Math. Z. 174, 135–139 (1980)

Mathematische Zeitschrift

The Curvature of Most Convex Surfaces Vanishes Almost Everywhere

Tudor Zamfirescu

Abteilung Mathematik der Universität, Postfach 500500, D-4600 Dortmund, Federal Republic of Germany

At least since 1904 we know about the existence of singular monotone functions, i.e. continuous monotone functions of one real variable with vanishing derivative a.e. In that year Lebesgue [7] and Minkowski [8] gave their famous examples. In 1910 Faber [4], in 1916 Sierpinski [10] and then many others greatly enriched the variety of known examples. By integrating anyone of these functions we get a differentiable convex function with vanishing second derivative a.e. Thus we obtain smooth convex curves with vanishing curvature a.e. A very beautiful and simple example is due to de Rham [3]: Take a convex polygon, consider on each side the two points dividing it into three equal parts and take the convex polygon with all these division points as vertices. Repeat the procedure. As a limit we get a smooth, strictly convex curve with a.e. vanishing curvature!

The purpose of this paper is to show that, in the sense of Baire categories, most convex curves have the above property. More generally, most convex surfaces in $\mathbb{R}^n$ have a.e. a vanishing sectional curvature in every tangent direction.

We start with some definitions.

By a *convex surface* we shall always understand a closed convex surface in Busemann's sense (see [1], p. 3). A *convex curve* is a one-dimensional convex surface.

A point x of a convex surface S is called *smooth* if S is differentiable at x. S is *smooth* if each of its points is smooth.

Any half-line in a supporting plane (this is a hyperplane, see [1], p. 4) of a convex surface S, originating at a point $x \in S$ will be called *supporting direction* at x. A supporting direction at a smooth point is called *tangent direction.*

The union of a circular with a square 2-cell, such that the radius and centre of the circle coincide with the side-length and a vertex of the square, is called *corner-disk.* The two points lying on both the circle and the square are called *touching points* of the corner-disk. A 2-cell having the union of half a circle with a segment as boundary is called *semidisk.* The points of the boundary of a semidisk which are not smooth are called *corners* of the semidisk. The *radius* of a corner-disk or a semidisk is the radius of the circle appearing in their definitions.

0025-5874/80/0174/0135/$01.00

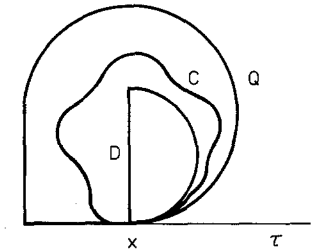

Fig. 1

Let x be a point of a convex curve C. A corner-disk Q with x as touching point is called *supporting corner-disk* of C at x if $C \subset Q$. A semidisk D with x as corner is called *supporting semidisk* of C at x if $D \subset \operatorname{conv} C$.

Let τ be a supporting direction of C at x. The supporting corner-disk Q of C at x is said to be *τ-oriented* if the line through τ supports Q and $\tau \cap Q = \{x\}$. The supporting semidisk D of C at x is said to be *τ-oriented* if the line λ through the corners of D is orthogonal to τ and both D and τ lie in the same closed half-plane determined by λ.

Now let τ be a supporting direction of the convex surface S at x, such that some normal section $S(\tau)$ of S along τ is a convex curve. S is called *τ-round* at x if such an $S(\tau)$ has both a τ-oriented supporting corner-disk and a τ-oriented supporting semidisk at x.

If x is a smooth point of the convex surface S and τ a tangent direction at x, then $S(\tau)$, which is now unique, is a convex curve; let $\rho_i(\tau)$ and $\rho_s(\tau)$ be the lower and upper radii of curvature (see [1], p. 14 for a definition) of $S(\tau)$ in the direction τ. In the case of an arc A (convex or not) differentiable and having an osculating plane at x, $\rho_i(\tau)$ and $\rho_s(\tau)$ will also mean the lower and upper radii of curvature of A at x in the tangent direction τ. If $\rho_i(\tau) = \rho_s(\tau)$, we also write $\rho(\tau)$ for the common value.

The word *most* will always be used in a space of second Baire category, in the sense of "all, except those in a set of first Baire category".

Klee [6] and Gruber [5] showed independently that most convex surfaces are smooth and strictly convex. The following theorem completes the description of most convex surfaces.

Theorem 1. *Most convex surfaces are smooth and strictly convex, and for each point x and tangent direction τ at x, $\rho_i(\tau) = 0$ or $\rho_s(\tau) = \infty$ (or both).*

Proof. Let $\mathscr{S}$ be the space of all convex surfaces in $\mathbb{R}^n$. With the usual Hausdorff metric, $\mathscr{S}$ is isometric to the subspace $\mathscr{K}^n$ of all n-dimensional compact convex sets in $\mathbb{R}^n$ of the space $\mathscr{K}$ of all compact convex sets in $\mathbb{R}^n$. Since $\mathscr{K}$ is complete and $\mathscr{K} - \mathscr{K}^n$ is nowhere dense in $\mathscr{K}$, $\mathscr{K}^n$ and hence $\mathscr{S}$ too are of second Baire category. We first suppose $n \geqq 3$.

Let $\mathscr{S}_n$ be the family of all surfaces S in $\mathscr{S}$ that possess at some point x a tangent hyperplane P and a tangent direction τ, such that $S(\tau)$ has a τ-oriented supporting corner-disk Q of radius n and a τ-oriented supporting semidisk D of radius n^{-1} at x.

Let $S_0 \in \bar{\mathscr{S}}_n$, $\{S_m\}_{m=1}^{\infty}$ be a sequence of surfaces from $\mathscr{S}_n$ such that $S_m \to S_0$, and x_m, P_m, τ_m, Q_m, D_m be corresponding points, tangent hyperplanes, tangent directions, supporting corner-disks, and supporting semidisks, resulting from the definition of $\mathscr{S}_n$. By taking if necessary a convenient subsequence, we can arrange that $x_m \to x_0$, $P_m \to P_0$ and $\tau_m \to \tau_0$, where $x_0 \in S_0$, P_0 is a supporting plane and τ_0 a supporting direction of S_0 at x_0. (Since we consider the closure of $\mathscr{S}_n$ in $\mathscr{S}$, S_0 is a nondegenerate convex surface.) Clearly, the sequences $\{Q_m\}_{m=1}^{\infty}$ and $\{D_m\}_{m=1}^{\infty}$ converge, the first to a corner-disk Q_0 of radius n, the second to a semidisk of radius n^{-1}, both lying in a (2-dimensional) plane N_0 through τ_0 orthogonal to P_0. We consider $S_0(\tau_0) = S_0 \cap N_0$.

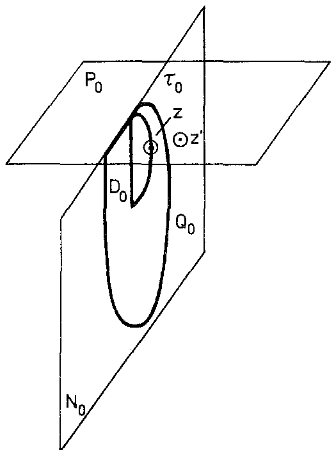

Fig. 2

Suppose $D_0 - \operatorname{conv} S_0 \neq \emptyset$. Then we have a point z in D_0 and an entire ball B around z, which does not meet $\operatorname{conv} S_0$. But for some $m \geqq 1$, D_m meets B and $\operatorname{conv} S_m$ does not meet B. This means that the inclusion $D_m \subset \operatorname{conv} S_m(\tau_m)$ does not hold, which contradicts the fact that D_m is a supporting semidisk of $S_m(\tau_m)$ at x_m. Hence $D_0 \subset \operatorname{conv} S_0$. Since $D_0 \subset N_0$, it follows that $D_0 \subset \operatorname{conv} S_0(\tau_0)$.

Suppose $\operatorname{int} \operatorname{conv} S_0 \cap N_0 - Q_0 \neq \emptyset$. Then we have a point z' in N_0 and an entire ball B' around z' lying in $\operatorname{int} \operatorname{conv} S_0 - Q_0$. Again, for some $m \geqq 1$, Q_m does not meet B' and $\operatorname{conv} S_m(\tau_m)$ meets B'. This means that the inclusion $S_m(\tau_m) \subset Q_m$ does not hold, which contradicts the fact that Q_m is a supporting corner-disk of $S_m(\tau_m)$ at x_m. Hence $\operatorname{conv} S_0(\tau_0) - Q_0 \subset S_0$. Now it is easily seen that $S_0(\tau_0) \subset Q_0$ or $\operatorname{conv} S_0(\tau_0) \subset S_0$ (or both). The second inclusion yields the existence of a supporting plane of S_0 including N_0.

In conclusion, S_0 is τ_0-round at x_0, or P_0 and some hyperplane including N_0 are orthogonal supporting planes of S_0 at x_0.

Let $\mathscr{S}^*$ be the family of all surfaces S in $\mathscr{S}$, which are τ-round for some point $x \in S$ and supporting direction τ at x, or have two orthogonal supporting planes at some point. We saw that $\bar{\mathscr{S}}_n \subset \mathscr{S}^*$.

Let $\mathcal{O}$ be an open set in $\mathscr{S}$. Let W be a polytopal surface in $\mathcal{O}$. If the polytopal surface V approximates a hypersphere well enough, then there is no point on

V with orthogonal supporting planes through it. For $\varepsilon>0$ small enough, $Y=\operatorname{bd}\operatorname{conv}(W+\varepsilon V)$ still belongs to $\mathcal{O}$. Y has no point lying on two orthogonal supporting planes and, since every normal planar section is a polygon, is not τ-round for any point x and supporting direction τ at x. Thus $Y\in\mathcal{O}-\mathcal{S}^*$. It results that the complement of $\mathcal{S}^*$ is dense and therefore $\mathcal{S}_n$ nowhere dense in $\mathcal{S}$.

Let $\mathcal{S}^\dagger$ be the family of all surfaces in $\mathcal{S}$, which have a smooth point x and a tangent direction τ in x satisfying

$$0<\rho_i(\tau)\leqq\rho_s(\tau)<\infty .$$

It is easy to see that these inequalities are verified if and only if the surface is τ-round at x. Thus

$$\mathcal{S}^\dagger=\bigcup_{n=1}^{\infty}\mathcal{S}_n$$

and $\mathcal{S}^\dagger$ is of first Baire category in $\mathcal{S}$.

Thus, on most convex surfaces, for every smooth point x and tangent direction τ at x, $\rho_i(\tau)=0$ or $\rho_s(\tau)=\infty$. Combining this with the mentioned result of Klee and Gruber, we get the theorem.

After having read the proof for $n\geqq 3$, the case $n=2$ becomes almost trivial.

Since Meusnier's theorem holds for arbitrary convex surfaces (see [1], p. 15 and [2]), we may consider any kind of sections in the next theorem, which follows from Theorem 1.

Theorem 2. *On most convex surfaces (which are smooth), in all points where a sectional curvature along some tangent direction exists and is finite, hence a.e. in all directions, this curvature vanishes.*

On an arbitrary convex surface S, the set E of all points where Euler's theorem holds can be strictly included in the set C of those points where a sectional curvature in every direction exists, but the measure of $S-E$ is zero ([1], p. 23 and [2]). Theorem 2 yields:

Theorem 3. *On most convex surfaces, $E=C$ and the Dupin indicatrix at $x\in E$ is the whole tangent hyperplane in x.*

Combining Theorem 1 with Meusnier's theorem we also get:

Theorem 4. *On most convex surfaces (which are smooth) the following happens: on each differentiable arc having at every point an osculating plane not lying in the tangent hyperplane of the surface at that point, we have $\rho_i(\tau)=0$ or $\rho_s(\tau)=\infty$ (or both) at every point x and tangent direction τ in x. If the curve is planar, then* $\rho(\tau)=\rho(-\tau)=\infty$ a.e., *where $-\tau$ is the half-line opposite to τ.*

Before closing the paper, we mention a few other results of the same nature. First, we have Gruber's result [5] that most convex surfaces are not of class C^2, which follows now immediately from Theorem 2. Schneider [9] proved recently that on most convex surfaces there is a dense set of smooth points x such that for every tangent direction τ in x, $\rho_i(\tau)=0$ and $\rho_s(\tau)=\infty$. Now, it turns out that most points of these surfaces are such points x [11]. On most convex curves there is a dense set F of smooth points, such that $\rho(\tau)=0$ for every $x \in F$ and tangent direction τ at x. Also, there is a dense set G of smooth points, such that

$$0=\rho_i(\tau)<\rho_s(\tau)=\rho_i(-\tau)<\rho_s(-\tau)=\infty$$

for every $x \in G$ and some tangent direction τ at x. There are extensions to higher dimensions. These results will be published elsewhere.

References

1. Buseman, H.: Convex Surfaces. New York: Interscience 1958
2. Buseman, H., Feller, W.: Krümmungseigenschaften konvexer Flächen. Acta Math. **66**, 1–47 (1935)
3. de Rham, G.: Sur quelques courbes définies par des équations fonctionnelles. Rend. Sem. Mat. Univ. e Politec. Torino. **16**, 101–113 (1956/57)
4. Faber, G.: Über stetige Funktionen II. Math. Ann. **69**, 372–443 (1910)
5. Gruber, P.: Die meisten konvexen Körper sind glatt, aber nicht zu glatt. Math. Ann. **229**, 259–266 (1977)
6. Klee, V.: Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. **139**, 51–63 (1959)
7. Lebesgue, H.: Leçons sur l'intégration et la recherche des fonctions primitives. Paris: Gauthier-Villars 1904
8. Minkowski, H.: Zur Geometrie der Zahlen. In: Verhandlungen des dritten Internationalen Mathematiker-Kongresses (Heidelberg 1904), pp. 164–173. Leipzig: Teubner 1905. Reprint: Nendeln, Liechtenstein: Kraus 1967
9. Schneider, R.: On the curvatures of convex bodies. Math. Ann. **240**, 177–181 (1979)
10. Sierpiński, W.: Un example élémentaire d'une fonction croissante qui a presque partout une dérivée nulle. Giorn. Mat. Battaglini **54**, 314–344 (1916)
11. Zamfirescu, T.: Nonexistence of curvature in most points of most convex surfaces. Math. Ann. (to appear)

Received July 4, 1979