \frac{\alpha}{r-s}\ (a_{l-r+1}+\,.\,.+a_{l-s})$$

$$a_l < \frac{\alpha}{r+r^*-s-s^*}\ (a_{l-r}+\dots+a_{l-s}+a_{l-r^*+1}+\dots+{}_{l-s^*-1})$$

$(a_{l-r^*+1}+\dots+a_{l-s^*-1}$ means 0 if $r^*=s^*+1)$. Now, using the induction hypothesis, the first inequality yields

$$a_l < \alpha a_{l-r} \leqq \alpha a_m \alpha^{\left[\frac{l-r}{k}\right]} = a_m \alpha^{\left[\frac{l+k-r}{k}\right]} \leqq a_m \alpha^{\left[\frac{l}{k}\right]}\ .$$

The second inequility yields

$$a_l < \frac{\alpha}{r-s}\ (a_{l-r}+\dots+a_{l-s-1}) \leqq \frac{\alpha}{r-s}\ a_m(\alpha^{\left[\frac{l-r}{k}\right]}+\dots+\alpha^{\left[\frac{l-s-1}{k}\right]})$$

$$\leqq \frac{\alpha}{r-s}\ a_m(r-s)\,\alpha^{\left[\frac{l-r}{k}\right]} = a_m\alpha^{\left[\frac{l+k-r}{k}\right]} \leqq a_m\alpha^{\left[\frac{l}{k}\right]}\ .$$

The third inequality can be treated like the second in the case $s>0$. If $s=0$,

$$a_l < \frac{\alpha}{r}\ (a_{l-r+1}+\dots+a_l)$$

implies

$$a_l < \frac{\beta}{r-1}\ (a_{l-r+1}+\dots+a_{l-1})$$

with $\beta=(r-1)\alpha/(r-\alpha)<\alpha$. Further

$$a_l < \frac{\beta}{r-1}\ a_m(\alpha^{\left[\frac{l-r+1}{k}\right]}+\dots+\alpha^{\left[\frac{l-1}{k}\right]}) < \beta a_m \alpha^{\left[\frac{l-r+1}{k}\right]}$$

$$< a_m\alpha^{\left[\frac{l+1+k-r}{k}\right]} \leq a_m\alpha^{\left[\frac{l}{k}\right]}\ .$$

The fourth inequality can be handled like the second one if $s > 0$ and like above if $s = 0$.

Since $\Sigma_n a_m a^{\left[\frac{n}{k}\right]}$ is convergent, $\Sigma_n a_n$ is convergent too, whence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Now, consider also $y_0 \in M$ and put $y_n = f^n(y_0)$ $(n \in \mathfrak{N})$. Since $\{y_n\}_{n=0}^{\infty}$ is Cauchy too and each Cauchy sequence is bounded, the diameter of the union of the orbits of $x_0$ and $y_0$ is a real number $\Delta$. We intend to show that

$$d(x_i, y_j) \leqq \Delta a^{\left[\frac{\min[i,j]}{k}\right]}.$$

$$d(x_i, x_j) \leqq \Delta a^{\left[\frac{\min[i,j]}{k}\right]},$$

and

$$d(y_i, y_j) \leqq \Delta a^{\left[\frac{\min[i,j]}{k}\right]}$$

These inequalities are of course satisfied if $\min\{i, j\} < k$. We proceed by induction on $\min\{i, j\}$. Thus we take $i, j \geqq k$ and assume the above incqualities hold for each couple of indices, the minimum of which is smaller than $\min\{i, j\}$.

At least one of the following inequalities is verified.

$$d(x_i, y_j) < a d(x_{i-r}, y_{j-r})$$

$$d(x_i, y_j) < \frac{a}{r-s} d(x_{i-r}, x_{i-s})$$

$$d(x_i, y_j) < \frac{a}{r-s} d(y_{j-r}, y_{j-s})$$

$$d(x_i, y_j) < \frac{a}{r+r^*-s-s^*} (d(x_{i-r}, y_{j-s}) + d(x_{i-s^*}, y_{j-r^*})).$$

Together with the induction hypothesis, these inequalities yield:

$$d(x_i, y_j) < a\Delta a^{\left[\frac{\min[i-r,\, j-r]}{k}\right]} = \Delta a^{\left[\frac{k-r+\min[i,j]}{k}\right]}$$

$$\leqq \Delta a^{\left[\frac{\min[i,j]}{k}\right]}$$

$$d(x_i, y_j) < \frac{\alpha}{r-s} \Delta\alpha^{\left[\frac{\min[i-r,\ i-s]}{k}\right]}$$

$$\leqq \Delta\alpha^{\left[\frac{k-\max[r,\ s[+i}{k}\right]} \leqq \Delta\alpha^{\left[\frac{\min[i,\ j]}{k}\right]}$$

$$d(x_i, y_j) < \frac{\alpha}{r-s} \Delta\alpha^{\left[\frac{\min[i-r,\ i-s]}{k}\right]} \leqq \Delta\alpha^{\left[\frac{\min[i,\ j]}{k}\right]}$$

$$d(x_i, y_j) < \frac{\alpha}{r+r^*-s-s^*} \left(\Delta\alpha^{\left[\frac{\min[i-r,\ j-s]}{k}\right]} + \Delta\alpha^{\left[\frac{\min[i-s^*,\ j-r^*]}{k}\right]}\right)$$

$$\leqq \frac{\alpha}{2} \left(\Delta\alpha^{\left[\frac{\min[i,\ j]-\max[r,\ s]}{k}\right]} + \Delta\alpha^{\left[\frac{\min[i,\ j]-\max[r^*,\ s^*]}{k}\right]}\right)$$

$$\leqq \alpha\Delta\alpha^{\left[\frac{\min[i,\ j)-\max[r,\ r^*,\ s,\ s^*]}{k}\right]} \leqq \Delta\alpha^{\left[\frac{\min[i,\ j]}{k}\right]}.$$

Similarly one can prove that also $d(x_i, x_j)$ and $d(y_i, y_j)$ are smaller than $\Delta\alpha^{[\min[i,j]/k]}$.

It follows that

$$d(x_n, y_n) \leqq \Delta\alpha^{\left[\frac{n}{k}\right]},$$

whence $d(x_n, y_n) \to 0$.

## 3. On $\alpha$-precontractions and $\alpha$-contractions.

Theorem 1. *Let $M$ be a subspace of the metric spaxe $X$ and $f$ an $\alpha$-precontraction of $M$ with $\alpha < 1$. If some orbital sequence has a limit point $z \in X$, then all orbital sequences converge to $z$.*

*Proof.* If the orbital sequence $\{x_n\}_{n=0}^{\infty}$ of $x_0$, which is Cauchy by the preceding Lemma, has $z$ as a limit point, then $x_n \to z$. By the same Lemma, for each orbital sequence $\{y_n\}_{n=0}^{\infty}$, $d(x_n, y_n) \to 0$, which implies $y_n \to z$.

Theorem 2. *Let $M$ be a metric space and $f$ an $\alpha$-precontraction with $\alpha < 1$. If each orbit is either finite or nonclosed, then $f$ has a unique fixed point and all orbital sequences converge to this point.*

*Proof.* Let $x_0$ be an arbitrary point in $M$. If the orbit of $x_0$ is finite, then its orbital sequence $\{x_n\}_{n=0}^{\infty}$ must have a periodic subsequence $\{x_n\}_{n=n_0}^{\infty}$, and since $\{x_n\}_{n=0}^{\infty}$ is also Cauchy by the

Lemma, the period must be one, which yields $x_n \to x_{n_0}$; if the orbit of $x_0$ is infinite, then it must be nonclosed, i.e. $\{x_n\}_{n=0}^{\infty}$ possesses a limit point, and since it is also Cauchy, it is convergent. Thus, in any case, there is $z \in M$ such that $x_n \to z$.

Suppose now $z$ is not a fixed point of $f$. We have then both $z \neq f(z)$ and, by Theorem 1, $f^n(z) \to z$, which imply that the orbit of $z$ is infinite and closed. This contradiction proves that $f$ has at least the fixed point $z$.

To prove the uniqueness of the fixed point, suppose $z'$ is a fixed point of $f$. By Theorem 1, the orbital sequence of $z'$ converges to $z$, which means $z = z'$. The proof is achieved.

THEOREM 3. *Suppose $M$ is a subspace of a complete metric space $X$ and $f$ an $\alpha$-precontraction in $M$ with $\alpha < 1$. Then all orbital sequences converge to the same point of $X$.*

*Proof.* In view of Theorem 1, it suffices to obtain that some orbital sequence has a limit point in $X$. This is the case indeed, since each orbital sequence is, by the Lemma, Cauchy and therefore convergent in $X$.

THEOREM 4. *Let $M$ be a metric space and $f$ an $\alpha$-contraction with $\alpha < 1$. If some orbital sequence has a limit point $z$, then $z$ is a unique fixed point of $f$ and all orbital sequences converge to $z$.*

*Proof.* First we prove that each orbit is finite or not closed. Suppose, on the contrary, the orbit of a certain point $y$ is infinite and closed. By Theorem 1, the orbital sequence of $y$ converges to $z$. It follows that $z$ belongs to the orbit of $y$ and all the members of the orbital sequence of $z$ are distinct. Obviously $z_n \to z$, where $z_n = f^n(z)$.

Let us consider the (supplementary) inequalities which define our $\alpha$-contraction, in the special case of the couple of points $z, z_n$ $(n \geqq t)$. We have at least one of the following inequalities

$$d(z_{r_n}, z_{n+s_n}) < \tau d(z, z_n)$$

$$d(z_{r_n}, z_{n+s_n}) < \alpha(d(z, z_{r_n}) + d(z_n, z_{n+s_n}))$$

$$d(z_{r_n}, z_{n+s_n}) < \alpha(d(z, z_{n+s_n}) + d(z_{r_n}, z_n)),$$

where $r_n = r(z, z_n)$, $s_n = s(z, z_n)$.

By taking $\liminf_{n\to\infty}$, we get on the left side $d(z_{r_0}, z)$ for some $r_0 \in \mathfrak{N}$, since $r$ is bounded, and on the right side respectively 0, $ad(z, z_{r_0})$, $ad(z_{r_0}, z)$, which implies $z = z_{r_0}$, and a contradiction is found.

Following Theorem 2, $f$ has a unique fixed point $z'$ and all orbital sequences converge to $z'$. By Theorem 1 the orbital sequence of $z'$ converges to $z$. Hence $z = z'$ and the theorem is proved.

THEOREM 5. *Suppose $M$ is a complete metric space and $f$ an $\alpha$-contraction with $\alpha < 1$. Then $f$ has a unique fixed point and all orbital sequences converge to this point.*

*Proof*. By Theorem 3, all orbital sequences converge; now the conclusion follows from Theorem 4.

Any $\alpha$-pseudocontraction [13], i.e. any function $f: M \to M$ such that for each couple of distinct points $x, y \in M$, at least one of the following inequalities holds

$$d(f(x), f(y)) < ad(x, y)$$

$$d(f(x), f(y)) < ad(x, f(x))$$

$$d(f(x), f(y)) < ad(y, f(y))$$

$$d(f(x), f(y)) < \frac{a}{2}\,(d(x, f(y)) + d(f(x), y)),$$

is also an $\alpha$-contraction in our sense: choose $r(x, y) = r^*(x, y) = 1$ and $s(x, y) = s^*(x, y) = 0$ in the definition of an $\alpha$-precontraction and choose $r(x, y) = s(x, y) = 1$ in the additional part of the definition of an $\alpha$-contraction.

Thus Theorem 5 generalizes Theorem 2 in [14] and implicitely Theorem 2 of Tiberio Bianchini [11], Theorem 1 in [12], Theorem 3 of Reich [8] which coincides with Theorem 1 of Rus [9], Theorem 2 of Kannan [5] and the classical fixed point theorem of Banach [1].

Theorem 4 generalizes Theorem 3 in [14] and implicitely Theorem 2 in [12] and Theorem 1 of Kannan [6].

Since a fortiori an $\alpha$-pseudocontraction is an $\alpha$-precontraction, Theorem 3 generalizes Theorem 4 in [14] and also Theorem 3 in [12].

If $f$ is such that $f^p$ is $\alpha$-contractive in the classical sense, then $f$ itself is a $\sqrt{\alpha}$-contraction: choose $r(x, y) = s(x, y) = p$ (in both the definitions of an $\alpha$-precontraction and the rest of the definition of an $\alpha$-contraction). Thus the Cacciopoli [3] - Bonsall [2] idea of replacing $f$ by $f^p$ is included in our development.

Finally, Pittnauer's condition

$$d(f(x_1), f(x_2)) \leqq \alpha (d(x_1, f^p(x_3)) + d(x_2, f^p(x_3))) \ ,$$

which has to be fulfilled at each triple of points $x_1, x_2, x_3 \in M$ implies that $f$ is acting on couples of points in $M \times f^p(M)$ $\alpha$-contractively in the classical sense, whence $f$ is a $\sqrt{\alpha}$-contraction: choose $r(x, y) = 1$ for all $x, y \in M$ (respectively $r(x, y) = s(x, y) = 1$) and $t = p$. Thus his Satz 4 in [7] is also a corollary of our Theorem 2.

## 4. On 1-precontractions.

THEOREM 6. *Let $M$ be a metric space and $f$ a continuous* 1-*precontraction. If some orbital sequence has a limit point $z$, then $z$ is a unique fixed point of $f$.*

*Proof.* Suppose $z$ is a limit point of the orbital sequence $\{x_n\}_{n=0}^{\infty}$ of $x_0$ and put $a_n = d(x_n, x_{n+1})$ $(n \in \mathbb{Z}_+)$. Let again

$$a_m = \max\{a_0, \ldots, a_{k-1}\} \ ,$$

where $k \geqq t$ is a common upper bound for $r$ and $r^*$.

First we prove the following «monotony»-property: for each $n \geqq k$, there exists $l \leqq k$ such that $a_n \leqq a_{n-l}$. This is clear if $a_n = 0$. If $a_n \neq 0$, we can see what the eight conditions of the definition of an $\alpha$-precontraction say for the couple of points $(x_n, x_{n+1})$ and $\alpha = 1$:

$$(1^0) \qquad a_n < \gamma a_{n-r}$$

$$(2^0) \qquad a_n < \frac{\gamma}{r-s}\,(a_{n-r} + \ldots + a_{n-s-1}) \leqq \gamma \max\{a_{n-r}, \ldots, a_{n-s-1}\}$$

$$(3^0) \qquad a_n < \frac{\gamma}{r-s}\,(a_{n-r+1} + \cdots + a_{n-s}) \leqq \gamma \max\{a_{n-r+1}, \ldots, a_{n-s}\} \qquad (s > 0)$$

or

$$a_n < \frac{\gamma}{r} \left(a_{n-r+1} + \dots + a_n\right)$$

which implies

$$a_n < \frac{\gamma}{r-1} \left(a_{n-r+1} + \dots + a_{n-1}\right)$$

$$\leqq \gamma \max\{a_{n-r+1}, \dots, a_{n-1}\}$$

(4⁰)

$$a_n < \frac{\gamma}{r+r^*-s-s^*} \left(a_{n-r} + \dots + a_{n-s} + a_{n-r^*+1} + \dots + a_{n-s^*-1}\right)$$

$$\leqq \gamma \max\{a_{\min[n-r,\, n-r^*+1]}, \dots, a_{\max[n-s,\, n-s^*-1]}\} \qquad (s>0)$$

or

$$a_n < \frac{\gamma}{r+r^*-s^*} \left(a_{n-r} + \dots + a_n + a_{n-r^*+1} + \dots + a_{n-s^*-1}\right)$$

which similarly implies

$$a_n < \gamma \max\{a_{\min[n-r,\, n-r^*+1]}, a_{n-1}\}$$

(5⁰)

$$a_n < a_{n-1}$$

(6⁰)

$$a_n < \frac{1}{r} \left(a_{n-r} + \dots + a_{n-1}\right)$$

$$\leqq \max\{a_{n-r}, \dots, a_{n-1}\}$$

(7⁰)

$$a_n < \frac{1}{r} \left(a_{n-r+1} + \dots + a_n\right)$$

which yields

$$a_n < \max\{a_{n-r+1}, \dots, a_{n-1}\}$$

(8⁰)

$$a_n < \frac{1}{r+r^*-s} \left(a_{n-r} + \dots + a_{n-s} + a_{n-r^*+1} + \dots + a_{n-1}\right)$$

which implies

$$a_n < \max\{a_{\min[n-r,\, n-r^*+1]}, \dots, a_{n-1}\} .$$

A first consequence of this « monotony »-property is that $\{a_n\}_{n=0}^{\infty}$ is bounded: each $a_n$ lies in $[0, a_m]$. Now we want to show that this sequence is convergent. Suppose, on the contrary, $b = \liminf_{n\to\infty} a_n$ an $c = \limsup_{n\to\infty} a_n$ are distinct. Choose arbitrarily $b' \in (b, c)$ and let

$$c'' \in \left(c, \min\left\{\frac{c}{\gamma}, \frac{kc - b'}{k-1}\right\}\right)$$

$$c' = \max\left\{\gamma c'', \frac{b' + (k-1)c''}{k}\right\} .$$

Then $b' \leqq c' < c < c''$. There exists $n_1 \in \mathbb{N}$ such that $a_n < c''$ for all $n \geqq n_1$. There also exists $n_2 \geqq n_1 + k - 1$ with $a_{n_2} < b'$. Let $n_3 = n_2 + 1$. The inequalities (1°) - (4°) obtained above show that $a_{n_3} < \gamma c'' \leqq c'$. The inequality (5°) implies $a_{n_3} < a_{n_2} < c'$. (6°) implies

$$a_{n_3} < \frac{1}{r}\left((r-1)\max\{a_{n_3-r}, \dots, a_{n_3-2}\} + a_{n_3-1}\right)$$
$$\leqq \frac{(r-1)c'' + b'}{r} \leqq c' .$$

From (7°) it follows

$$a_{n_3} < \frac{1}{r}\left((r-1)\max\{a_{n_3-r+1}, \dots, a_{n_3-2}, a_{n_3}\} + a_{n_3-1}\right)$$
$$\leqq \frac{(r-1)c'' + b'}{r} \leqq c' .$$

Analogously (8°) implies $a_{n_2} < c'$.

In a similar way we can also show that $a_{n_3+1} < c'$: let us treat, for example, case 6°:

$$a_{n_3+1} < \frac{1}{r}\left(a_{n_3+1-r} + \dots + a_{n_3}\right)$$
$$\leqq \frac{1}{r}\left((r-1)\max\{a_{n_3+1-r}, \dots, a_{n_3-2}, a_{n_3}\} + a_{n_3-1}\right)$$
$$\leqq \frac{(r-1)c'' + b'}{r} \leqq c' .$$

Further, for each $l<k$, we can similarly prove that $a_{n_3+l} \leqq c'$. Now, a second consequence of the « monotony »-property of $\{a_n\}_{n=0}^{\infty}$ is that $a_n \leqq c'$ for all $n \geqq n_3$. But this contradicts the fact that $c$ is a limit point; thus it is shown that there exists $a \in [0, a_m]$ such that $a_n \to a$.

Let $\{x_{n_i}\}_{i=1}^{\infty}$ be a subsequence of the orbital sequence of $x_0$, which converges to $z$. From the continuity of $f$ it follows that for each $l \leqq k$,

$$x_{n_i+l} \to f^l(z) .$$

Thus, for all these $l$'s,

$$a_{n_i+l} \to d(f^l(z), f^{l+1}(z)) .$$

On the other hand $a_{n_i+l} \to a$, whence

$$d(f^l(z), f^{l+1}(z)) = a \qquad (l \leq k) .$$

Suppose $a \neq 0$. Then we can apply again the inequalities defining an $\alpha$-precontraction to the couple $(f^k(z), f^{k+1}(z))$ and for $\alpha = 1$. Each of these implies $d(f^k(z), f^{k+1}(z)) < a$ and this contradiction shows that $a = 0$, i.e. $z$ is a fixed point of $f$.

Suppose $z'$ is another fixed point of $f$. Then, by applying any one of the eight inequalities defining a 1-precontraction in $(z, z')$ we get a contradiction.

A slight generalization of the preceding theorem is the following

THEOREM 7. *Let M be a metric space and f a 1-precontraction which is continuous on* $\overline{f^t(M)}$ *for some* $t \in \mathbb{N}$. *If some orbital sequence has a limit point z, then z is a unique fixed point of f.*

*Proof.* To verify the theorem it suffices to put on the sequence $\{x_{n_i}\}_{i=1}^{\infty}$ in the proof of Theorem 6 the additional condition $n_1 \geqq t$ and to remark that $z \in \overline{f^t(M)}$. Then it follows $x_{n_i+l} \to f^l(z)$ for each $l \leqq k$, like in the mentioned proof.

A special case, in which the use of Theorem 7 is evident, is given below.

Theorem 8. *Let $M$ be a metric space and $f$ a 1-precontraction which is continuous on $f^t(M)$ for some $t \in \mathbb{N}$. If $f^t(M)$ is compact, then $f$ has a unique fixed point.*

*Proof.* Remark that $f^t(M)$ is closed and each orbital sequence has a subsequence in $f^t(M)$, hence it has a limit point there; now Theorem 7 can be applied.

A function $f: M \to M$ satisfying at each couple of distinct points $x, y \in M$ at least one of the following inequalities

$$d(f(x), f(y)) < d(x, y)$$
$$d(f(x), f(y)) < d(x, f(x))$$
$$d(f(x), f(y)) < d(y, f(y))$$
$$d(f(x), f(y)) < \frac{1}{2}\,(d(x, f(y)) + d(f(x), y))$$

is also a 1-contraction: choose $r(x,y) = r^*(x,y) = 1$ and $s(x,y) = s^*(x,y) = 0$ (in the definition of a 1-precontraction), and choose $r(x,y) = s(x,y) = 1$ (in the remaining part of the definition of a 1-contraction).

Thus Theorem 6 generalizes Theorem 5 in [14] and implicitely Theorem 5 in [12], Theorem 1 of Edelstein [4] and Theorem 1 of Singh [10].

We finally mention that each function $f: M \to M$ satisfying the inequality

$$d(f(x_1), f(x_2)) < d(x_1, f^p(x_3)) + d(x_2, f^p(x_3))$$

for all triples $x_1, x_2, x_3 \in M$ with $x_1 \neq x_2$ (Pittnauer [7]) is also a 1-contraction: the fifth condition is satisfied for $t = p$ and choose for the additional part of the definition of a 1-contraction $r(x,y) = s(x,y) = 1$. Thus Korollar 5 in [7] follows from our Theorem 8 since if $f$ satisfies the above condition, it is obviously continuous on $f^p(M)$.

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