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# INSCRIBED AND CIRCUMSCRIBED CIRCLES TO CONVEX CURVES

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ABSTRACT. A convex planar curve may have 2, 3, . . . , $\mathfrak{c}$ contact points with its inscribed or circumscribed circle. One of these numbers appears in most cases: 3.

Let $\mathcal{C}$ be the space of all closed convex curves in the plane (see [**1**, p. 3] for a precise definition of a closed convex surface, in particular curve). Several pathological properties of most curves in $\mathcal{C}$ (in the sense of Baire categories) are described in [**4**] and [**5**]. We shall show here that most curves in $\mathcal{C}$ have the (expected?) number, 3, of contact points with their inscribed and circumscribed circles. It seems that mainly local properties may be pathological for most curves in $\mathcal{C}$.

We say that *most* elements of a space of second Baire category have a certain property if those elements which do not have it form a set of first Baire category. Now, $\mathcal{C}$ is of second Baire category (see for instance [**5**]) if we endow it with the Hausdorff metric, so it makes sense to speak about most curves in $\mathcal{C}$.

Let $C \in \mathcal{C}$, $D$ be the convex domain with boundary $C$, $K_C$ the *circumscribed circle* of $C$, i.e. the smallest circle surrounding $D$, and $k_C$ an *inscribed circle* of $C$, i.e. a largest circle included in $D \cup C$. The circle $k_C$ is not unique only if $C$ contains parallel segments. Since most curves in $\mathcal{C}$ are strictly convex (see [**3**] or [**2**]), they admit a unique inscribed circle.

Clearly, $\operatorname{card}(C \cap K_C)$ may be any cardinal number between 2 and $\mathfrak{c}$. We prove

THEOREM 1. *For most curves* $C \in \mathcal{C}$, $\operatorname{card}(C \cap K_C) = 3$.

We will use the following elementary Lemma, that we give without proof.

LEMMA. *Let $P$ be a polygon such that $P \cap K_P$ consists of precisely three points $x_1$, $x_2$, $x_3$ determining an acute triangle. Let $N_1$, $N_2$, $N_3$ be neighborhoods of $x_1$, $x_2$, $x_3$. Then there is a neighborhood $\mathfrak{N}$ of $P$ in $\mathcal{C}$ such that, for each $C \in \mathfrak{N}$,*

$$C \cap K_C \cap N_i \neq \varnothing \qquad (i = 1, 2, 3)$$

*and*

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3.$$

PROOF OF THEOREM 1. We first show that the set $\mathcal{C}_2$ of all curves in $\mathcal{C}$ satisfying $\operatorname{card}(C \cap K_C) = 2$ is nowhere dense in $\mathcal{C}$.

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Let $\mathcal{O}$ be an open set in $\mathcal{C}$. We choose a polygon $P$ in $\mathcal{O}$.

If $\operatorname{card}(P \cap K_P) = 2$, let $x$ and $y$ be the vertices of $P$ lying on $K_P$. Let $x_1$, $x_2$ be two points such that the segment $x_1x_2$ contains $x$ and is orthogonal on $xy$. Let $K'$ be the circle through $x_1, x_2, y$. The boundary $P'$ of the convex hull of $P \cup \{x_1, x_2\}$ has $K'$ as circumscribed circle. If $x_1$ and $x_2$ are close enough to $x$, $P'$ still lies in $\mathcal{O}$. Obviously

$$P' \cap K' = \{x_1, x_2, y\}.$$

If $\operatorname{card}(P \cap K_P) \geqslant 3$, there are three points $x_1$, $x_2$, $x_3$ in $P \cap K_P$ determining a triangle with all angles of measure at most $\pi/2$. By gently cutting all the other vertices of $P$ and slightly moving $x_1$ if necessary, we obtain a polygon $P'$ still belonging to $\mathcal{O}$ such that $P' \cap K_{P'}$ is the vertex-set of an acute triangle. Now, by the Lemma, for a neighborhood $\mathfrak{N}$ of $P'$ in $\mathcal{C}$, each curve $C \in \mathfrak{N}$ meets $K_C$ in at least three points. Thus

$$\mathcal{O} \cap \mathfrak{N} \cap \mathcal{C}_2 = \varnothing,$$

which proves that $\mathcal{C}_2$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{C}_{(n)}$ be the set of all curves $C$ in $\mathcal{C}$ such that

(i) $\operatorname{card}(C \cap K_C) \geqslant 4$, and

(ii) there exist four points $x_1$, $x_2$, $x_3$, $x_4 \in C \cap K_C$ such that the side-lengths of the convex quadrangle with vertices $x_1$, $x_2$, $x_3$, $x_4$ are at least $n^{-1}$ ($n \in \mathbf{N}$).

We show that $\mathcal{C}_{(n)}$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{O}$ be an open set in $\mathcal{C}$. We choose like before a polygon $P'$ in $\mathcal{O}$ such that $P' \cap K_{P'}$ is the vertex-set of an acute triangle. Now let $N_i$ be a disk of centre $x_i$ and radius less than $(2n)^{-1}$. By the Lemma, there is a neighborhood $\mathfrak{N}$ of $P'$ such that, for each curve $C \in \mathfrak{N}$,

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3,$$

and therefore we cannot find 4 points in $C \cap K_C$ determining a convex quadrangle with side-lengths at least $n^{-1}$. Thus

$$\mathcal{O} \cap \mathfrak{N} \cap \mathcal{C}_{(n)} = \varnothing$$

and $\mathcal{C}_{(n)}$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{C}_3$ be the set of all curves $C \in \mathcal{C}$ verifying $\operatorname{card}(C \cap K_C) = 3$. Every curve of $\mathcal{C}$ not belonging to $\mathcal{C}_2$ or $\mathcal{C}_3$ must be in $\mathcal{C}_{(n)}$ for some $n \in \mathbf{N}$. Thus

$$\mathcal{C} - \mathcal{C}_3 = \mathcal{C}_2 \cup \bigcup_{n=1}^{\infty} \mathcal{C}_{(n)},$$

where $\mathcal{C}_2$ and $\mathcal{C}_{(n)}$ ($n = 1, 2, 3, \ldots$) are nowhere dense; therefore $\mathcal{C} - \mathcal{C}_3$ is of first Baire category, which proves the theorem.

Surprisingly enough, the proof of Theorem 2 which follows is so similar to the preceding one, that we do not need to give it separately.

Like in the case of $K_C$, $C \cap k_C$ may be any cardinal number between 2 and $\mathfrak{c}$.

THEOREM 2. *For most curves* $C \in \mathcal{C}$, $\operatorname{card}(C \cap k_C) = 3$.

The above results extend to higher dimensions. Since no technical difficulties appear in connection with increased dimension, we choose to present the planar case as a typical one.

Let $\mathbb{S}^d$ be the space of all ($d$-dimensional) closed convex surfaces $S$ in $\mathbf{R}^{d+1}$. Let $K_S$ and $k_S$ be the circumscribed and an inscribed hypersphere of $S \in \mathbb{S}^d$.

THEOREM 3. *For most surfaces* $S \in \mathbb{S}^d$,

$$\operatorname{card}(S \cap K_S) = \operatorname{card}(S \cap k_S) = d + 2.$$

The proof parallels that of Theorems 1 and 2.

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