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INTERSECTING DIAMETERS IN CONVEX BODIES

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1. Introduction

We consider here spreads in the sense of Grünbaum (see [3, 4]). Let C be a closed Jordan curve in the plane, D_C the simply connected domain bounded by C, and $x \mapsto -x$ a continuous fixpoint free involution on C. A family $\mathscr{L}$ of Jordan arcs (called *curves*) is said to be a *spread* provided:

(i) for each $x \in C$ there is one curve $L(x) \in \mathscr{L}$ joining x with $-x$ $(L(x) = L(-x))$;

(ii) $\operatorname{inn} \Gamma \subset D_C$ for each $\Gamma \in \mathscr{L}$ ($\operatorname{inn} \Gamma$ means Γ minus its endpoints);

(iii) $\Gamma_1 \cap \Gamma_2$ is a single point for each pair of distinct curves $\Gamma_1, \Gamma_2 \in \mathscr{L}$;

(iv) $L: C \to \mathscr{L}$ is continuous, the topology of $\mathscr{L}$ being that induced by the Hausdorff metric in the space of compact plane sets.

Let cA be the family of all connected conponents of A, and let

$$M_\alpha = \{x \in D_C : \operatorname{card}\{\Gamma \in \mathscr{L} : x \in \Gamma\} \geq \alpha\},$$

$$P_\alpha = \{x \in D_C : \operatorname{card} c\{\Gamma \in \mathscr{L} : x \in \Gamma\} \geq \alpha\},$$

$$T_\alpha = \{x \in D_C : \operatorname{card}\{\Gamma \in \mathscr{L} : x \in \Gamma\} = \alpha\}.$$

Watson [8] proved that there exist spreads for which $M_{\aleph_0} = D_C$, whence $T_1 = \emptyset$. However, in [10] it is proved that, for spreads satisfying certain additional continuity conditions, the boundary of M_2 is a closed Jordan curve different from C. Then $\operatorname{int} T_1 \neq \emptyset$. Such a spread is, for example, that of area bisectors of a planar convex body. Also, it is known that for every *straight* spread (i.e. a spread the curves of which are line-segments) $T_1 \neq \emptyset$ [8, 12]. What, in general, does T_1 look like for straight spreads? Can M_2 be dense in D_C? Can even $M_{\aleph_0}$ be dense in D_C? The last two questions will be answered in this paper.

The word *most* will always be used in the sense of *those in a residual set*, or *all, except those in a set of first Baire category.*

The following result on general spreads will be used below.

Proposition 1. *If $P_{\aleph_0}$ is dense in D_C, then $M_{\aleph_0}$ is residual in D_C.*

Proof. Let $O \subset D_C$ be an open set and $p \in O \cap P_{\aleph_0}$. Since $p \in P_{2n}$ $(n \in \mathbb{N})$, by

Corollary 4 in [11] (see also Theorem 2 of [9]), $p \in \overline{\operatorname{int} M_{2n+1}}$, which shows that there is an open set

$$O' \subset M_{2n+1} \cap O \subset M_n \cap O.$$

Thus, $\complement M_n$ is nowhere dense in D_C and

$$\complement M_{\aleph_0} = \complement \bigcap_n M_n = \bigcup_n \complement M_n$$

is of first Baire category. The proof is complete.

If C is a planar, smooth (differentiable), strictly convex curve, then the family of all *diameters* (i.e. chords admitting parallel tangent lines at their endpoints) of C constitutes a spread. In the rest of the paper we shall study (and, without contrary mention, always consider) spreads of diameters. The main attention will be focused on the strange properties that $M_{\aleph_0}$ may have.

Consider such a curve C and let $x, y \in C$, $y \neq \pm x$. Let $d(xy, z)$ denote the distance from $L(x) \cap L(y)$ to z and put

$$\gamma_i^-(x) = \liminf \frac{d(xy, x)}{d(xy, -x)},$$

where y converges to x from the left. $\gamma_s^-(x)$, $\gamma_i^+(x)$, and $\gamma_s^+(x)$ are defined analogously. The following technical lemma, proved elsewhere, will be useful.

Lemma 1 [14]. *Suppose C includes an arc B of a circle and let $x \in C$ be such that $-x \in \operatorname{inn} B$. Then*

$$\gamma_i^{\pm}(x) \leq \frac{\rho_i^{\pm}(x)}{\rho^+(-x)}; \qquad \gamma_s^{\pm}(x) \geq \frac{\rho_s^{\pm}(x)}{\rho^+(-x)}.$$

(For a definition of the left radius of curvature $\rho^-(x)$ and left lower and upper radii of curvature $\rho_i^-(x)$ and $\rho_s^-(x)$ — and analogously of the right ones — see for instance [1].)

We denote by $\mathscr{C}$ the space of all smooth, strictly convex curves in the plane. By results proved independently by Klee [6] and Gruber [2], $\mathscr{C}$ is residual in the Baire space of all convex curves of the plane, the topology being again induced by the Hausdorff metric.

The rest of the paper is organized as follows. In the next section we present an example for which $M_{\aleph_0}$ is residual and null-swept (the definition follows in Section 2). Then we investigate, for most convex curves (and always the spread of diameters), the set $M_{\aleph_0}$ from Baire categories' point of view. Finally, we prove some connectivity properties of $M_{\aleph_0}$ for most convex curves.

2. $M_{\aleph_0}$ can be residual and null-swept

Let μ be the Lebesgue measure in the plane and λ the (one-dimensional) Hausdorff measure on convex curves.

For $C \in \mathscr{C}$, we say that a set $V \subset D_C$ is *null-swept* if there is $H \subset C$ with $\lambda(H) = 0$ such that $V \subset \bigcup_{x \in H} L(x)$.

Following Hammer and Sobczyk [5], a *turning point* is a limit point of $L(x) \cap L(y)$ for $y \to x$. Let U be the set of all turning points in D_C. It is easy to see that U may consist of finitely many points, but may also not be null-swept (see, for instance, the outwardly simple line family constructed at the end of [5]). We have $M_{\aleph_0} \subset U$ [5].

Theorem 1. *There exist curves in $\mathscr{C}$ such that $M_{\aleph_0}$ and U are simultaneously residual and null-swept.*

Proof. By Theorem 2 in [13] and the Theorem in [7], most convex curves $B \in \mathscr{C}$ have the property that:

(i) at each point x of a set $F \subset B$ with $\lambda(B - F) = 0$ of curvature exists and vanishes; and

(ii) at each point y of a dense set $E \subset B$ the lower and upper radii of curvature (from left) satisfy

$$\rho_i^-(y) = 0 \quad \text{and} \quad \rho_s^-(y) = \infty.$$

Take such a curve B, an arc $A \subset B$ at the endpoints of which the tangent lines are parallel, and a semicircle S such that $A \cup S$ is a convex curve C. First, we show that, in D_C, $M_{\aleph_0}$ is residual. Since, for every x in inn $A \cap E$, $\gamma_i(x) = 0$ and $\gamma_s^-(x) = \infty$, it follows from Lemma 1 that all points of $L(x)$ are limit points of $L(y) \cap L(x)$ for $y \to x$ from the left. Thus, inn $L(x) \subset P_{\aleph_0}$, whence $P_{\aleph_0}$ is dense in D_C. By Proposition 1, $M_{\aleph_0}$ is residual in D_C.

Secondly, we prove that

$$U \subset \bigcup_{x \in H} L(x),$$

where $H = A - F$. Let $p \in U$. Clearly, p is a limit point of $L(y) \cap L(x)$ for some point $x \in A$ and $y \to x$ from left or from right, say from left. Then $x \notin F$, since for each $z \in F$ the intersection $L(u) \cap L(z)$ converges to $-z$ when $u \to z$. Hence, $p \in \bigcup_{x \in H} L(x)$ and the theorem is proved.

Hammer and Sobczyk [5] have shown that for every straight spread $\mu(M_{\aleph_0}) = 0$. Also, simple examples show that $M_{\aleph_0}$, even M_4, may be empty, but the preceding example showed that $M_{\aleph_0}$ can also be residual. What does $M_{\aleph_0}$ look like for most convex curves? The answer is given in the next section.

3. In most cases $M_{\aleph_0}$ is residual

Let $\mathscr{C}_n$ be the set of all curves C in $\mathscr{C}$ having an arc A of length n^{-1} such that, for each $x \in \operatorname{inn} A$, there is an endpoint e of $L(x)$ and a component A^* of $A-\{x\}$ such that, for every $y \in A^*$, $d(xy, e) \geqslant n^{-1}$ $(n \in \mathbb{N})$.

Lemma 2. *$\mathscr{C}_n$ is closed in $\mathscr{C}$.*

Proof. Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of curves in $\mathscr{C}_n$ converging to a curve $C \in \mathscr{C}$. By choosing, if necessary, a subsequence, we arrange that the corresponding sequence of arcs $\{A_i\}_{i=1}^{\infty}$ converges to some arc A of length n^{-1} on C. Let x be a point of $\operatorname{inn} A$. We can choose $x_i \in A_i$ such that $x_i \to x$. The corresponding endpoints e_i coincide with x_i or with $-x_i$ for infinitely many indices i, say with x_i. Then we choose $e = x$. In the same way we may suppose that the sequence of the corresponding components A_i^* converges to some component A^* of $A-\{x\}$. Suppose now there is a point $y \in A^*$ with $d(xy, x) < n^{-1}$. We choose $y_i \in A_i^*$ such that $y_i \to y$. For i large enough, the Hausdorff distance from $L(x_i)$ to $L(x)$ and from $L(y_i)$ to $L(y)$ are so small, that $d(x_iy_i, x_i) < n^{-1}$, which is a contradiction.

Lemma 3. *$\mathscr{C}_n$ is nowhere dense in $\mathscr{C}$.*

Proof. It is well known that an arbitrary curve $C \in \mathscr{C}$ can be approximated as well as we like by a polygon. We can replace the sides of the polygon by arcs of very large circles and the vertices of the polygon by arcs of very small circles, the resulting curve, $C' \in \mathscr{C}$, still remaining near enough to C. On the other hand, C can be approximated well enough by a curve $C'' \in \mathscr{C}$ with $\rho_i^{\pm}(x) = 0$ and $\rho_s^{\pm}(x) = \infty$ on a dense set E of points $x \in C''$ [7]. It is an easy matter now to construct a smooth convex curve C^* such that on one side of a certain diameter of C^*, C^* coincides with half a curve C' and on the other side with half a curve C''. Since

$$\rho_i^{\mp}(x) = 0, \qquad \rho_s^{\pm}(x) = \infty,$$

for every point $x \in C^* \cap E$, by Lemma 1 all points of $L(x)$ are limit points of $L(y) \cap L(x)$ for $y \to x$ from left as well as from right. It follows that $C^* \notin \mathscr{C}_n$. Hence, $\mathscr{C} - \mathscr{C}_n$ is dense in $\mathscr{C}$ and, by Lemma 2, $\mathscr{C}_n$ is nowhere dense in $\mathscr{C}$.

Proposition 2. *For most curves $C \in \mathscr{C}$ the following holds: For every arc A on C and every number $\varepsilon > 0$, there exists a point $x \in \operatorname{inn} A$ such that, for any component A^* of $A-\{x\}$, there are $y, y' \in A^*$ verifying $d(xy, x) < \varepsilon$ and $d(xy', -x) < \varepsilon$.*

Proof. Let $\mathscr{C}^*$ be the set of those curves of $\mathscr{C}$ enjoying the property of the statement. It is easily checked that

$$\mathscr{C} - \mathscr{C}^* = \bigcup_{n=1} \mathscr{C}_n ;$$

hence $\mathscr{C} - \mathscr{C}^*$ is, by Lemma 3, of first Baire category, which proves Proposition 2.

Theorem 2. *For most convex curves C, most points of D_C belong to infinitely many diameters.*

Proof. Let C be a curve enjoying the property of Proposition 2. Put

$$E = \{x \in C\colon \operatorname{inn} L(x) \subset P_{\aleph_0}\}.$$

We show that E is dense on C. Let A_0 be an arc on C; we find $x_1 \in \operatorname{inn} A_0$ and $y_1, y_1' \in \operatorname{inn} A_0$ on the same side of x_1 such that $d(x_1y_1, x_1) < 1$ and $d(x,y_1', -x_1) < 1$. We construct the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{y_n'\}_{n=1}^{\infty}$ inductively as follows. By continuity, there is an arc $A_n \subset A_{n-1}$ containing x_n in its interior, such that for all $x \in A_n$, $d(xy_n, x) < n^{-1}$ and $d(xy_n', -x) < n^{-1}$. Again, we can find a point $x_{n+1} \in A_n$ separating x_n from y_n and y_n', and two points $y_{n+1}, y_{n+1}' \in A_n$ separating x_{n+1} from y_n and y_n' such that $d(x_{n+1}y_{n+1}, x_{n+1}) < (n+1)^{-1}$ and $d(x_{n+1}y_{n+1}', -x_{n+1}) < (n+1)^{-1}$. It is equally guaranteed that $d(x_{n+1}y_m, x_{n+1}) < m^{-1}$ and $d(x_{n+1}y_m', -x_{n+1}) < m^{-1}$ for all $m \leq n$.

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to some point $x \in A$ verifying:

$$d(xy_n, x) < n^{-1} \quad \text{and} \quad d(xy_n', -x) < n^{-1},$$

for every $n \in \mathbb{N}$. Also, y_n' lies between y_{n-1} and y_{n+1}, and y_n lies between y_{n-1}' and y_{n+1}'. Thus, it is clear that $\operatorname{inn} L(x) \subset P_{\aleph_0}$ and that E is dense on C. Then $P_{\aleph_0}$ is dense in D_C and the theorem follows from Proposition 1.

4. In most cases $M_{\aleph_0}$ is connected

Simple examples show that, for every cardinal number $\alpha > 3$, M_α may be disconnected, and this seems to be the rule. That — from the point of view of Baire categories — this is not the case, will prove the next result.

Theorem 3. *For most convex curves, the components of T_1 are line-segments (without one or both endpoints), $M_2 - M_{\aleph_0}$ is totally disconnected, and M_α is connected for every $\alpha \leq \aleph_0$.*

Proof. Let $p \in T_1$ and $x \in C$ with $p \in L(x)$. Clearly, either $\operatorname{inn} xp$ or $\operatorname{inn} -xp$ lies in T_1. Say $\operatorname{inn} xp \subset T_1$. There is a maximal line-segment S ending in x such that $\operatorname{inn} S \subset T_1$. We show that $\operatorname{inn} S$ plus, possibly, the endpoint of S different from x is, for most $C \in \mathscr{C}$, the component K of T_1 containing p. Suppose $q \in K - S$. Since, for most convex curves, the set E considered in the proof of Theorem 2 is dense on C, there exists $y \in E$ such that $L(y)$ separates p from q. Since $\operatorname{inn} L(y) \subset M_{\aleph_0}$, $q \notin K$, a contradiction.

To see that $M_2 - M_{\aleph_0}$ is totally disconnected, let $s \in M_2 - M_{\aleph_0}$ and let B be a disk around s. We consider $x, y \in C$ such that $L(x) \cap L(y) = \{s\}$. We use again the above set E and see that there are four points, x', x'', y', and y'' in E, such that s lies within a quadrangle $Q \subset B$, with sides on $L(x')$, $L(x'')$, $L(y')$, and $L(y'')$. Since the sides of Q lie in $M_{\aleph_0}$, the component of $M_2 - M_{\aleph_0}$ containing s is a subset of B.

Finally, we show that M_α is connected for every $\alpha \leq \aleph_0$. Let

$$Z = \bigcup_{x \in E} \operatorname{inn} L(x).$$

Clearly, Z is arcwise connected (even an $L_2(\mathscr{L})$-set in the terminology of [9, Section 3]). Since Z is dense in D_C and $Z \subset M_{\aleph_0} \subset M_\alpha$, the assertion follows.

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