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Spreads

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The investigation of spreads is mainly motivated by their applications in the theory of convex sets (see Grünbaum [22]). They may be considered as a continuous pendant to the discrete arrangements of pseudolines (Grünbaum [23]).

Many authors have investigated special families of chords of planar convex bodies. Zindler studied several of them systematically in [44], [45]. Hammer–Sobczyk [24], Smith [32], Ceder [11] considered diameters. Spreads of midcurves have been discussed by Brunn [7], Emch [18], Steinhaus [33]; see also Viet [35], Chakerian–Stein [14], Ceder [13], Ehrhart [17]. Grünbaum [23] points out that, using another special type of spread, one can establish the existence of an inscribed affinely regular hexagon (see Besicovitch [3], Fáry [19], Fejes Tóth [20]). That each planar convex body with straight-line midcurves is an ellipse was under various additional conditions proved by Bertrand [2], Blaschke [4], [5], Nakajima [30], Berger [1], Kubota [28], Kneser [27], Busemann–Kelly [10], Busemann [9], Danzer–Laugwitz–Lenz [15], Süss–Viet–Berger [34]. Brennan [6], Zitronenbaum [46] considered bisectors of area or perimeter. However, this list is far from complete.

The concept of a spread can be generalized further (see Zamfirescu [36]). But the object of the present paper does not involve this generalization; on the contrary, here we shall be mainly interested in particular types of spreads. These are spreads under additional continuity conditions. Their treatment is mainly motivated by the theory of smooth convex bodies in the plane: each one of the above examples of spreads becomes of one of these particular types if the convex body is smooth enough.

Introduction

Let C be a Jordan closed curve in the Euclidean plane and D be the bounded domain with boundary C. A family $\mathfrak{L}$ of simple arcs in $\bar{D}$, further called *curves*, is a *spread* provided [22]:

(i) each curve $L \in \mathfrak{L}$ (except its endpoints) lies in D and its endpoints belong to C,

(ii) each point $p \in C$ is the endpoint of exactly one curve $L(p) \in \mathfrak{L}$,

(iii) if L_1, L_2 are two different curves in $\mathfrak{L}$, then $L_1 \cap L_2$ contains a single point,

(iv) the curve $L(p)$ depends continuously on $p \in C$.

We notice that the topology of $\mathfrak{L}$ is that induced by Hausdorff's metric. A spread, the curves of which are line-segments, is said to be *straight*. Following [37], a maximal connected subset of $\mathfrak{L}$, the elements of which are concurrent curves, is called a *pencil*. By taking $\mathfrak{B} \subset \mathfrak{L}$ instead of $\mathfrak{L}$, we get the notion of a $\mathfrak{B}$-*pencil*. Let $a \in D$, $\mathfrak{B} = \{L \in \mathfrak{L} : a \in L\}$, $\mathfrak{A}_a$ be the family of all $\mathfrak{B}$-pencils, and $\mathfrak{x}$ a cardinal number not greater than $\mathfrak{c}$. We denote:

$$P_\mathfrak{x}(\mathfrak{L}) = \{a \in D : \text{card } \mathfrak{A}_a \geqslant \mathfrak{x}\},$$
$$V_\mathfrak{x}(\mathfrak{L}) = \{a \in D : \text{card } \mathfrak{A}_a = \mathfrak{x}\}.$$

We also recall [37], [39]:

$$M_\mathfrak{x}(\mathfrak{L}) = \{a \in D : \text{card } \{L \in \mathfrak{L} : a \in L\} \geqslant \mathfrak{x}\},$$
$$T_\mathfrak{x}(\mathfrak{L}) = \{a \in D : \text{card } \{L \in \mathfrak{L} : a \in L\} = \mathfrak{x}\}.$$

Points in $T_\mathfrak{x}(\mathfrak{L})$ are said to have *multiplicity* $\mathfrak{x}$. Obviously,

$$T_\mathfrak{x}(\mathfrak{L}) \subset V_\mathfrak{x}(\mathfrak{L}) \qquad (\mathfrak{x} \leqslant \aleph_0),$$
$$V_\mathfrak{x}(\mathfrak{L}) \subset T_\mathfrak{x}(\mathfrak{L}) \cup T_\mathfrak{c}(\mathfrak{L}),$$
$$P_\mathfrak{x}(\mathfrak{L}) \subset M_\mathfrak{x}(\mathfrak{L}).$$

If card $M = 1$, then $\langle M \rangle$ denotes the single point of M.

The first result established on spreads in their general form was the following:

Theorem A (Grünbaum [22]). *For all $L \in \mathfrak{L}$ with at most one exception,*

$$L \cap M_3(\mathfrak{L}) \neq \varnothing.$$

This theorem admits as a corollary the assertion that $\mathfrak{L}$ itself must be a pencil if card $M_3(\mathfrak{L}) = 1$. This implies further the symmetry of a planar convex body in which $\mathfrak{L}$ is the spread of area-bisectors, perimeter-bisectors, or midcurves, and card $M_3(\mathfrak{L}) = 1$ (Zarankiewicz [43], Viet [35]; see also Piegat [31], Menon [29], Chakerian–Stein [14]). Theorem A also holds for the mentioned generalized spreads (Zamfirescu [36]). In the case of usual spreads, Theorem A has been completed by:

Theorem A′ (Zamfirescu [37]). *If $\mathfrak{L}$ is not a pencil, then for all $L \in \mathfrak{L}$ with at most three exceptions*

$$\text{relint } (L \cap M_3(\mathfrak{L})) \neq \varnothing.$$

We notice that relint I means the interior of I in the topology of $L \in \mathfrak{L}$, where $L \supset I$.

Corollary 1 will show that in our more particular spreads such exceptions cannot occur. Further results on relint $(L \cap M_2(\mathfrak{L}))$ can be found in Zamfirescu [38] and Ivan [26].

We also recall:

Theorem B (Zamfirescu [36]). *Let n be finite and even. Then*
(1) int $T_n(\mathfrak{L}) = \varnothing$,
(2) if $M_n(\mathfrak{L}) \neq M_{\aleph_0}(\mathfrak{L})$, *then* int $M_{n+1}(\mathfrak{L}) \neq \varnothing$.

Chakerian–Stein [14], Hammer–Sobczyk [25], Ceder [12] (see also [11]) obtained stronger variants of Theorem B in the case of spreads of midcurves and of diameters. A strengthening of Theorem B and a related result in the general case are Theorems 2 and 3, respectively.

Before we state Theorems C and D, let us recall two definitions [22]: A set $E \subset D$ is said to be

$\mathfrak{L}$-*convex* if, for each $L \in \mathfrak{L}$, $L \cap E$ is either empty or connected,

an $L_2(\mathfrak{L})$-*set* if every pair of points in E may be joined within E by a simple arc composed by two subarcs of curves in $\mathfrak{L}$.

Theorem C (Grünbaum [22]) $M_2(\mathfrak{L})$ *is* $\mathfrak{L}$-*convex.*

Theorem D (Grünbaum [22], Zamfirescu [36]). $M_2(\mathfrak{L})$ *and* $M_3(\mathfrak{L})$ *are* $L_2(\mathfrak{L})$-*sets.*

Related to Theorems C and D is a conjecture of Grünbaum [23] which says that, for each $j \geqslant 1$, $M_j(\mathfrak{L})$ is an $\mathfrak{L}$-convex $L_2(\mathfrak{L})$-set. Theorem 6 will disprove this conjecture. The $\mathfrak{L}$-convexity is also studied in [41], [42].

Theorem E (Zamfirescu [36]). *If* f_1 *and* f_2 *are continuous maps of* $\mathfrak{L}$ *into itself, then there exists* $L \in \mathfrak{L}$ *such that*

$$L \cap f_1(L) \cap f_2(L) \neq \varnothing .$$

This theorem and corollaries of it can be used to derive the existence of six-partite points for each planar convex body (Buck–Buck [8]) and other more general results (Eggleston [16], Grünbaum [21], Ceder [12]). Applied to midcurves Theorem E implies a result of Steinhaus [33] concerning bisecting chords.

Let T be the union of all triangles (a triangle is the bounded component of the complement of the union of three non-concurrent curves in $\mathfrak{L}$ [40]). It is easy to verify that

$$T \subset \operatorname{int} M_2(\mathfrak{L}) \subset \operatorname{int} \overline{T}.$$

More precisely, the mutual position of T and $M_2(\mathfrak{L})$ is given by

Theorem F (Zamfirescu [40]). $T = \operatorname{int} M_2(\mathfrak{L})$.

Theorem 8 will strengthen Theorem F in the case of a spread of curves of one of our particularized forms.

Special Spreads and Relationships Between Them

For $p \in C$, denote by $-p$ the other endpoint of $L(p)$. Let ω be a certain orientation on C. We write $p < q$ if by walking on C in the orientation ω one meets the distinct points p, q, $-p$ in this order. A sequence $\{p_n\}_{n=1}^{\infty}$ of points on C is *monotone* if $p_n < p_{n+1}$ for all n in the set $\mathbb{N}$ of the natural numbers or $p_{n+1} < p_n$ for all $n \in \mathbb{N}$. We consider the following properties that a spread $\mathfrak{L}$ may possess at some point $p \in C$:

C_1: $p_n \in C - \{p\}(n \in \mathbb{N}) \wedge \{p_n\}_{n=1}^{\infty}$ is monotone $\wedge\, p_n \to p \Rightarrow \{\langle L(p_n) \cap L(p)\rangle\}_{n=1}^{\infty}$ converges.

C_2: $p_n \in C - \{p\}(n \in \mathbb{N}) \wedge p_n \to p \Rightarrow \{\langle L(p_n) \cap L(p)\rangle\}_{n=1}^{\infty}$ converges.

C_3: $p_n, q_n \in C$, $L(p_n) \neq L(q_n)$ $(n \in \mathbb{N}) \wedge p_n, q_n \to p \Rightarrow \{\langle L(p_n) \cap L(q_n)\rangle\}_{n=1}^{\infty}$ converges.

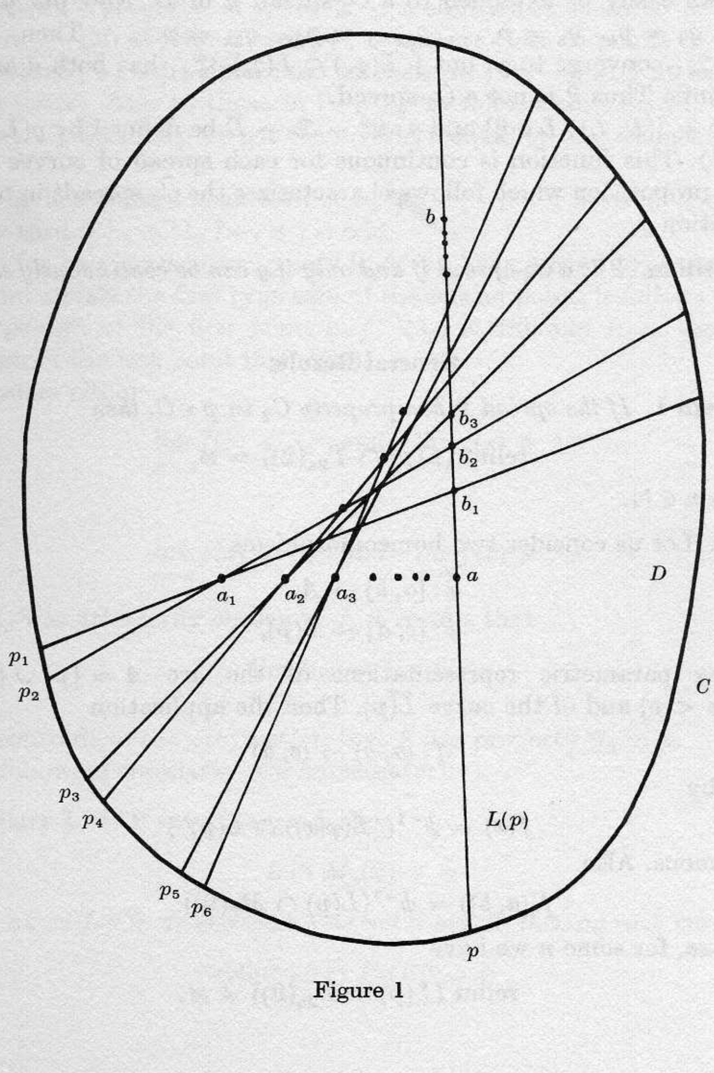

Figure 1

A usual spread of curves as defined in the Introduction will also be called a C_0*-spread.* A C_0-spread with property C_i at every point of C will be called a C_i*-spread* ($i = 1, 2, 3$). Of course, each C_i-spread is also a C_{i-1}-spread.

To see that not all C_{i-1}-spreads are C_i-spreads is easy, with perhaps the exception $i = 3$. Here is an example of a straight C_2-spread which is not a C_3-spread:

On Figure 1 $\{a_n\}_{n=1}^{\infty}$ is a horizontal sequence of points converging to $a \in L(p)$ and $\{b_n\}_{n=1}^{\infty}$ a sequence of points on $L(p)$ converging to b. Joining by straight lines a_1 with b_1 and b_2, a_2 with b_3 and b_4, and so on, we get intersection points p_1 and p_2, p_3 and $p_4, \ldots$ with C. The intersections of these lines with $\bar{D}$ will be curves $L(p_1)$ and $L(p_2)$, $L(p_3)$ and $L(p_4), \ldots$ of a spread. Clearly, $\{\langle L(p_n) \cap L(p)\rangle\}_{n=1}^{\infty}$ converges to b and $\{L(p_n) : n \in \mathbb{N}\} \cup \{L(p)\}$ can easily be extended to a C_2-spread $\mathfrak{L}$ in $\bar{D}$. Now put $q_1 = p_2$, $q_2 = p$, $q_3 = p_4$, $q_4 = p, \ldots, q_{2k-1} = p_{2k}$, $q_{2k} = p, \ldots$. Then $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ converge to p, but $\{\langle L(p_n) \cap L(q_n)\rangle\}_{n=1}^{\infty}$ has both a and b as limit points. Thus $\mathfrak{L}$ is not a C_3-spread.

Let $\mathfrak{D} = \{(L, L) : L \in \mathfrak{L}\}$ and $g : \mathfrak{L}^2 - \mathfrak{D} \to \bar{D}$ be defined by $g(L, L') = \langle L \cap L'\rangle$. This function is continuous for each spread of curves $\mathfrak{L}$. The obvious proposition which follows characterizes the C_3-spreads in terms of the function g.

Proposition. *$\mathfrak{L}$ is a C_3-spread if and only if g can be continuously extended to $\mathfrak{L}^2$.*

General Results

Theorem 1. *If the spread $\mathfrak{L}$ has property C_2 in $p \in C$, then*

$$\operatorname{relint}\,(L(p) \cap T_{2n}(\mathfrak{L})) = \varnothing$$

for every $n \in \mathbb{N}$.

Proof. Let us consider two homeomorphisms

$$\varphi : [a, b] \to A,$$
$$\psi : [c, d] \to L(p),$$

providing parametric representations of the arc $A = \{p\} \cup \{-p\} \cup \{x \in C : x < p\}$ and of the curve $L(p)$. Then the application

$$f : (a, b) \to (c, d)$$

defined by

$$f(x) = \psi^{-1}(\langle L(\varphi(x)) \cap L(p)\rangle)$$

is continuous. Also,

$$f((a, b)) = \psi^{-1}(L(p) \cap M_2(\mathfrak{L})).$$

Suppose, for some n we have

$$\operatorname{relint}\,(L(p) \cap T_{2n}(\mathfrak{L})) \neq \varnothing.$$

Then there exists an open interval $J \subset f((a, b))$, such that, for each $\lambda \in J$, $\operatorname{card} f^{-1}(\lambda) = 2n - 1$. It follows that for such a λ, each point $x \in f^{-1}(\lambda)$ is a local extremum of $f|_{(a,x]}$ and of $f|_{[x,b)}$. This kind of argument was already used in [38] and [39]. Like there, we fix a point $\lambda_0 \in J$ and say $x \in f^{-1}(\lambda_0)$ is of type $(+, +)$ if x is a local maximum for both $f|_{(a,x]}$ and $f|_{[x,b)}$, of type $(+, -)$ if x is a local maximum for $f|_{(a,x]}$ and a local minimum for $f|_{[x,b)}$, etc. Let $[+, +], [+, -], [-, +], [-, -]$ be the set of points in $f^{-1}(\lambda_0)$ of type $(+, +)$, $(+, -)$, $(-, +)$, $(-, -)$ respectively.

Since

$$\operatorname{card} f^{-1}(\lambda) = 2n - 1$$

not only for $\lambda = \lambda_0$ but for every $\lambda \in J$, we can show that

$$\operatorname{card}[+, +] = \operatorname{card}[-, -].$$

By Darboux's property of f, for $\lambda' < \lambda$ sufficiently close to λ, we find indeed in $f^{-1}(\lambda')$ at least 2 card $[+, +]$ points close to points in $[+, +]$, at least card $[+, -]$ points close to those in $[+, -]$, and at least card $[-, +]$ points close to those in $[-, +]$. It follows that card $[-, -] \geqslant$ card $[+, +]$. Similarly we get the inverse inequality too, whence equality holds.

We further deduce that card $([+, +] \cup [-, -])$ is even and consequently card $([+, -] \cup [-, +]$ is odd.

Since for two consecutive points in $f^{-1}(\lambda_0)$ the second type-sign of the first point equals the first type-sign of the second point, it follows that the first type-sign of the first point in $f^{-1}(\lambda_0)$ is different from the second type-sign of the last point in $f^{-1}(\lambda_0)$.

Therefore either

$$\lim_{x \to a} f(x) \leqslant \lambda_0 \quad \text{and} \quad \lim_{x \to b} f(x) \geqslant \lambda_0$$

or

$$\lim_{x \to a} f(x) \geqslant \lambda_0 \quad \text{and} \quad \lim_{x \to b} f(x) \geqslant \lambda_0.$$

Since λ_0 was arbitrarily chosen in J, it results that

$$\lim_{x \to a} f(x) \neq \lim_{x \to b} f(x),$$

which contradicts the assumption that $\mathfrak{L}$ has property C_2 in p.

The following corollaries are immediate:

Corollary 1. *If $\mathfrak{L}$ is a C_2-spread, then*

$$L \cap M_3(\mathfrak{L}) \neq \varnothing$$

on each curve $L \in \mathfrak{L}$. If moreover $\mathfrak{L}$ is not a pencil, then on each curve $L \in \mathfrak{L}$

$$\operatorname{relint}(L \cap P_3(\mathfrak{L})) \neq \varnothing.$$

Corollary 2. *If $n \in \mathbb{N}$ and $\mathfrak{L}$ is a C_2-spread, but not a pencil, then*

$$\text{relint } (L \cap M_2(\mathfrak{L})) \subset T_{2n}(\mathfrak{L})$$

on no curve $L \in \mathfrak{L}$.

Compare [26].

Theorem 2. *If $\mathfrak{L}$ is a C_0-spread and n is finite and even, then $P_n(\mathfrak{L}) \subset \text{int } P_{n+1}(\mathfrak{L})$. Consequently $V_n(\mathfrak{L})$ is rare and if $P_n(\mathfrak{L}) \neq \varnothing$, then $\text{int } P_{n+1}(\mathfrak{L}) \neq \varnothing$.*

Since the proof is essentially the same as that of Theorem 2 in [36] (Theorem B of the Introduction), we omit it.

Question. *Is $\cup_{n \text{ even}} T_n(\mathfrak{L})$ rare?*

Notice that from Theorem 2 it follows that the set of all points of even multiplicity in D is of the first Baire category. Hence, if $M_{\aleph_0}(\mathfrak{L}) = \varnothing$, then most of the points in D are of odd multiplicity in the sense of Baire categories. The situation may change if $M_{\aleph_0}(\mathfrak{L}) \neq \varnothing$. We have examples of spreads for which the whole set of points of finite multiplicity is of first Baire category, but this will be published elsewhere.

An infinite analogue of Theorem 2 does not exist. We only have

Theorem 3. *Let $\mathfrak{L}$ be a C_0-spread with $P_{\aleph_0}(\mathfrak{L}) \neq \varnothing$. Then $\text{int } P_n(\mathfrak{L}) \neq \varnothing$ for all finite n.*

Proof. Let $n < \aleph_0$. $P_{\aleph_0}(\mathfrak{L}) \neq \varnothing$ implies $P_{2n}(\mathfrak{L}) \neq \varnothing$. By Theorem 2, $\text{int } P_{2n+1}(\mathfrak{L}) \neq \varnothing$, which yields $\text{int } P_n(\mathfrak{L}) \neq \varnothing$.

The following formulation includes both Theorem 3 and the last implication of Theorem 2.

Proposition. *In a C_0-spread $\mathfrak{L}$, $P_{2\mathfrak{x}}(\mathfrak{L}) \neq \varnothing$ implies $\text{int } P_{\mathfrak{y}}(\mathfrak{L}) \neq \varnothing$ for each pair of cardinal numbers $\mathfrak{x} \leqslant \aleph_0$ and $\mathfrak{y} < 2(\mathfrak{x} + 1)$.*

The next theorem shows that commonly encountered spreads are all in fact C_1-spreads.

Theorem 4. *If the C_0-spread $\mathfrak{L}$ is not a C_1-spread, then $\text{card } P_{\aleph_0}(\mathfrak{L}) = \mathfrak{c}$. More precisely, if $\mathfrak{L}$ has not property C_1 at $p \in C$, then*

$$\text{relint } (L(p) \cap P_{\aleph_0}(\mathfrak{L})) \neq \varnothing.$$

Proof. If $\mathfrak{L}$ has not property C_1 at p, then there is a monotone sequence $\{p_n\}_{n=1}^{\infty}$ convergent to p, such that $\{< L(p_n) \cap L(p) >\}_{n=1}^{\infty}$ is divergent. Employing the functions φ, ψ, f, and the arc A defined in the proof of Theorem 1, we can say without loss of generality that all p_n's belong to A. Suppose $\varphi(a) = -p$. Then $\varphi^{-1}(p_n) \to \varphi^{-1}(p) = b$. But $\{f(\varphi^{-1}(p_n))\}_{n=1}^{\infty}$ does not converge. Let j, k $(j < k)$ be two limit points of $\{f(\varphi^{-1}(p_n))\}_{n=1}^{\infty}$. It follows from Darboux's property of f that for each point $x \in (j, k)$, the set $f^{-1}(x)$ has infinitely many components. This yields $\psi(x) \in P_{\aleph_0}(\mathfrak{L})$. Since

$$L(p) \cap P_{\aleph_0}(\mathfrak{L}) \supset \{\psi(x) : x \in (j, k)\}$$

and since the last set is open and nonempty, we obtain

$$\text{relint}\,(L(p) \cap P_{\aleph_0}(\mathfrak{L})) \neq \varnothing.$$

Until now we have not considered $V_c(\mathfrak{L})$; one might get the false impression that this set must be void.

Theorem 5. *There are straight C_3-spreads $\mathfrak{L}$ with $V_c(\mathfrak{L}) \neq \varnothing$.*

Proof. Let C be a circle with centre c, the points p and $-p$ be diametrically opposite on C, and qr an arc on C such that $-p < q < r < p$, relative to a certain sense on C. We consider a Cantor set S on qr. Let $L(p)$ be the segment joining p with $-p$. Let s, t $(q < s < t < r)$ be the endpoints of one of the (open) components of the complement of S in qr. Let $L(s)$ and $L(t)$ be the diameters of C through s and t. Also, let τ be a curved triangle consisting of three convex differentiable arcs α, β, γ such that α is tangent at c to $L(s)$, α and β have the same tangent at $\langle\alpha \cap \beta\rangle$, β and γ have the same tangent at $\langle\beta \cap \gamma\rangle$, and γ is tangent at c to $L(t)$, and such that the distance between c and any point of τ is smaller than that from s to t, see Figure 2. Now let $L(x)$ be the (unique) chord of C passing through x and tangent to τ, for each $x \in st$. Repeat this construction for all components of $qr - S$. It is clear that

$$\bigcap_{x \in S} L(x) = \{c\},$$

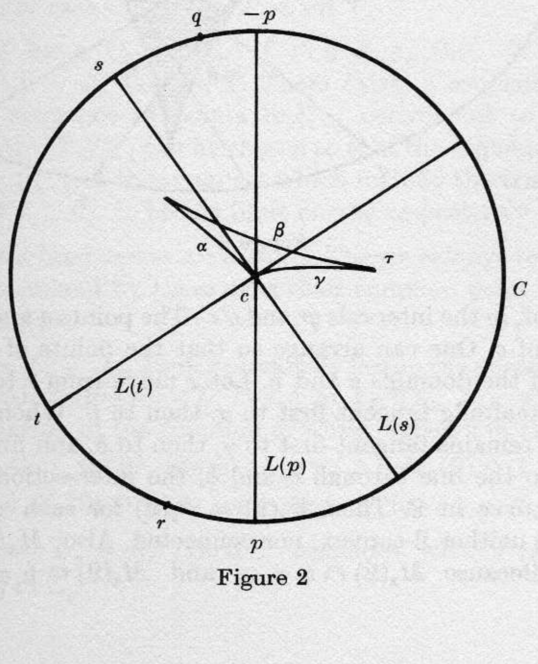

Figure 2

which implies $c \in V_c(\mathfrak{L})$. The conditions that $\mathfrak{L}$ must satisfy in order to be a C_3-spread are also fulfilled for all points in qr and it is obvious that the construction can be extended conveniently to the complement of qr.

The above process can be repeated in $\aleph_0$-many different intervals, so we can have card $V_c(\mathfrak{L}) = \aleph_0$.

Question. *Do there exist spreads* $\mathfrak{L}$ *such that*

$$\text{card } V_c(\mathfrak{L}) = \mathfrak{c}\,?$$

We study now connectedness properties of the sets $M_{\mathfrak{x}}(\mathfrak{L})$ and $P_{\mathfrak{x}}(\mathfrak{L})$.

Theorem 6. *There are straight* C_3*-spreads* $\mathfrak{L}$ *such that* $M_{\mathfrak{x}}(\mathfrak{L})$ *and* $P_{\mathfrak{x}}(\mathfrak{L})$ *are not* $\mathfrak{L}$*-convex for any* $\mathfrak{x} \geqslant 3$ *and not connected for any* $\mathfrak{x} \geqslant 4$.

Proof. Look at Figure 3. We make constructions similar to that of the

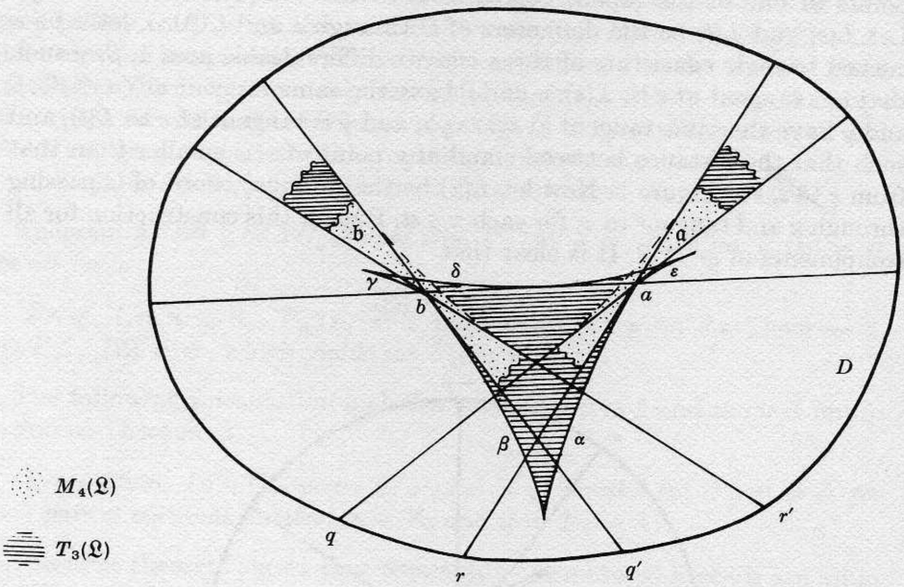

Figure 3

preceding proof, in the intervals qr and $q'r'$. The points a and b respectively play the role of c. One can arrange so that the points of $M_4(\mathfrak{L})$ lie only in the union of the domains $\mathfrak{a}$ and $\mathfrak{b}$. Let x move from r to q' on C. Then $L(x)$ moves remaining tangent first to α, then to β. When x moves from r' to $-q$, $L(x)$ remains tangent first to γ, then to δ, and finally to ε. Since δ is tangent to the line through a and b, the intersection L of this line with $\bar{D}$ is a curve in $\mathfrak{L}$. Thus $M_{\mathfrak{x}}(\mathfrak{L}) = P_{\mathfrak{x}}(\mathfrak{L})$ for each $\mathfrak{x}$ and $M_{\mathfrak{c}}(\mathfrak{L}) = \{a, b\}$, which is neither $\mathfrak{L}$-convex, nor connected. Also, $M_x(\mathfrak{L}) \subset \mathfrak{a} \cup \mathfrak{b}$ for each $\mathfrak{x} \geqslant 4$. Because $M_{\mathfrak{x}}(\mathfrak{L}) \cap \mathfrak{a} \neq \varnothing$ and $M_{\mathfrak{x}}(\mathfrak{L}) \cap \mathfrak{b} \neq \varnothing$, $M_{\mathfrak{x}}(\mathfrak{L})$ is disconnected (for $x \geqslant 4$). To see that $M_x(\mathfrak{L})$ is $\mathfrak{L}$-convex for no $x \geqslant 3$, it suffices to notice that $a, b \in M_c(\mathfrak{L})$ but $\delta \cap L \cap M_3(\mathfrak{L}) = \varnothing$.

Also the set $\overline{M_x(\mathfrak{L})}$ is not necessarily $\mathfrak{L}$-convex for any $x \geqslant 4$. However, for $x = 3$ we have

Theorem 7. *For every C_2-spread $\mathfrak{L}$, $\overline{M_3(\mathfrak{L})}$ is $\mathfrak{L}$-convex.*

Since the proof is easy we leave it to the reader.

Theorem 8. *There are straight C_2-spreads without locally connected T.*

Proof. Consider the example of a C_2-spread which is not a C_3-spread, given in the preceding section. We arrange that when extending $\{L(p_n): n \in \mathbb{N}\} \cup \{L(p)\}$ to $\mathfrak{L}$, no new curve (chord) should pass through a. We also arrange that $L(p)$ be a line of symmetry for D and $\mathfrak{L}$; then the chord of C symmetrical to any $L \in \mathfrak{L}$ lies in $\mathfrak{L}$. Let

$$\gamma = \cup \{L(p) \cap L : L \in \mathfrak{L} - \{L(p)\}\}.$$

We have $a \notin \gamma$. Since γ has a nearest point to p, we can choose a point $a' \notin \gamma$ different from a and lying between a and γ on $L(p)$. For each $n \in \mathbb{N}$, the (open) triangles $a_n b_{2n-1} b_{2n}$ and $a'_n b_{2n-1} b_{2n}$, where a'_n and a_n are symmetric with respect to $L(p)$, lie in T.

Now, we easily see that $a' \in \overline{T}$, but T is not locally connected in a'.

Since every Jordan domain is locally connected, Theorem 8 implies the existence of straight C_2-spreads such that T is not a Jordan domain.

Conjecture. *For each C_3-spread, T is a Jordan domain.*

We can prove only

Theorem 9. *For each C_3-spread, $T = \operatorname{int} \overline{T}$.*

Proof. Let $\mathfrak{L}$ be a C_3-spread. If $\cap \mathfrak{L} \neq \varnothing$, then $T = \operatorname{int} \overline{T} = \varnothing$. Suppose now $\cap \mathfrak{L} = \varnothing$. Let $x \in \overline{T}$. There exists a sequence of triangles $\{T_n\}_{n=1}^{\infty}$ and a sequence of points $\{y_n\}_{n=1}^{\infty}$ convergent to x, such that $y_n \in T_n$. Evidently $\{T_n\}_{n=1}^{\infty}$ can be chosen so that the sequences $\{L(p_n)\}_{n=1}^{\infty}$, $\{L(q_n)\}_{n=1}^{\infty}$, $\{L(r_n)\}_{n=1}^{\infty}$ of those curves which include the triangle-sides are convergent. Let L_p, L_q, L_r be the limit curves respectively.

Case I. all of the limit curves are distinct. Then x belongs to the closure of the triangle determined by them or is their common point. By Lemma 2 from [22], $x \in M_2(\mathfrak{L})$.

Case II. $L_p = L_q \neq L_r$. Then x obviously lies on the possibly degenerate subarc of L_p with endpoints $\langle L_p \cap L_r \rangle$ and $g^*(L_p, L_p)$, where g^* is the continuous extension of the function g, which appears in the Proposition of the Introduction, to $\mathfrak{L}^2$. Since $\langle L_p \cap L_r \rangle \in \overline{M_2(\mathfrak{L}) \cap L_p}$, $g^*(L_p, L_p) \in \overline{M_2(\mathfrak{L}) \cap L_p}$, and $M_2(\mathfrak{L}) \cap L_p$ is connected, we have $x \in \overline{M_2(\mathfrak{L}) \cap L_p}$.

Case III. $L_p = L_q = L_r$. Then x coincides with $g^*(L_p, L_p)$, hence again $x \in \overline{M_2(\mathfrak{L}) \cap L_p}$.

Thus in any case $x \in \overline{M_2(\mathfrak{L}) \cap L}$ for some curve $L \in \mathfrak{L}$.

Let now $x \in \text{int } \overline{T}$. We have $x \in \overline{M_2(\mathfrak{L}) \cap \text{L}}$ for some $L \in \mathfrak{L}$. Suppose $x \notin M_2(\mathfrak{L})$. Then x is one of the endpoints of the arc $\alpha = \overline{M_2(\mathfrak{L}) \cap L}$. Let U be a neighbourhood of x lying entirely in $\overline{T}$, and let $y \in U \cap L - \alpha$. Since $y \in \overline{T}$, there exists a curve $L' \in \mathfrak{L}$, such that $y \in \overline{M_2(\mathfrak{L}) \cap L'}$. But from $y \notin \alpha$, it follows that $L' = L$ and consequently $y \in \alpha$, which is impossible.

Hence int $\overline{T} \subset M_2(\mathfrak{L})$. By Theorem 2 from [40] (Theorem F in the Introduction), int $M_2(\mathfrak{L}) = T$; therefore int $\overline{T} = T$.

In a first version of this paper we conjectured that $T = \text{int } \overline{T}$ for each straight spread. We now know that this is not true. The mentioned example preceding Theorem 3 is a straight spread $\mathfrak{L}$ and not only $M_{\aleph_0}(\mathfrak{L})$ but also $P_{\aleph_0}(\mathfrak{L})$ is dense in D. It follows that $P_2(\mathfrak{L})$ is dense in D. Then, by Theorem 2, int $P_3(\mathfrak{L})$ is dense in D too, whence, by Theorem F, $\overline{T} = \overline{\text{int } M_2(\mathfrak{L})} \supset \overline{\text{int } P_3(\mathfrak{L})} = \overline{D}$. Thus int $\overline{T} = D$. On the other hand, by Theorem 11 (in the next section), $M_2(\mathfrak{L}) \neq D$, hence $T = \text{int } M_2(\mathfrak{L}) \neq D = \text{int } \overline{T}$.

Results on Straight Spreads

In this Section we study straight spreads $\mathfrak{L}$. Let ω be a point on the unit circle S^1 of the plane; there exists a unique line-segment L_ω in $\mathfrak{L}$ with direction ω. We say that $\mathfrak{L}$ has property C_i in direction ω, if $\mathfrak{L}$ has property C_i at the endpoints of L_ω.

Theorem 10 (Hammer-Sobczyk [25]). *Every straight spread has property C_2 in almost every direction.*

Proof. We extend the curves of the given straight spread $\mathfrak{L}$ to lines in the plane, choose two orthogonal lines among them, and consider a square *abcd* surrounding D and having its diagonals *ac* and *bd* on these lines (see Figure 4). The square cuts on all the lines segments forming a new spread $\mathfrak{L}'$, which has property C_2 in direction $\omega \in S^1$ if and only if $\mathfrak{L}$ has property C_2 in direction ω. To prove the theorem it will obviously suffice to show that $\mathfrak{L}'$ has property C_2 in almost every point of *ab*.

For each $p \in ab$, let $-p \in cd$ be the other endpoint of $L(p) \in \mathfrak{L}'$. Let α be the distance from a to b and

$$f: [0, \alpha] \to [0, \alpha]$$

be the strictly monotone function associating to each $x \in [0, \alpha]$ the distance between c and $-p$, where $p \in ab$ is at distance x from a. From an early theorem of Lebesgue we know that f is almost everywhere differentiable, which yields $\mathfrak{L}'$ has property C_2 almost everywhere on *ab*.

Theorem 11. *For every straight spread $\mathfrak{L}$ and for almost every direction $\omega \in S^1$,*

$$\text{relint } (L_\omega \cap T_1(\mathfrak{L})) \neq \varnothing.$$

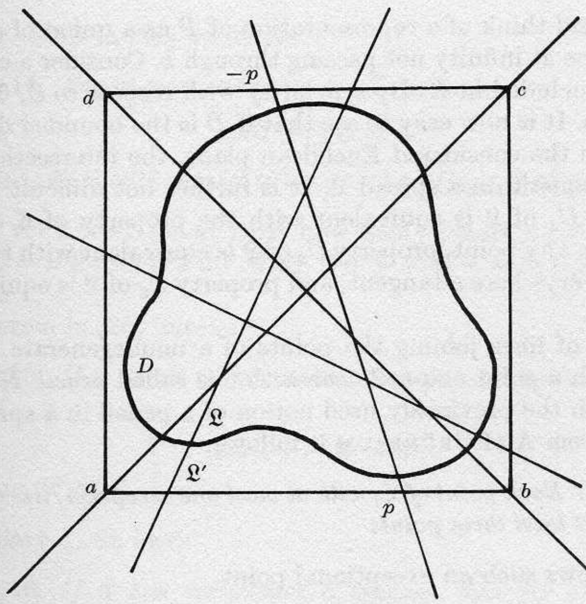

Figure 4

Proof. Consider $\omega \in S^1$ such that $\mathfrak{L}$ has property C_2 in direction ω. It follows that the connected set $L_\omega \cap M_2(\mathfrak{L})$ is not open on the line $L \supset L_\omega$ and therefore contains at least one of its boundary points (on L), say ε. Thus, at least two curves of $\mathfrak{L}$ pass through ε, whence $\varepsilon \notin C$. It results that

$$\operatorname{relint}(L_\omega - M_2(\mathfrak{L})) \neq \varnothing.$$

Since $\mathfrak{L}$ has property C_2 in almost every direction, the theorem follows.

Theorem 11 proves, in the case of straight spreads, GRÜNBAUM's conjecture that $T_1(\mathfrak{L}) \neq \varnothing$ (part of Conjecture 4.1 in [23]).

Order-Geometrical Properties of Starshaped Curves

This section contains a short order-geometrical study of strictly starshaped curves in the projective plane. It is included in this paper, because spreads will be the only tool used.

Let $\mathfrak{C}$ be a curve in the projective plane P and K the set of all points $x \in P$, such that every line through x meets $\mathfrak{C}$ in precisely one point. $\mathfrak{C}$ is said to be *starshaped* if $K \neq \varnothing$, and *strictly starshaped* if $\operatorname{int} K \neq \varnothing$. Of course, any starshaped curve is Jordan and the complement of it is connected. In the following, $\mathfrak{C}$ will always be a strictly starshaped curve, different from a line.

We can derive properties of $\mathfrak{C}$ from results on spreads in the following way:

Let $k \in K$ and think of a representation of P as a union of a Euclidean plane and a line at infinity not passing through k. Consider a circle C with centre k and included in K. By a polarity with respect to C, $\mathfrak{C}$ becomes a family of lines. It is now easy to see that if D is the bounded domain with boundary C in the considered Euclidean plane, the intersections of these lines with $\bar{D}$ constitute a spread $\mathfrak{L}$. It is further not difficult to establish that property C_1 of $\mathfrak{L}$ is equivalent with the property of $\mathfrak{C}$ of having a half-tangent in any point, property C_2 of $\mathfrak{L}$ is equivalent with the property of $\mathfrak{C}$ having everywhere a tangent, and property C_3 of $\mathfrak{L}$ is equivalent with class C^1 for $\mathfrak{C}$.

The family of lines joining the points of a nondegenerate closed line-segment σ with a point non-collinear with σ is called *pencil*. No confusion is possible with the previously used notion of a pencil in a spread.

From Theorem A of Grünbaum it follows:

Theorem 12. *Each point of $\mathfrak{C}$, with at most one exception, lies on some line meeting $\mathfrak{C}$ in at least three points.*

Figure 5 shows such an exceptional point.

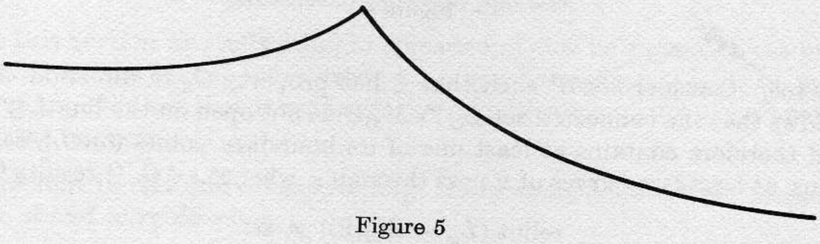

Figure 5

Let $\mathscr{K}(x)$ be the set of all lines through $x \in P$.

Theorem A′ yields:

Theorem 13. *For all points $x \in \mathfrak{C}$, with at most three exceptions, there exists a pencil in $\mathscr{K}(x)$, each line of which meets $\mathfrak{C}$ in at least three points.*

In Figure 6 we present a curve $\mathfrak{C}$ with three exceptional points.

Let

$$\mathscr{M}(x) = \{G \in \mathscr{K}(x) : \text{card } G \cap \mathfrak{C} \geqslant 2\}.$$

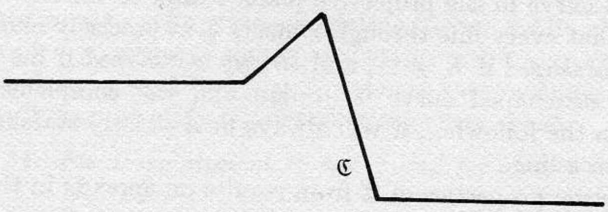

Figure 6

From the result in [38] we derive

Theorem 14. *For at most one point* $x \in \mathfrak{C}$ *we have*

$$\operatorname{card} G \cap \mathfrak{C} = 4$$

for all $G \in \operatorname{int} \mathscr{M}(x)$ *with respect to the usual topology of* $\mathscr{K}(x)$.

It is not yet known for at most how many points $x \in \mathfrak{C}$ we may have $\operatorname{card} G \cap \mathfrak{C} = n$ for all $G \in \operatorname{int} \mathscr{M}(x)$, if $n \geqslant 5$. If $n = 3$, this upper bound (the best) is $\mathfrak{c}$.

The Theorem in [26] implies:

Theorem 15. *If* n *is an even number, then for at most countably many points* $x \in \mathfrak{C}$,

$$\operatorname{card} G \cap \mathfrak{C} = n$$

for all $G \in \operatorname{int} \mathscr{M}(x)$.

By Corollary 1, we have:

Theorem 16. *If* $\mathfrak{C}$ *has everywhere a tangent, then for all points of* $\mathfrak{C}$, *without exceptions, there exists a pencil in* $\mathscr{K}(x)$, *each line of which meets* $\mathfrak{C}$ *in at least three points.*

Compare Theorem 16 with Theorems 12 and 13.

Corollary 2 implies:

Theorem 17. *If* n *is finite and even and* $\mathfrak{C}$ *has everywhere a tangent, then for no point* $x \in \mathfrak{C}$ *we have*

$$\operatorname{card} G \cap \mathfrak{C} = n$$

for all $G \in \operatorname{int} \mathscr{M}(x)$.

Compare Theorem 17 with Theorem 15. Theorem 10 leads to the observation that $\mathfrak{C}$ must have almost everywhere a tangent, which also follows in a direct way from a corresponding property of Lipschitz functions.

From Theorem E we get:

Theorem 18. *If* $f, g: \mathfrak{C} \to \mathfrak{C}$ *are continuous functions, then there exists* $x \in \mathfrak{C}$ *such that* $x, f(x)$ *and* $g(x)$ *lie on a line.*

We conclude this last section with the remark that its theorems are only a small selection of properties of $\mathfrak{C}$ obtainable by using systematically the theory of spreads.

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