Arch. Math., Vol. 42, 151–156 (1984) 0003-889X/84/4202-0151 $ 2.70/0

Typical monotone continuous functions

By

Tudor Zamfirescu

It is usual to say that a "typical" element of some Baire space has a certain property if the set of those elements which do not enjoy that property is of first Baire category. Then we also say that "most" elements have the property.

A rather complete description of typical real continuous functions of one real variable is known (see for example Bruckner [2]). For instance, it is known that they have no finite unilateral derivative at any point (Banach [1], Mazurkiewicz [4]), have no infinite derivative at any point (Jarnik [3]), but have an infinite unilateral derivative at uncountably many points (Saks [6]). In this paper we find properties in the same spirit of typical monotone continuous functions.

Let $I = [0, 1]$. It is well-known that the space $\mathscr{C}$ of all continuous functions $f: I \to \mathbb{R}$, endowed with the usual distance

$$\sup_{x \in I} |f(x) - g(x)|$$

between $f, g \in \mathscr{C}$, is a Baire space. Also the subspace $\mathscr{M} \subset \mathscr{C}$ of all increasing functions in $\mathscr{C}$ is a Baire space.

For any function $f \in \mathscr{C}$, let f_i' be the lower, f_s' the upper, f_i^- the left lower, f_s^- the left upper, f_i^+ the right lower and f_s^+ the right upper Dini derivative of f.

We recall the following results.

Theorem A (Jarnik [3]). *For a typical function $f \in \mathscr{C}$, at each point $x \in I$,*

$$f_i'(x) = -\infty \quad \textit{and} \quad f_s'(x) = \infty.$$

Theorem B (Banach [1]). *For a typical function $f \in \mathscr{C}$, at each point $x \in (0, 1]$,*

$$f_i^-(x) = -\infty \quad \textit{or} \quad f_s^-(x) = \infty$$

and, at each point $x \in [0, 1)$,

$$f_i^+(x) = -\infty \quad \textit{or} \quad f_s^+(x) = \infty.$$

Theorem C [7]. *For a typical function $f \in \mathscr{M}$, at each point $x \in (0, 1]$,*

$$f_i^-(x) = 0 \quad \textit{or} \quad f_s^-(x) = \infty$$

and, at each point $x \in [0, 1)$,

$$f_i^+(x) = 0 \quad or \quad f_s^+(x) = \infty.$$

This theorem has the following immediate consequences.

Corollary A [7]. *For a typical function* $f \in \mathcal{M}$,

$$f' = 0 \quad a.e.$$

Corollary B. *For a typical function* $f \in \mathcal{M}$,

$$f'(x) = \infty$$

at densely, uncountably many points $x \in I$.

Proof. Since f is strictly increasing, there exists f^{-1} which, being monotone, is differentiable a.e. But f has in no point of I a finite derivative different from zero, by Theorem C. Hence $(f^{-1})' = 0$ a.e., whence $f'(x) = \infty$ at densely, uncountably many points $x \in I$.

Notice that Corollary B also follows from Corollary A, because each strictly increasing function $f \in \mathcal{M}$ with $f' = 0$ a.e. satisfies $f'(x) = \infty$ at densely, uncountably many points x (see the proof of Theorem 2 in [8]).

A function $f \in \mathcal{C}$ is called *nonangular* if

$$f_i^- \leqq f_s^+ \quad \text{and} \quad f_i^+ \leqq f_s^-.$$

The following result is known (Theorem 2.3 in [2]).

Theorem D. *A typical function in* $\mathcal{C}$ *is nonangular.*

We constatate that precisely the same is true for monotone continuous functions.

Theorem 1. *A typical function in* $\mathcal{M}$ *is nonangular.*

Since the proof parallels that of Theorem D, we omit it.

There is a far reaching analogy between the properties of typical continuous and those of typical monotone continuous functions. This is already evident by comparing Banach's Theorem B with Theorem C and Theorem D with Theorem 1. Corollary B may be considered as a monotone pendant of Saks' theorem. However, it seems that Corollary A has no corresponding result for continuous functions and Jarnik's Theorem A seems to have no proper monotone analog.

Theorem 2. *For a typical function* $f \in \mathcal{C}$,

$$f_i^-(x) = f_i^+(x) = -\infty$$

and

$$f_s^-(x) = f_s^+(x) = \infty$$

at most points $x \in I$.

Proof. Neugebauer [5] proved that, for any function $g \in \mathscr{C}$,

$$g_i^-(x) = g_i^+(x) \quad \text{and} \quad g_s^-(x) = g_s^+(x)$$

at most points $x \in I$. Now, Jarnik's Theorem A yields the theorem.

Since the above argument uses Jarnik's result, the question arises whether there is a monotone analog for Theorem 2. Our main result answers affirmatively this question.

Theorem 3. *For a typical function $f \in \mathscr{M}$,*

$$f_i^-(x) = f_i^+(x) = 0$$

and

$$f_s^-(x) = f_s^+(x) = \infty$$

at most points $x \in I$.

Proof. Let $f \in \mathscr{M}$ and

$$A = \{x \in (0,1) : f_i^+(x) = 0 \quad \text{and} \quad f_s^+(x) = \infty\}.$$

For $x \in (0,1)$, put

$$f_+(x) = \sup_{y > x} \frac{f(y) - f(x)}{y - x}; \quad f_-(x) = \inf_{y > x} \frac{f(y) - f(x)}{y - x}.$$

We see that

$$A = \{x \in (0,1) : f_-(x) = 0 \quad \text{and} \quad f_+(x) = \infty\}.$$

Let $\mathscr{F}$ be the family of all functions $f \in \mathscr{M}$ such that A is not residual. Consider $f \in \mathscr{F}$ and write

$$A_n = \{x \in (0,1) : f_-(x) < n^{-1} \quad \text{and} \quad f_+(x) > n\}.$$

Obviously $x_m \to x$ and $f_+(x_m) \leqq n$ imply $f_+(x) \leqq n$, and $x_m \to x$ and $f_-(x_m) \geqq n^{-1}$ imply $f_-(x) \geqq n^{-1}$. Therefore, for $x_m \to x$,

$$f_+(x_m) \leqq n \quad \text{or} \quad f_-(x_m) \geqq n^{-1} \quad (m \in \mathbb{N})$$

implies

$$f_+(x) \leqq n \quad \text{or} \quad f_-(x) \geqq n^{-1}.$$

Hence $I - A_n$ is closed. Since

$$A = \bigcap_{n=1}^{\infty} A_n$$

is not residual,

$$I - A = \bigcup_{n=1}^{\infty} (I - A_n)$$

is of second category, whence $I - A_n$ is not nowhere dense for some $n \in \mathbb{N}$. Thus, $I - A_n$ includes an interval.

Let $\mathscr{F}_{m,n}$ be the set of all functions $f \in \mathscr{F}$ such that $I - A_n$ includes, for some index n, an interval of length m^{-1} ($m, n \in \mathbb{N}$). One verifies by routine arguments that $\mathscr{F}_{m,n}$ is closed. Let us prove that $\mathscr{M} - \mathscr{F}_{m,n}$ is dense: One can approximate any function $f \in \mathscr{M}$ by piecewise linear, strictly increasing functions $g: I \to \mathbb{R}$. By replacing appropriately linear restrictions $g|_{[a,b]}$ of g (with $b - a < m^{-1}$) by increasing functions $h: [a, b] \to \mathbb{R}$, defined such that

$$h(a) = g(a), \quad h(b) = g(b),$$
$$h_i^+(a) = 0, \quad h_s^+(a) = \infty,$$

with $\max|h - g|([a, b])$ small enough, we find functions in $\mathscr{M} - \mathscr{F}_{m,n}$ approximating f. Now it is shown that $\mathscr{F}_{m,n}$ is nowhere dense and

$$\mathscr{F} = \bigcup_{m,n} \mathscr{F}_{m,n}$$

of first category.

Hence, for typical $f \in \mathscr{M}$, A is residual; analogously

$$B = \{x \in (0, 1): f_i^-(x) = 0 \quad \text{and} \quad f_s^-(x) = \infty\}$$

is residual too, which shows that $A \cap B$ is residual.

Hence, contrary to the measure-theoretical point of view (Lebesgue's theorem), we have the following

Corollary 1. *A typical function $f \in \mathscr{M}$ is not differentiable at most points of I.*

For the last two results we need the following simple

Lemma. *Let $-\infty \leqq \alpha < \beta \leqq \infty$. If $f \in \mathscr{C}$ and*

$$f_i^+(x) = \alpha, \quad f_s^+(x) = \beta$$

at a dense set of points x, then, for each $k \in (\alpha, \beta)$, there are two dense sets A_k and B_k such that

$$f_s^-(x) \leqq k \leqq f_i^+(x)$$

for $x \in A_k$ and

$$f_s^+(x) \leqq k \leqq f_i^-(x)$$

for $x \in B_k$.

Proof. We prove the existence of B_k. Let $z \in I$. There is a point $x \in [0, 1)$ as close to z as we want such that

$$f_i^+(x) = \alpha, \quad f_s^+(x) = \beta.$$

Let $\alpha < a < k < b < \beta$. We can choose x_1, x_2 arbitrarily close to x such that $x < x_1 < x_2$ and

$$\frac{f(x_1) - f(x)}{x_1 - x} \leqq a, \quad \frac{f(x_2) - f(x)}{x_2 - x} \geqq b.$$

Then there exists $x_3 \in (x_1, x_2)$ such that

$$\frac{f(x_3) - f(x)}{x_3 - x} = k.$$

Let y be an absolute maximum of $g|_{[x, x_3]}$, where

$$g(t) = f(t) - kt.$$

Obviously,

$$g_i^-(y) \geqq 0, \quad g_s^+(y) \leqq 0,$$

whence

$$f_i^-(y) \geqq k, \quad f_s^+(y) \leqq k.$$

The existence of A_k is verified similarly.

Theorem 4. *For a typical function $f \in \mathscr{C}$ and any $k \in \mathbb{R}$, there exist two dense sets A_k and B_k such that*

$$f_i^-(x) = -\infty, \quad f_s^-(x) = f_i^+(x) = k, \quad f_s^+(x) = \infty$$

for $x \in A_k$ and

$$f_i^+(x) = -\infty, \quad f_s^+(x) = f_i^-(x) = k, \quad f_s^-(x) = \infty$$

for $x \in B_k$.

Proof. By Theorem 2 and by the Lemma with $\alpha = -\infty$, $\beta = \infty$, for each $k \in \mathbb{R}$ there exist dense sets A_k, B_k such that

$$f_s^-(x) \leqq k \leqq f_i^+(x)$$

for $x \in A_k$ and

$$f_s^+(x) \leqq k \leqq f_i^-(x)$$

for $x \in B_k$. Now, using Theorem B and Theorem D, it is evident that the equalities of the theorem hold.

Theorem 5. *For a typical function $f \in \mathscr{M}$ and any $k > 0$, there exist two dense sets A_k and B_k such that*

$$f_i^-(x) = 0, \quad f_s^-(x) = f_i^+(x) = k, \quad f_s^+(x) = \infty$$

for $x \in A_k$ and

$$f_i^+(x) = 0, \quad f_s^+(x) = f_i^-(x) = k, \quad f_s^-(x) = \infty$$

for $x \in B_k$.

Proof. The argument parallels the preceding one and uses Theorem 3, the Lemma with $\alpha = 0$, $\beta = \infty$, Theorem C and Theorem 1.

Concluding, we remark that the Dini derivatives of a typical monotone continuous function are exactly determined at almost all points, at most points, and at some other dense sets of points.

References

[1] S. BANACH, Über die Baire'sche Kategorie gewisser Funktionenmengen. Studia Math. **3**, 174–179 (1931).
[2] A. M. BRUCKNER, Differentiation of Real Functions. LNM **659**, Berlin-Heidelberg-New York 1978.
[3] V. JARNIK, Über die Differenzierbarkeit stetiger Funktionen. Fund. Math. **21**, 48–58 (1933).
[4] S. MAZURKIEWICZ, Sur les fonctions non-dérivables. Studia Math. **3**, 92–94 (1931).
[5] C. NEUGEBAUER, A theorem on derivatives. Acta Sci. Math. (Szeged) **23**, 79–81 (1962).
[6] S. SAKS, On the functions of Besicovitch in the space of continuous functions. Fund. Math. **19**, 211–219 (1932).
[7] T. ZAMFIRESCU, Most monotone functions are singular. Amer. Math. Monthly **88** (1), 47–49 (1981).
[8] T. ZAMFIRESCU, Curvature properties of typical convex surfaces. Manuscript.

Eingegangen am 8.12.1982 *)

Anschrift des Autors:

Tudor Zamfirescu
Abteilung Mathematik
Universität Dortmund
August-Schmidt-Straße
D-4600 Dortmund

*) Eine Neufassung ging am 5.4.1983 ein.