INTERIORS OF UNIFORM SIZE IN STEINITZ'S THEOREM

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ABSTRACT.

If 0 lies in the interior of any convex hull conv $S \subset \mathbb{R}^d$, then Steinitz's Theorem implies that $0 \in B_r(0) \subset \text{conv } U$ for some set U of at most 2d points of S, and some positive r. If we always assume that the biggest ball about 0 in conv S has radius one, then it is of interest to ask for lower bounds on the size of r, and how they depend on the set S or on the dimension d of the space. Lower bounds for r are found for certain cases, and examples are presented which verify the sharpness.

1. INTRODUCTION.

Suppose a point p lies in the interior of the convex hull of a finite d-dimensional set S (denoted $p \in \text{int conv } S$). Steinitz's theorem [1] asserts that there is some subset U of S of at most 2d points whose convex hull contains a ball $B_r(p)$ around p. We will investigate how large the radius r may be. In particular, we may assume without loss of generality that p is the origin 0 of $\mathbb{R}^d$, $S \subset \mathbb{R}^d$, and the unit ball is the largest ball centered at 0 which is contained in conv S.

The example of a cross basis for $\mathbb{R}^d$ (that is, a set $S = B \cup -B$, where B is a linear basis for $\mathbb{R}^d$) shows that it may be necessary for the subset U in Steinitz's theorem to have 2d points. We first show that r may be bounded below by a number greater than zero when we allow larger subsets U of S.

THEOREM 1. Suppose $S \subset \mathbb{R}^d$ is finite and conv S contains the unit ball $B_1(0)$. Then there exists a subset U of at most $d^2 + 1$ points of S for which $B_{1/d}(0) \subset \text{conv } U$.

Proof: Let T be the vertex set of the convex polytope conv S. Then each point of T lies at a distance greater than one from 0. Choose some point p_o of T and let $\{p_o, p_1, \ldots, p_d\}$ be the vertices of a regular simplex centered at 0. Let v_i be the point where the ray $[0, p_i[$ meets the boundary of conv T. Then $v_i \in H_i \cap \text{conv } S = \text{conv } (S \cap H_i)$, where H_i is a supporting hyperplane to the compact set conv S at v_i. Applying Caratheodory's theorem in this hyperplane, we can find $U_i \subset T$ such that $v_i \in \text{conv } U_i$ and $|U_i| \leq d$. Define $U = \{p_o\} \cup \bigcup_{i=1}^{d} U_i$. Then $U \subset S$, and $|U| \leqq 1 + d^2$, and since conv U contains a regular simplex inscribed in $B_1(0)$, it also must contain $B_{1/d}(0)$, the largest ball contained in this simplex.

PROPOSITION. Suppose S is finite and conv $S \supset B_1(0)$. Then, in $\mathbb{R}^2$, some subset U of at most 5 points of S must have $B_{1/2}(0) \subset \text{conv } U$; in $\mathbb{R}^3$, some subset U of at most 9 points of S must have $B_{1/3}(0) \subset \text{conv } U$.

Proof: In the plane, $d^2 + 1 = 5$. In $\mathbb{R}^3$, rotate the regular simplex (tetrahedron) in the proof of theorem 1 about the line through $[0,p_o]$ until the ray $[0,p_1[$ meets an edge (or better yet a vertex) of conv S. If such an rotation may be found, we may assume that the set U_1 contains at most 2 points. Thus the cardinality of U will be at most $1 + 2 + (2)(3) = 9$ rather than the $1 + d^2 = 10$ of the theorem. If each point v_i $(i = 1,2,3)$ lies in the interior of one face F of conv U, then such a rotation may not exist. In this case we may rotate the tetrahedron, keeping 0 as the center, until v_1 becomes a vertex of F, while keeping v_2 in F. If we choose the original vertex p_o to be the vertex of F at v_1, then the desired rotation of the simplex about $[0,p_o[$ may be found. Similarly in higher dimensions.

Let $\mathcal{F}$ be a family of finite sets in $\mathbb{R}^d$. We define $b_{\mathcal{F}}$ to be the largest number (a priori possibly zero) such that, whenever S is a set belonging to $\mathcal{F}$ with $B_1(0) \subset$ conv S, then there exists a subset U of at most 2d points of S with $B_{b_{\mathcal{F}}}(0) \subset$ conv U. Equivalently,

$$b_{\mathcal{F}} = \inf_{S \in \mathcal{F}} \max_{U \in \mathcal{U}(S)} \{r \mid B_r(0) \subset \text{conv } U\}$$

where $\mathcal{U}(S)$ is the collection of all subsets U of S of at most 2d points. If $\mathcal{F}$ is the family of all finite sets in $\mathbb{R}^d$, we put $b(d) = b_{\mathcal{F}}$.

THEOREM 2. The lower bound b(d) is a monotone decreasing function of the dimension d.

Proof: For a fixed dimension d and for each $\varepsilon > 0$ we can find a finite set $S \subset \mathbb{R}^d$ whose convex hull contains the unit ball, and for which

$$b(d) + \epsilon/2 \;>\; \max_{U \in \mathcal{U}(S)} \{ r \mid B_r(0) \subset \text{conv } U \}.$$

Embed this d-dimensional set in a d-dimensional subspace H of $\mathbb{R}^{d+1}$. Choose a point $y \in H^{\perp}$ and let $T = \{y, -y\} \cup S$. By choosing $\|y\|$ sufficiently large, conv T will contain a ball about 0 in $\mathbb{R}^{d+1}$ of radius at least $1 - \epsilon/2$. Any subset V of at most $2(d+1)$ points of T with $0 \in \text{int conv } V$ must contain both y and $-y$. Hence V can contain at most 2d points of S. Thus $B_r(0) \subset H \cap \text{conv } (V \cap H)$ implies that $r < b(d) + \epsilon/2$. It follows that $b(d+1) < b(d) + \epsilon$. ∎

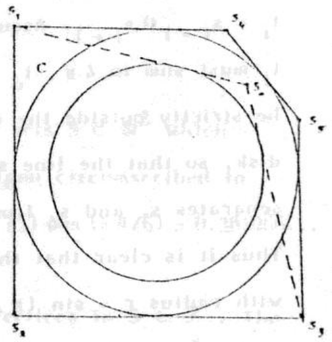
The main problems are:

1. Is $b(d) > 0$ for every d?
2. If yes, determine b(d).

We will answer here the first question for $d = 2$ and do a few more steps toward resolving 1. and 2.

2. PLANAR RESULTS.

Example 1. Let S consist of the 5 vertices of a regular pentagon with an inscribed unit circle whose center is at 0. Then letting U denote any $2d = 4$ of the vertices, we see $B_r(0) \subset \text{conv } U$ where $r = \sec(\pi/5)\cos(2\pi/5) = 0.381966\ldots$. This gives an upper bound on b(2), which we conjecture is sharp.



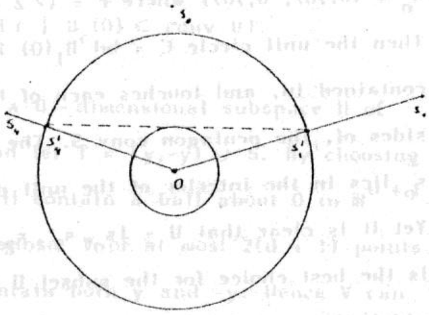
Example 2. Let S consist of the six points $s_1 = (-1,1)$, $s_2 = (-1,-1)$, $s_3 = (1,-1)$, $s_4 = (c,1)$, $s_5 = (1,c)$, $s_o = (0,707,\ 0,707)$ where $c = (\sqrt{2} - 1)$. Then the unit circle $C = \text{bd } B_1(0)$ is contained in, and touches each of the 5 sides of, the pentagon conv S. The point s_o lies in the interior of the unit disk. Yet it is clear that $U = \{s_o, s_1, s_2, s_3\}$ is the best choice for the subset U for which the maximum in the definition of $b(d)$ is attained. Thus the search for such a best U may not, in general, be limited to the vertices of conv S, or even to the points of S that lie outside the unit disk with center 0.

The next results will show that $b(2) > 0$.

LEMMA. If S is a set of 5 points in the plane for which $B_1(0) \subset \text{conv } S$ then we can find $s_o \in S$ and $r \geq \sin (\pi/10) = 0{,}309\ldots$ such that $B_r(0) \subset \text{conv } (S - \{s_o\})$.

Proof: Assume that conv S is a pentagon, since if some $s_o \in S$ lies in conv $(S - \{s_o\})$ then the whole unit disk is contained in conv $(S - \{s_o\})$. Label the 5 vertices s_i of conv S so that their projections s_i' on $B_1(0)$ are in clockwise order, and identifying subscipts mod 5, choose an indes i which minimizes the angle

$t_i = s_{i-1} 0\, s_{i+1}$. Assume 0 is such an index. Since the five angles t_i must sum to 4π, t_o can be at most $4\pi/5$. Each vertex s_i must be strictly outside the unit disk, so that the line $s_1's_4'$ separates s_1 and s_4 from 0. Thus it is clear that the disk with radius $r = \sin(\pi/10)$ and center 0 must be in the interior of conv $(S - \{s_o\})$.



THEOREM 3. $0.154\ldots = (1/2)\sin(\pi/10) < b(2) < \sec(\pi/5)\cos(2\pi/5) = 0.381\ldots$

Hence, if the convex hull of any set S in the plane contains the unit disk, then the convex hull of some 4 points of S must contain a disk about 0 of positive radius r, where r does not depend on the set S.

Proof: Assume S is a finite set, since otherwise for each $\epsilon > 0$ we may choose a finite set F of points on the boundary of $B_1(0)$ so that $B_{1-\epsilon}(0) \subset$ conv F. Apply Caratheodory's theorem to each point of F, and $B_{1-\epsilon}(0)$ lies in the convex hull of a finite subset of S.

By Theorem 1 there is a subset U of at most 5 points of S for which $B_{1/2}(0) \subset$ conv U. By the Lemma, the disk about 0 of radius $(1/2)\sin(\pi/10)$ must lie in the convex hull of some 4 points of U. Example 1 proves the upper bound.

The lower bounds of the Lemma and Theorem 3 are obviously not sharp. It would be interesting to know if the d-dimensional analog of Theorem 3 is true.

The next result supports the conjecture (see Example 1) that $b(2) = \sec(\pi/5)\cos(2\pi/5)$.

THEOREM 4. Let $\mathcal{F}$ be the collection of all sets $S \subset \mathbb{R}^2$ which contain the vertices of a pentagon circumscribed to $C = \mathrm{bd}\, B_1(0)$. Then $b_{\mathcal{F}} = \sec(\pi/5)\cos(2\pi/5) = 0.381966\ldots$

Proof: Let $x_1x_2x_3x_4x_5$ be the pentagon with vertices in $S \in \mathcal{F}$. The indices will be numbered modulo 5, such that increasing i induces a direct sense rotation of the ray $0x_i$.

Case 1. For some index j, the line $x_{j-1}x_{j+1}$ does not strictly separate 0 from x_j. Let $y_i = C \cap 0x_i$ for all i. One of the angles $x_{j-1}0x_j$ and x_j0x_{j+1}, say x_j0x_{j+1}, must measure at least $\pi/2$. Then the sum of the lengths of the arcs $y_{j+1}y_{k+3}$ and $y_{j+3}y_{j+5}$ (where $j+5 \equiv j$) is at most $3\pi/2$. Hence one of them, say $y_{j+1}y_{j+3}$ has length at most $3\pi/4$. Thus the distance from 0 to the line $y_{j+1}y_{j+3}$ is at least $\cos(3\pi/8)$. Since the distance from 0 to $x_{j+1}x_{j+3}$ is even larger and $\cos(3\pi/8) < \sec(\pi/5)\cos(2\pi/5)$, this case is settled.

Case 2. For all indices i, the line $x_{i-1}x_{i+1}$ strictly separates 0 from x_i. We apply the polar transformation determined by C. The polygon $x_1x_2x_3x_4x_5$ becomes a polygon $z_1z_2z_3z_4z_5$ inscribed to C, the diagonal lines $x_{i-1}x_{i+1}$ become the points $z_i' = z_{i-2}z_{i-1} \cap z_{i+1}z_{i+2}$.

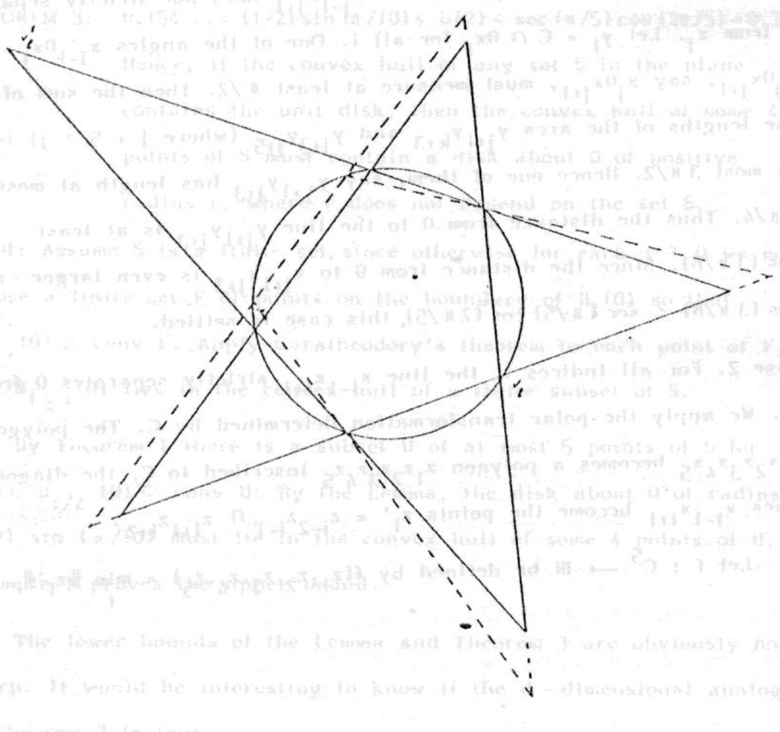
Let $f : C^5 \longrightarrow \mathbb{R}$ be defined by $f(z_1,z_2,z_3,z_4,z_5) = \min_i \|z_i'\|$.

The theorem will be completely proved if we can show that f attains its maxima exactly for those $(z_1,\ldots,z_5)$ corresponding to regular pentagons inscribed to C.

Clearly, f is continuous and one sees immediately that $\sup\limits_{z_i \in C} f(z_1,\ldots,z_5)$ is finite. We prove now that

$$\sup_{z_i \in C} f(z_1,\ldots,z_5) = f(v_1,\ldots,v_5)$$

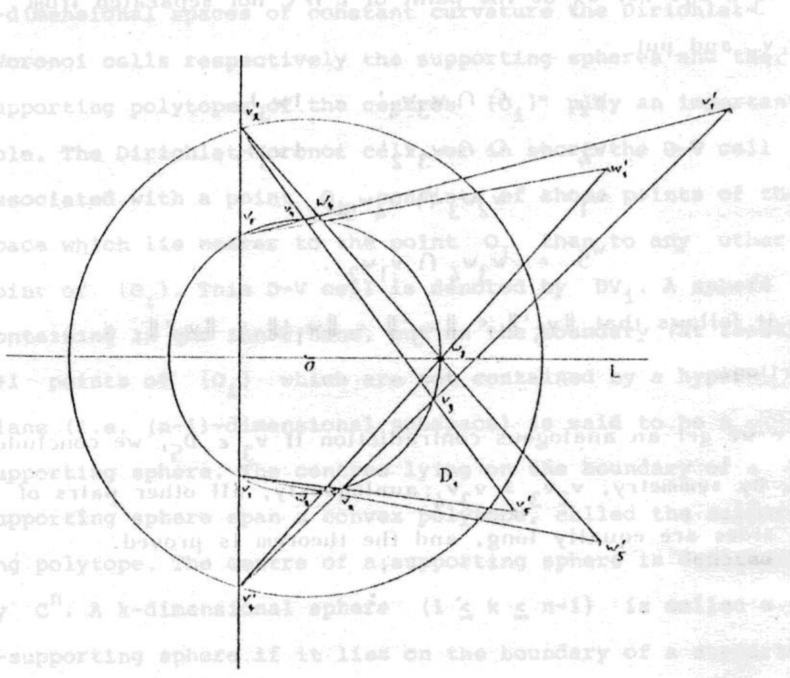
yields $\|v_1'\| = \ldots = \|v_5'\|$, where again, $v_i' = v_{i-2}v_{i-1} \cap v_{i+1}v_{i+2}$ for every i.



Suppose on the contrary $\|v_k'\| + \varepsilon = \|v_\ell'\|$ for some indices k,ℓ and some $\varepsilon > 0$. Then move slightly $v_{\ell-2}$ and $v_{\ell+2}$ away from v_ℓ, as shown in the Figure. It is obvious then, that all $\|v_i'\|$'s will increase except $\|v_\ell'\|$, which will decrease. Perform this movement such that the modifications of $\|v_i'\|$ remain less than $\varepsilon/2$ for all i. Then

$$\min_i \|v_i'\|$$

will clearly increase, which contradicts



$$f(v_1,\dots,v_5) \geq f(z_1,\dots,z_5) \quad (z_i \in C).$$

Hence $\|v_1'\| = \dots = \|v_5'\|$ as stated.

Now we show that this implies the regularity of the pentagon $v_1v_2v_3v_4v_5$. First, notice that $\|v_2'\| = \|v_4'\|$ implies $\|v_2' - v_5\| = \|v_4' - v_1\|$. Now take the line L orthogonal to v_5v_1 through 0. $\mathbb{R}^2 - L$ consists of two domains D_1 containing v_1, and D_5 containing v_5. Suppose $v_3 \in D_1$. Let w_3 be the point of $L \cap C$ not separated from v_3 by v_5v_1 and put

$$w_2 = C \cap w_3w_4' - \{w_3\},$$
$$w_4 = C \cap w_3v_2' - \{w_3\},$$
$$w_1' = w_2w_3 \cap w_4w_5,$$
$$w_5' = w_3w_4 \cap v_1w_2.$$

Clearly, it follows that $\|v_5'\| < \|w_5'\| = \|w_1'\| < \|v_1'\|$, a contradiction.

Since we get an analogous contradiction if $v_3 \in D_5$, we conclude $v_3 = w_3$. By symmetry, $v_2v_3 = v_3v_4$; analogously, all other pairs of adjacent sides are equally long, and the theorem is proved.

Reference:

[1] E. Steinitz, Bedingt konvergente Reihen und konvexe Systeme.
I - II - III
J. Reine Angew. Math. 143 - 144 - 146 (1913, 1914, 1916)