Canad. Math. Bull. Vol. **27** (3), 1984

CONTINUOUS FAMILIES OF SMOOTH CURVES AND GRÜNBAUM'S CONJECTURE

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ABSTRACT. First we construct spreads consisting of analytic curves (circular arcs and segments), without points of finite multiplicity. Then we see that, in the sense of Baire categories, most such spreads have no points of finite multiplicity.

Introduction. Let C be the unit circle in the Euclidean plane and D its interior. A family $\mathscr{L}$ of simple arcs (homeomorphs of segments) in the closure $\bar{D}$ of D is called a *spread* and its elements *curves* if there exists a continuous function

$$L: C \to \mathscr{L},$$

$\mathscr{L}$ carrying the Hausdorff metric, such that $L(p)$ joins p with $-p$ and $L(p)-\{p,-p\} \subset D$ for all $p \in C$, and two distinct curves of $\mathscr{L}$ meet in exactly one point. It follows that $L(p)=L(-p)$.

The image of a spread through a homeomorphism of the plane will be called a *g-spread.*

In 1971 B. Grümbaum [3] conjectured that every g-spread admits points of order 2 called *double points*, that is points lying on precisely two curves of $\mathscr{L}$. This conjecture was verified for g-spreads consisting exclusively of line segments ([5], [6]). On the other hand K. Watson [5] produced a counterexample in which the elements of a g-spread were polygonal arcs (consisting of 2 segments).

Two natural questions arise:

1. Do there exist spreads consisting of smooth (differentiable) curves without double points?
2. How "exceptional" are spreads without double points?

We answer these questions here by constructing spreads consisting of circular arcs and segments, without points of *finite order*, that is points lying on finitely many curves. It will then follow that in the Baire category sense most spreads of this type have no points of finite order. The precise sense of the word "most" is "all, except those in a set of first Baire category" or, equivalently, "those in a residual set".

Received by the editors April 28, 1982 and, in revised form, October 21, 1983.

AMS Classification: 52A37.

The construction. Let

$$M_\alpha = \{x \in D : \operatorname{card}\{\Gamma \in \mathscr{L} : x \in \Gamma\} \geqslant \alpha\}.$$

The following lemmas will be useful for our construction.

LEMMA 1 ([4]). *For most continuous functions* $f:[0,\pi]\to\mathbf{R}$,

$$f_i'(\alpha) = \liminf_{\beta\to\alpha} \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = -\infty$$

and

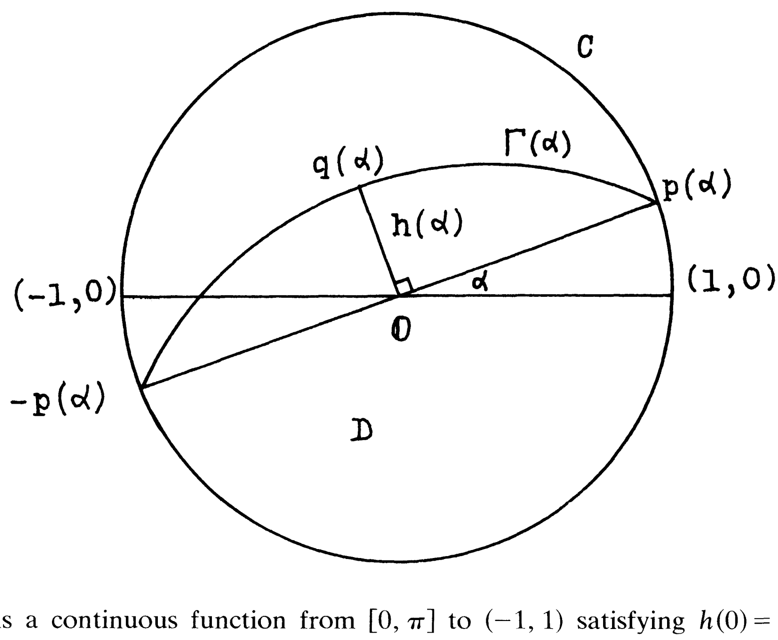
$$f_s'(\alpha) = \limsup_{\beta\to\alpha} \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \infty,$$

for all $\alpha\in[0,\pi]$.

LEMMA 2 ([1], [5]). *If* $M_2 = D$, *then* $M_{\aleph_0} = D$.

THEOREM 1. *There is a spread* $\mathscr{L}$ *consisting of circular arcs and segments without points of finite order.*

Proof. For each $\alpha\in[0,\pi]$, consider the point $p(\alpha)$ on C of coordinates $(\cos\alpha, \sin\alpha)$, the curve (arc of circle or segment) $L(p(\alpha))$, the middle point $q(\alpha)$ of $L(p(\alpha))$ at distance $\|q(\alpha)\| = h(\alpha)$ from $\mathbf{0}$, and the diameter $q(\alpha)q^*(\alpha)$ of the circle $\Gamma(\alpha)$ including $L(p(\alpha))$, if $h(\alpha)\neq 0$.



If h is a continuous function from $[0,\pi]$ to $(-1,1)$ satisfying $h(0) = -h(\pi)$, then the family $\mathscr{L}$ constitutes a spread.

From an arbitrary function f satisfying the conditions of Lemma 1 we easily derive an h satisfying those conditions and the preceding conditions as well: take for instance

$$g(\alpha)=f(\alpha)-\frac{f(0)+f(\pi)}{\pi}\alpha$$

and

$$h(\alpha)=\frac{g(\alpha)}{2\max\limits_{t\in[0,\pi]}|g(t)|}.$$

In constructing his counterexample in [5] Watson used such functions observing that their existence followed from the Baire category theorem.

There is no loss in generality in assuming, in case $L((1,0))=\Gamma(0)\cap\bar{D}$ is a non-degenerate arc of the circle and that $h(0)$ is positive. Note that $\|q(\alpha)\|\cdot\|q^*(\alpha)\|=1$, whence $\|q^*(\alpha)\|=1/h(\alpha)$. Thus $q(\alpha)$ has coordinates $(-h(\alpha)\sin\alpha, h(\alpha)\cos\alpha)$, $q^*(\alpha)$ has coordinates $((1/h(\alpha))\sin\alpha, -(1/h(\alpha))\cos\alpha)$ and the coordinates of the center of $\Gamma(\alpha)$ are

$$\left(\frac{1}{2}\left(\frac{1}{h(\alpha)}-h(\alpha)\right)\sin\alpha, \frac{1}{2}\left(h(\alpha)-\frac{1}{h(\alpha)}\right)\cos\alpha\right).$$

Thus the equation of $\Gamma(\alpha)$ is

$$\left[x-\frac{1}{2}\left(\frac{1}{h(\alpha)}-h(\alpha)\right)\sin\alpha\right]^2+\left[y-\frac{1}{2}\left(h(\alpha)-\frac{1}{h(\alpha)}\right)\cos\alpha\right]^2=\frac{1}{4}\left[h(\alpha)+\frac{1}{h(\alpha)}\right]^2$$

and this simplifies to

$$x^2+y^2-x\sin\alpha\left(\frac{1}{h(\alpha)}-h(\alpha)\right)-y\cos\alpha\left(h(\alpha)-\frac{1}{h(\alpha)}\right)=1.$$

The equation of $\Gamma(0)$ is thus

$$x^2+y^2-y\left(h(0)-\frac{1}{h(0)}\right)=1$$

and if the coordinates of $w(\alpha)\equiv\Gamma(0)\cap\Gamma(\alpha)\cap\bar{D}$ are $(u(\alpha), v(\alpha))$, then

$$u(\alpha)\sin\alpha\left(\frac{1}{h(\alpha)}-h(\alpha)\right)+v(\alpha)\cos\alpha\left(h(\alpha)-\frac{1}{h(\alpha)}\right)=v(\alpha)\left(h(0)-\frac{1}{h(0)}\right).$$

So

$$\frac{v(\alpha)}{u(\alpha)}=\frac{-\sin\alpha}{\dfrac{h(0)-\dfrac{1}{h(0)}}{h(\alpha)-\dfrac{1}{h(\alpha)}}-\cos\alpha}$$

Let

$$H(\alpha)=\frac{h(0)-\dfrac{1}{h(0)}}{h(\alpha)-\dfrac{1}{h(\alpha)}}$$

By the properties of h, there are sequences $\{\beta_i\}_{i=1}^{\infty}$ and $\{\gamma_i\}_{i=1}^{\infty}$ convergent to zero such that

$$\lim_{\beta_i\to 0}\frac{h(\beta_i)-h(0)}{\beta_i}=\infty \quad \text{and} \quad \lim_{\gamma_i\to 0}\frac{h(\gamma_i)-h(0)}{\gamma_i}=-\infty.$$

We will now show that $v(\beta_i)/u(\beta_i)$ approaches zero through negative values and hence $w(\beta_i)\to(-1,0)$ while $v(\gamma_i)/u(\gamma_i)$ approaches zero through positive values and $w(\gamma_i)\to(1,0)$. But

$$\frac{v(\beta_i)}{u(\beta_i)}=\frac{-\sin(\beta_i)}{H(\beta_i)-\cos\beta_i}=\frac{-\dfrac{\sin\beta_i}{\beta_i}}{\dfrac{H(\beta_i)-1}{\beta_i}+\dfrac{1-\cos\beta_i}{\beta_i}},$$

$$\lim_{\beta_i\to 0}\frac{\sin\beta_i}{\beta_i}=1 \quad \text{and} \quad \lim_{\beta_i\to 0}\frac{1-\cos\beta_i}{\beta_i}=0,$$

so it only remains to show that

$$\lim_{\beta_i\to 0}\frac{H(\beta_i)-1}{\beta_i}$$

approaches ∞. Since

$$\frac{H(\beta_i)-1}{\beta_i}=\frac{1}{\beta_i}\left(\frac{h(0)-\dfrac{1}{h(0)}}{h(\beta_i)-\dfrac{1}{h(\beta_i)}}-1\right)=\frac{(h(0)-h(\beta_i))(h(0)h(\beta_i)+1)}{\beta_i\left(h(\beta_i)-\dfrac{1}{h(\beta_i)}\right)h(0)h(\beta_i)},$$

$$\lim_{\beta_i\to 0}\frac{H(\beta_i)-1}{\beta_i}=\infty.$$

Similarly

$$\lim_{\gamma_i\to 0}\frac{v(\gamma_i)}{u(\gamma_i)}=0.$$

Thus, we see that each point of any curve of $\mathscr{L}$ which is a circular arc, except the endpoints, lies between two points in M_2. An analogous calculation settles the case of a curve which is a line segment. Since $D=M_1$ (Theorem 4.1 in [3]) and M_2 is $\mathscr{L}$-convex, i.e. $M_2\cap L(p)$ is connected for any $p\in C$ (Theorem

3 in [2]), we have $D = M_2$. Hence, by Lemma 2,

$$D = M_{\chi_0}.$$

This completes the proof.

Most frequent spreads. Let now $\mathfrak{S}$ be the space of all spreads, the curves of which are circular arcs or line segments. Let L and K define the spreads $\mathscr{L}$ and $\mathscr{K}$. Then with the metric

$$d(\mathscr{L}, \mathscr{K}) = \sup_{p \in C} \delta(L(p), K(p)),$$

where δ is the Hausdorff distance, $\mathfrak{S}$ is a Baire space. Indeed, if $h_{\mathscr{L}}$ and $h_{\mathscr{K}}$ are the functions analogously associated to $\mathscr{L}$ and $\mathscr{K}$, then, for $p(\alpha) = (\cos \alpha, \sin \alpha)$,

$$\delta(L(p(\alpha)), K(p(\alpha))) = |h_{\mathscr{L}}(\alpha) - h_{\mathscr{K}}(\alpha)|,$$

hence $\mathscr{L} \mapsto h_{\mathscr{L}}$ is an isometry.

The well-known space $\mathscr{C}([0, \pi])$ of all continuous functions on $[0, \pi]$ is complete. The subspace $\mathscr{D}$ of all continuous functions $\rho : [0, \pi] \to [-1, 1]$ satisfying $\rho(0) = -\rho(\pi)$ is closed in $\mathscr{C}([0, \pi])$, hence again a complete space. By Baire's theorem, $\mathscr{D}$ is a Baire space. It is evident that

$$\mathscr{F} = \{\rho \in \mathscr{D} : \rho^{-1}(1) \cup \rho^{-1}(-1) \neq \varnothing\}$$

is a nowhere dense set in $\mathscr{D}$. Hence $\mathscr{D} - \mathscr{F}$ is again a Baire space. This one is isometric to $\mathfrak{S}$, which is therefore a Baire space.

THEOREM 2. *Most spreads consisting of circular arcs and segments have no points of finite order.*

Proof. An easy modification of Lemma 1 states that for most functions $\rho \in \mathscr{D}$,

$$\rho_i' = -\infty \quad \text{and} \quad \rho_s' = \infty.$$

Since for any set $\mathscr{A}$ of first category in $\mathscr{D}$, $\mathscr{A} \cap (\mathscr{D} - \mathscr{F})$ is of first category in $\mathscr{D} - \mathscr{F}$, we have

$$\rho_i' = -\infty \quad \text{and} \quad \rho_s' = \infty.$$

for most functions $\rho \in \mathscr{D} - \mathscr{F}$. Now the proof of Theorem 1 and the isometry between $\mathfrak{S}$ and $\mathscr{D} - \mathscr{F}$ yield Theorem 2.

Many thanks are due to the referee. He helped to simplify the calculations in the proof of Theorem 1.

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