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ESTRATTO

BIHOMOGENEOUSLY TRACEABLE ORIENTED GRAPHS

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Summary: *The main result of the paper is the construction of oriented graphs, such that at each point a hamiltonian path starts and another one ends. Besides an infinite sequence of such graphs, three special examples are given: the first is arc-minimal (has the smallest possible number of arcs) and has 7 vertices the second is planar and has 8 vertices, the third is both arc-minimal and planar and has 9 vertices.*

1. Results.

Most graphs considered in this paper will be oriented and without opposite arcs. Hence we use the word *graph* in the above sense and the word *digraph* in case opposite arcs are allowed. Z. Skupień [3] introduced the notion of homogeneous traceability. A graph is said to be *homogeneously traceable* if for each point x there exists a hamiltonian path starting at x. Of course, each hamiltonian graph is homogeneously traceable. Is conversely each homogeneously traceable graph hamiltonian? This question was answered in the negative by J.-C. Bermond, J.M.S. Simões-Pereira and C.M. Zamfirescu [2]. Moreover, they anounced to have proved that there exist homogeneously traceable nonhamiltonian graphs with n points, if and only if $n \geqslant 7$. G. Chartrand, R. Gould and S. Kapoor [4] studied the nonoriented case. In this paper we consider the following sharper notion.

Definition. *A graph is said to be bihomogeneously traceable if for each point x there exists a hamiltonian path starting at x and another one ending at x.*

Obviously, each bihomogeneously traceable graph is homogeneously traceable (but not conversely) and each hamiltonian graph is bihomogeneously

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traceable. Do there exist bihomogeneously traceable, nonhamiltonian graphs? The examples of homogeneously traceable nonhamiltonian graphs mentioned above are not suitable to provide an answer. In 2. we prove

Theorem 1. *For each $n \geqslant 7$ there exists a bihomogeneously traceable nonhamiltonian graph with n points.*

A homogeneously traceable nonhamiltonian graph with n points and precisely $2n$ arcs will be called *arc-minimal*. There exists an arc-minimal planar homogeneously traceable nonhamiltonian graph with 7 points. What can be said about bihomogeneously traceable graphs?

Before answering the preceding question, let $H(G)$ be the digraph having as point-set that of G and arcs (a, b) if and only if there is a hamiltonian path in G from a to b. Clearly G is homogeneously traceable if and only if $d_+ > 0$ everywhere in $H(G)$ (where $d_+(v)$ and $d_-(v)$ are the numbers of arcs leaving and entering the point v, respectively) and G is bihomogeneously traceable precisely when $d_+ d_- > 0$ everywhere in $H(G)$.

We get a special kind of bihomogeneously traceable graphs G on n points, if there exists a family $\mathscr{H}$ of n hamiltonian paths of G such that at each point of G a path of $\mathscr{H}$ starts and another path of $\mathscr{H}$ ends. This corresponds to the property that $H(G)$ is spanned by a union of disjoint circuits. It is obvious that we have a still more particular case if $H(G)$ is hamiltonian. We shall prove:

Theorem 2. *There exists an arc-minimal bihomogeneously traceable nonhamiltonian graph G on 7 points, with $H(G)$ hamiltonian.*

Theorems 1 and 2 also give partial answers to a question at the end of [2].

Theorem 3. *There exists a planar bihomogeneously traceable nonhamiltonian graph G on 8 points, with $H(G)$ hamiltonian, and there exists no planar bihomogeneously traceable nonhamiltonina graph on less that 8 points.*

Theorem 4. *There exists a planar arc-minimal bihomogeneously traceable nonhamiltonian graph G on 9 points and there exists no smaller such graph.*

2. Proofs.

Proof of Theorems 1 and 2. Let G' be the digraph of Figure 1. We show that G' is not hamiltonian. Suppose C is a hamiltonian circuit of G'. We write

$a \longrightarrow b$ whenever b follows a on C.

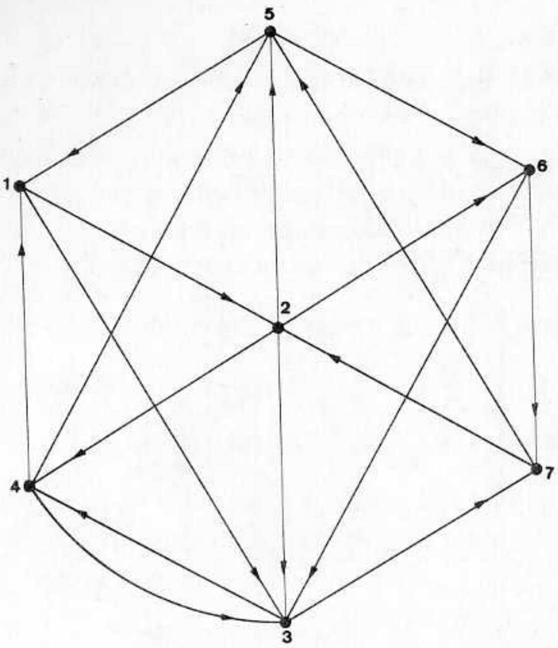

Fig. 1

Case 1: $5 \longrightarrow 1$.

In this case either $1 \longrightarrow 2$ or $1 \longrightarrow 3$. If $1 \longrightarrow 2$ then $2 \longrightarrow 6$, otherwise 6 would be lost, and then $6 \longrightarrow 3 \longrightarrow 4$ or $6 \longrightarrow 3 \longrightarrow 7$ or directly $6 \longrightarrow 7$ and the path ends each time too early. If $1 \longrightarrow 3$ then either C ends in 4 or $3 \longrightarrow 7 \longrightarrow 2$ and C ends in 4 or in 6.

Case 2: $5 \longrightarrow 6$.

In this case either $6 \longrightarrow 3$ or $6 \longrightarrow 7$. If $6 \longrightarrow 3$ then $3 \longrightarrow 4 \longrightarrow 1 \longrightarrow 2$ or $3 \longrightarrow 7 \longrightarrow 2 \longrightarrow 4 \longrightarrow 1$ and C would not be closed. If $6 \longrightarrow 7$, then necessarily $7 \longrightarrow 2$ and either $2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ or $2 \longrightarrow 4 \longrightarrow 1 \longrightarrow 3$, both just hamiltonian paths, or $2 \longrightarrow 4 \longrightarrow 3$, which is even shorter.

The diagraph G' minus the arcs 23, 25, 43 results in a graph G. Clearly also G is nonhamiltonian. The following paths are hamiltonian in G:

1372456,
2675134,
3451267,
4126375,
5637241,

6724513,
7563412.

Thus 16372451 is a hamiltonian circuit of $H(G)$. This proves Theorem 2.

To prove Theorem 1 it remains to construct for each $n \geqslant 8$ a bihomogeneously traceable nonhamiltonian graph with n points.

The digraph G' contains the graph of Figure 2a. By replacing this graph in G' by the graph of Figure 2b, we obtain a digraph G'' with the required

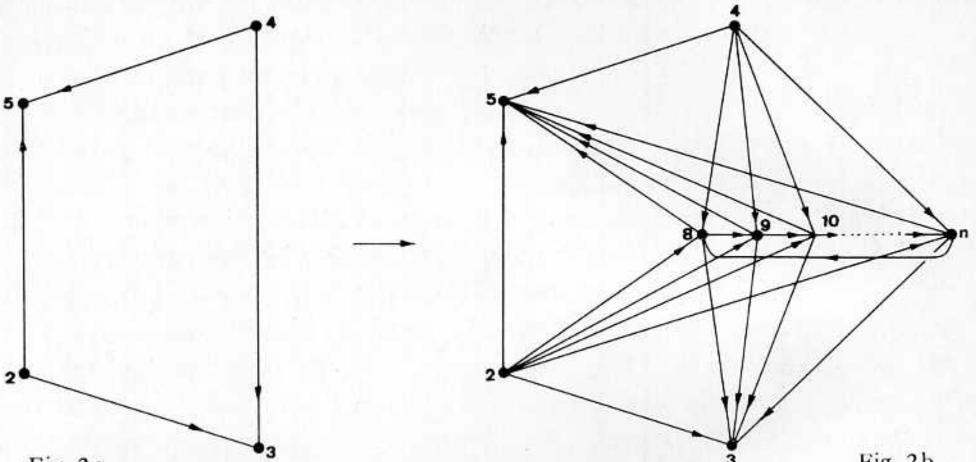

Fig. 2a   Fig. 2b

properties. Indeed, suppose G'' is hamiltonian. Then the hamiltonian circuit contains $2a3$ and $4b5$, or $2c5$ and $4d3$, where a and b, respectively c and d are complementary subpaths (one of them possibly empty) of the circuit $89 \ldots n8$. By replacing $2a3$, $4b5$, $2c5$, $4d3$ with the arcs 23, 45, 25, 43 we get a hamiltonian circuit of G', which is impossible. For each point x of $G' \cap G''$ there exists hamiltonian path of G' which starts in x and contains one of the arcs 23, 45 or ends in 4. There also exists for each point x in $G' \cap G''$ a hamiltonian path of G' which contains one of the arcs 23, 45 or begins in 3:

1372456,
2675134,
3451267,
4512637,
5672341,
6724513,
7263451,

1345672,
3412675,
6751234.

By replacing the arcs 23 and 45 with $289 \ldots n3$ and $489 \ldots n5$ and by extending 2675134 and 3412675 to $267513489 \ldots n$ and $89 \ldots n3412675$ respectively, we get hamiltonian paths of G'' starting or ending at the same points chosen in $G' \cap G''$. A hamiltonian path of G'' starting in m with $8 \leqslant m \leqslant n$ is

$$m(m+1) \ldots n89 \ldots (m-1)3412675;$$

a hamiltonian path of G'' ending in m is

$$2675134(m+1)(m+2) \ldots n89 \ldots (m-1)m.$$

The proof would be finished if G'' were a graph. But this is the case if and only if $n \neq 9$. For $n = 9$, we omit the arc 98. We only have to find a hamiltonian path beginning in 9 and one ending in 8:

934851267,
675129348,

and the proof of Theorem 1 is achieved.

Proof of Theorem 3. Let G be the graph of Figure 3a. We show that G is

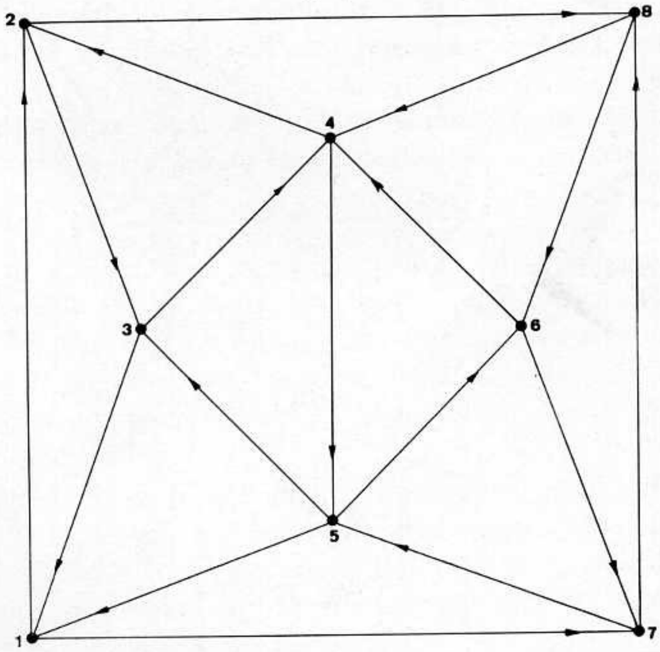

Fig. 3a

not hamiltonian. Suppose C is a hamiltonian circuit of G. We use again the notation $a \longrightarrow b$ from the preceding proof.

Case 1: $4 \longrightarrow 2$.

It follows either $2 \longrightarrow 3$ or $2 \longrightarrow 8$. If $2 \longrightarrow 3$ then $3 \longrightarrow 1 \longrightarrow 7$ and $7 \longrightarrow 5 \longrightarrow 6$ or $7 \longrightarrow 8 \longrightarrow 6$, and C ends there. If $2 \longrightarrow 8$ then $8 \longrightarrow 6 \longrightarrow 7 \longrightarrow 5$. Then either $5 \longrightarrow 1$ and C misses 3, or $5 \longrightarrow 3 \longrightarrow 1$ and C is not closed.

Case 2: $4 \longrightarrow 5$.

Suppose $5 \longrightarrow 1$. Then either $1 \longrightarrow 2$ and C, if it does not end in 3, goes $2 \longrightarrow 8 \longrightarrow 6 \longrightarrow 7$ and ends there, or $1 \longrightarrow 7 \longrightarrow 8 \longrightarrow 6$ and C ends again too early.

Suppose $5 \longrightarrow 3$. Then $3 \longrightarrow 1$ and $1 \longrightarrow 2 \longrightarrow 8 \longrightarrow 6 \longrightarrow 7$ and C is not closed or $1 \longrightarrow 7 \longrightarrow 8 \longrightarrow 6$, C missing 2.

Suppose finally $5 \longrightarrow 6$. Then $6 \longrightarrow 7 \longrightarrow 8$ and C is not a hamiltonian circuit.

To show that $H(G)$ is hamiltonian (see Figure 3b) it suffices to notice that G has the following hamiltonian paths:

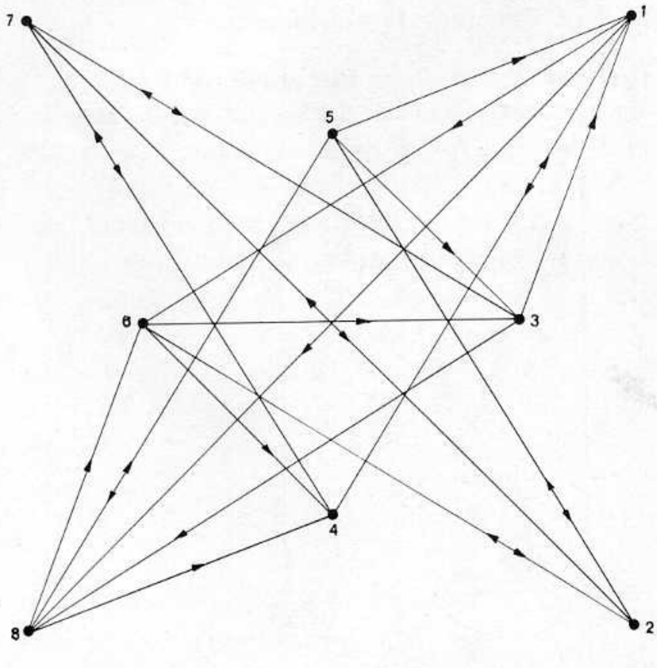

Fig. 3b

12345678,
84231756,
67845312,
23178645,
51786423,
31284567,
75312864,
42867531.

It remains to show that no planar bihomogeneously traceable nonhamiltonian graph on 7 points exists. Suppose F is a planar bihomogeneously traceable nonhamiltonian graph on 7 points. Then the degree $d = d_+ + d_-$ is at least 4 at each point of F, because in bihomogeneously traceable nonhamiltonian graphs $d_+ \geqslant 2$ and $d_- \geqslant 2$. Also,we remarked that $m \geqslant 14$, m being the number of arcs in F. On the other hand, if r is the number of regions, $3r \leqslant 2m$. Combining this with Euler's formula we get $m \leqslant 15$.

Case 1: $m = 14$.

In this case $d(v) = 4$ for all points v of F and F has exactly one arc less than a triangulation with 7 vertices. Thus F has only triangles as regions exept one quadrangle Q. Let v be a point of F which is not a point of Q. Then v has 4 neighbours v_1, v_2, v_3, v_4 and vv_1v_2, vv_2v_3, vv_3v_4, vv_4v_1 are regions. Let w_1 and w_2 be the remaining two points of F. Each of them must be adjacent with three of the points v_1, v_2, v_3, v_4. Thus, two of these four points must be adjacent to both points w_1 and w_2, which contradicts $d = 4$.

This proves that *there are no arc-minimal planar bihomogeneously traceable nonhamiltonian graphs on 7 points.*

Case 2: $m = 15$.

If there is a point v of degree 6, then all other points have degree 4 and v has 6 neighbours v_1, v_2, v_3, v_4, v_5, v_6 lying in this order around v. Without loss of generality we may assume that v_1 is adjacent to v_2 or v_3. Now, v_2 must be adjacent to v_4, v_5 or v_6 which is impossible. In conclusion there is no point of degree 6, but two points v, w have degree 5 and all other points have degree 4.

If v, w are adjacent, then F minus the arc joining them is in the situation treated by Case 1. Hence v and w are not adjacent and F looks like one of the four graphs in Figure 4. More precisely, F is isomorphic to one of those graphs after a suitable orientation of the nonoriented arcs in Figure 4 and by

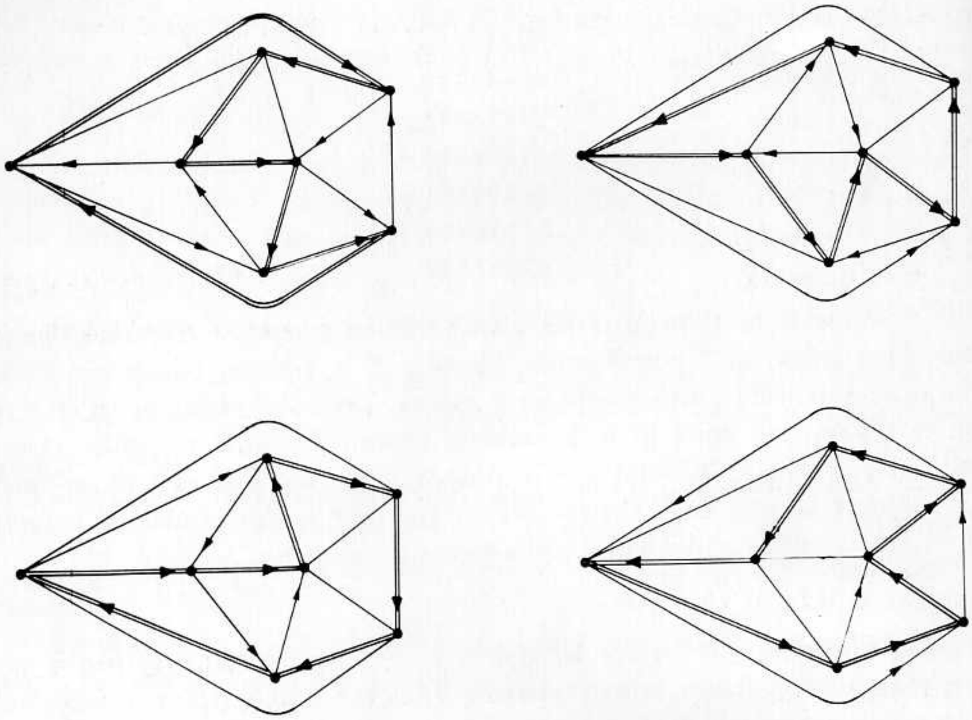

Fig. 4

inversing if necessary all arc orientations. In each case it is shown that a hamiltonian circuit exists (see Figure 4). Thus a contradiction is obtained and Theorem 3 is proved.

Proof of Theorem 4. The graph G is shown in Figure 5. The proof that G

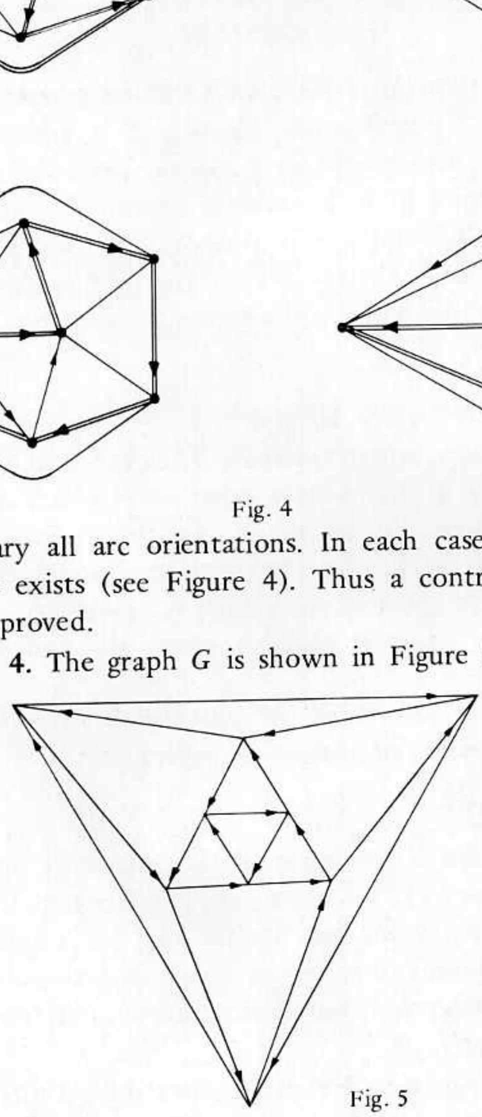

Fig. 5

has the stated properties is similar to the preceding one, so we omit it. We used a computer to verify that no graph with 8 points has the above properties: the computer found all arc-minimal bihomogeneously traceable nonhamiltonian graphs on 8 points and it happens that none of them is planar.

3. Open Problems.

Several questions concerning bihomogeneously traceable nonhamiltonian graphs remain unanswered. Among them the following are in our opinion the most interesting:

1. Do there exist infinitely many arc-minimal bihomogeneously traceable nonhamiltonian graphs?
2. Do there exist infinitely many planar bihomogeneously traceable nonhamiltonian graphs?
3. Do there exist infinitely many nonhamiltonian graphs G with $H(G)$ hamiltonian?

For each problem, in case of a negative answer it would be interesting to determine those graphs enjoying the demanded properties.

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Note added in proof. Z. Skupień kindly sent us his interesting paper "On homogeneously traceable nonhamiltonian digraphs and oriented graphs", which is appearing in The Proceedings of the 4th International Conference on the Theory and Applications of Graphs (Kalamazoo, 1980). He also finds our graphs G in Figures 1 and 5. Moreover he answers affirmatively our Problem 1.

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