



---

Inscribed and Circumscribed Circles to Convex Curves

Author(s): Tudor Zamfirescu

Source: *Proceedings of the American Mathematical Society*, Vol. 80, No. 3 (Nov., 1980), pp. 455-457

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2043739>

Accessed: 25/08/2009 09:02

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*.

<http://www.jstor.org>

## INSCRIBED AND CIRCUMSCRIBED CIRCLES TO CONVEX CURVES

TUDOR ZAMFIRESCU

**ABSTRACT.** A convex planar curve may have 2, 3, . . . ,  $c$  contact points with its inscribed or circumscribed circle. One of these numbers appears in most cases: 3.

Let  $\mathcal{C}$  be the space of all closed convex curves in the plane (see [1, p. 3] for a precise definition of a closed convex surface, in particular curve). Several pathological properties of most curves in  $\mathcal{C}$  (in the sense of Baire categories) are described in [4] and [5]. We shall show here that most curves in  $\mathcal{C}$  have the (expected?) number, 3, of contact points with their inscribed and circumscribed circles. It seems that mainly local properties may be pathological for most curves in  $\mathcal{C}$ .

We say that *most* elements of a space of second Baire category have a certain property if those elements which do not have it form a set of first Baire category. Now,  $\mathcal{C}$  is of second Baire category (see for instance [5]) if we endow it with the Hausdorff metric, so it makes sense to speak about most curves in  $\mathcal{C}$ .

Let  $C \in \mathcal{C}$ ,  $D$  be the convex domain with boundary  $C$ ,  $K_C$  the *circumscribed circle* of  $C$ , i.e. the smallest circle surrounding  $D$ , and  $k_C$  an *inscribed circle* of  $C$ , i.e. a largest circle included in  $D \cup C$ . The circle  $k_C$  is not unique only if  $C$  contains parallel segments. Since most curves in  $\mathcal{C}$  are strictly convex (see [3] or [2]), they admit a unique inscribed circle.

Clearly,  $\text{card}(C \cap K_C)$  may be any cardinal number between 2 and  $c$ . We prove

**THEOREM 1.** *For most curves  $C \in \mathcal{C}$ ,  $\text{card}(C \cap K_C) = 3$ .*

We will use the following elementary Lemma, that we give without proof.

**LEMMA.** *Let  $P$  be a polygon such that  $P \cap K_P$  consists of precisely three points  $x_1, x_2, x_3$  determining an acute triangle. Let  $N_1, N_2, N_3$  be neighborhoods of  $x_1, x_2, x_3$ . Then there is a neighborhood  $\mathcal{U}$  of  $P$  in  $\mathcal{C}$  such that, for each  $C \in \mathcal{U}$ ,*

$$C \cap K_C \cap N_i \neq \emptyset \quad (i = 1, 2, 3)$$

and

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3.$$

**PROOF OF THEOREM 1.** We first show that the set  $\mathcal{C}_2$  of all curves in  $\mathcal{C}$  satisfying  $\text{card}(C \cap K_C) = 2$  is nowhere dense in  $\mathcal{C}$ .

---

Received by the editors May 4, 1979.

AMS (MOS) subject classifications (1970). Primary 52A10, 54E52.

© 1980 American Mathematical Society  
0002-9939/80/0000-0566/\$01.75

Let  $\emptyset$  be an open set in  $\mathcal{C}$ . We choose a polygon  $P$  in  $\emptyset$ .

If  $\text{card}(P \cap K_P) = 2$ , let  $x$  and  $y$  be the vertices of  $P$  lying on  $K_P$ . Let  $x_1, x_2$  be two points such that the segment  $x_1x_2$  contains  $x$  and is orthogonal on  $xy$ . Let  $K'$  be the circle through  $x_1, x_2, y$ . The boundary  $P'$  of the convex hull of  $P \cup \{x_1, x_2\}$  has  $K'$  as circumscribed circle. If  $x_1$  and  $x_2$  are close enough to  $x$ ,  $P'$  still lies in  $\emptyset$ . Obviously

$$P' \cap K' = \{x_1, x_2, y\}.$$

If  $\text{card}(P \cap K_P) \geq 3$ , there are three points  $x_1, x_2, x_3$  in  $P \cap K_P$  determining a triangle with all angles of measure at most  $\pi/2$ . By gently cutting all the other vertices of  $P$  and slightly moving  $x_1$  if necessary, we obtain a polygon  $P'$  still belonging to  $\emptyset$  such that  $P' \cap K_P$  is the vertex-set of an acute triangle. Now, by the Lemma, for a neighborhood  $\mathcal{N}$  of  $P'$  in  $\mathcal{C}$ , each curve  $C \in \mathcal{N}$  meets  $K_C$  in at least three points. Thus

$$\emptyset \cap \mathcal{N} \cap \mathcal{C}_2 = \emptyset,$$

which proves that  $\mathcal{C}_2$  is nowhere dense in  $\mathcal{C}$ .

Let  $\mathcal{C}_{(n)}$  be the set of all curves  $C$  in  $\mathcal{C}$  such that

- (i)  $\text{card}(C \cap K_C) \geq 4$ , and
- (ii) there exist four points  $x_1, x_2, x_3, x_4 \in C \cap K_C$  such that the side-lengths of the convex quadrangle with vertices  $x_1, x_2, x_3, x_4$  are at least  $n^{-1}$  ( $n \in \mathbf{N}$ ).

We show that  $\mathcal{C}_{(n)}$  is nowhere dense in  $\mathcal{C}$ .

Let  $\emptyset$  be an open set in  $\mathcal{C}$ . We choose like before a polygon  $P'$  in  $\emptyset$  such that  $P' \cap K_{P'}$  is the vertex-set of an acute triangle. Now let  $N_i$  be a disk of centre  $x_i$  and radius less than  $(2n)^{-1}$ . By the Lemma, there is a neighborhood  $\mathcal{N}$  of  $P'$  such that, for each curve  $C \in \mathcal{N}$ ,

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3,$$

and therefore we cannot find 4 points in  $C \cap K_C$  determining a convex quadrangle with side-lengths at least  $n^{-1}$ . Thus

$$\emptyset \cap \mathcal{N} \cap \mathcal{C}_{(n)} = \emptyset$$

and  $\mathcal{C}_{(n)}$  is nowhere dense in  $\mathcal{C}$ .

Let  $\mathcal{C}_3$  be the set of all curves  $C \in \mathcal{C}$  verifying  $\text{card}(C \cap K_C) = 3$ . Every curve of  $\mathcal{C}$  not belonging to  $\mathcal{C}_2$  or  $\mathcal{C}_3$  must be in  $\mathcal{C}_{(n)}$  for some  $n \in \mathbf{N}$ . Thus

$$\mathcal{C} - \mathcal{C}_3 = \mathcal{C}_2 \cup \bigcup_{n=1}^{\infty} \mathcal{C}_{(n)},$$

where  $\mathcal{C}_2$  and  $\mathcal{C}_{(n)}$  ( $n = 1, 2, 3, \dots$ ) are nowhere dense; therefore  $\mathcal{C} - \mathcal{C}_3$  is of first Baire category, which proves the theorem.

Surprisingly enough, the proof of Theorem 2 which follows is so similar to the preceding one, that we do not need to give it separately.

Like in the case of  $K_C$ ,  $C \cap k_C$  may be any cardinal number between 2 and  $c$ .

**THEOREM 2.** *For most curves  $C \in \mathcal{C}$ ,  $\text{card}(C \cap k_C) = 3$ .*

The above results extend to higher dimensions. Since no technical difficulties appear in connection with increased dimension, we choose to present the planar case as a typical one.

Let  $\mathfrak{S}^d$  be the space of all ( $d$ -dimensional) closed convex surfaces  $S$  in  $\mathbf{R}^{d+1}$ . Let  $K_S$  and  $k_S$  be the circumscribed and an inscribed hypersphere of  $S \in \mathfrak{S}^d$ .

**THEOREM 3.** *For most surfaces  $S \in \mathfrak{S}^d$ ,*

$$\text{card}(S \cap K_S) = \text{card}(S \cap k_S) = d + 2.$$

The proof parallels that of Theorems 1 and 2.

#### REFERENCES

1. H. Busemann, *Convex surfaces*, Interscience, New York, 1958.
2. P. Gruber, *Die meisten konvexen Körper sind glatt, aber nicht zu glatt*, Math. Ann. **229** (1977), 259–266.
3. V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann. **139** (1959), 51–63.
4. R. Schneider, *On the curvatures of convex bodies*, Math. Ann. **240** (1979), 177–181.
5. T. Zamfirescu, *The curvature of most convex surfaces vanishes almost everywhere* (to appear).

ABTEILUNG MATHEMATIK, UNIVERSITÄT DORTMUND, 46 DORTMUND, FEDERAL REPUBLIC OF GERMANY