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USING BAIRE CATEGORIES IN GEOMETRY

Summary: *In this conference we present properties of most elements of several geometrically significant Baire spaces, i.e. of all elements of such spaces except those in a set of first Baire category. Sometimesç these properties are unixpectedly pathological.*

While the well-known Baire category theorem was immediately used in Analysis, its first application to Convex Geometry is of much more recent date: In 1959, V. Klee found in Banach spaces applications of accurately geometric significance. After another surprisingly long period, in 1977, P. Gruber opened again, independently, the Baire category suit-case, reproving those theorems of Klee and proving several new ones. Since 1977 a number of interesting geometrical results have been proved by using Baire's theorem. Almost all of them show that we think in a wrong way about several geometric objects, that we have a lot of prejudice, as we also have regarding people. One believes, for instance, that typical convex surfaces (and men) are not smooth; this turns out to be deeply erroneous.

Some of the results we shall mention present typical objects the existence of which was not easy to prove. In some other cases the existence itself of those objects proven to be typical was unknown. These are most

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interesting, pathological samples.

A set in a topological space is called *nowhere dense*, if its closure has empty interior. Any countable union of nowhere dense sets is said to be of *first category*. If a set is not of first category, then it is of *second category*. A topological space, each open set of which is of second category, is called a *Baire space*.

In $I\!R^d$ we use the Euclidean distance, in the space $\mathscr{C}$ of all convex surfaces (defined in the next section) the Hausdorff distance δ, on a surface the intrinsic metric. For some other spaces we shall use again Hausdorff's distance. By the Baire category theorem, all of these are Baire spaces.

Most or *typical* elements of a Baire space are those in a *residual* set, which means the complement of a set of first category.

Smoothness and strict convexity of convex surfaces.

We understand by a *convex surface* the boundary and by a *convex body* the closure of an open bounded convex set in $I\!R^d$. Clearly, a convex surface does not need to be smooth (differentiable). But, as Reidemeister [20] in 1921 (for $d = 3$) proved, each convex surface is smooth almost everywhere, with respect to the usual Hausdorff measure. For a more precise result and arbitrary dimension see [3].

The following result of Klee [9] seems to be the oldest in the field this article deals with. It was independenlty rediscovered 18 years later by P. Gruber [11].

Theorem 1. *Most convex surfaces are smooth and strictly convex.*

It is interesting to notice that P. McMullen [18] had asked for a measure in $\mathscr{C}$ with respect to which almost all convex surfaces are not smooth, which seems to be the more general case. Theorem 1 shows that from the point of view of Baire categories the contrary is true. R. Schneider and C. Bandt rejected the existence (proposed by P. Gruber) of any useful Hausdorff measure on $\mathscr{C}$.

In a metric space, a set M is called *porous* if for any point x, there exists a positive number α, such that, for any positive number γ, there is a point y in the ball $K(x, \gamma)$ of centre x and radius γ with the property that

$$K(y, \alpha\rho(x, y)) \cap M = \phi ,$$

ρ being here the distance function.

A countable union of porous sets is said to be *σ-porous*. We say that *nearly all* elements of a metric Baire space have a certain property, if those which do not enjoy it form a σ-porous set.

Since any σ-porous set is of first category, it is clear that the following result strengthens Theorem 1.

Theorem 2 [38]. *Nearly all convex surfaces are smooth and strictly convex.*

Every smooth convex surface belongs to the class C^1. Does a typical convex surface also belong to C^2 ? This question was answered negatively by Gruber [11]. On the other side, every convex surface has points where the curvature exists. Thus it makes a sense to ask about the curvature properties of typical convex surfaces.

The curvature of convex surfaces

Let x be a smooth point of a convex surface S. Since most convex surfaces are smooth everywhere, this means no restriction. At x we consider the tangent direction τ, the normal section of S in direction τ and the lower and upper radii of curvature $\rho_i^\tau(x)$ and $\rho_s^\tau(x)$ of the normal section (see [5], p. 14). The numbers

$$\gamma_i^\tau(x) = \rho_s^\tau(x)^{-1} , \quad \gamma_s^\tau(x) = \rho_i^\tau(x)^{-1}$$

are the *lower* and *upper curvatures* of S at x in direction τ. If they are equal, the common value $\gamma^\tau(x)$ is the *curvature* of S at x in direction τ.

By a theorem of Aleksandrov [1] (Buşemann and Feller [6] for $d = 3$), on every convex surface there exists a finite curvature a.e. in every tangent direction. How behaves the curvature of typical convex surfaces?

In 1957, G. de Rham treated in a conference here in Torino the following particular type of convex curves in $I\!R^2$: Take a convex polygon, then the two points on every side dividing it into three equal parts. Consider the convex hull of all these points and its boundary polygon. Repeat the procedure. The intersection of all these infinitely many convex sets is a convex set the frontier of which is a smooth convex curve with vanishing curvature a.e., as de Rham has shown [8].

There is another simple way of producing convex curves with vanishing curvature a.e.: Consider any *singular function* (also called *Vitali function*), i.e. a stricly increasing continuous function $f : [0, 1] \to \mathbb{R}$ with $f' = 0$ a.e. Then $\int f$ is a convex function, the graph of which has a vanishing curvature whereever $(\int f)'' = 0$, namely a.e.

For typical convex surfaces the following turns out to be true.

Theorem 3 [24]. *For most convex surfaces S, at each point $x \in S$ and every tangent direction τ at x,*

$$\gamma_i^\tau(x) = 0 \quad or \quad \gamma_s^\tau(x) = \infty .$$

Combining Theorem 3 with Aleksandrov's theorem we get the surprising conclusion that neither de Rham's convex curves, nor those obtained by integrating singular functions are too particular:

Theorem 4. *For most convex surfaces, $\gamma^\tau(x) = 0$ a.e. in every tangent direction τ.*

By Aleksandrov's theorem, on every convex surface a curvature exists a.e. in every tangent direction. Does it also exist at most points of an arbitrary convex surface? No explicit example of a surface without this property seems to be known. However, the next theorem shows that most convex surfaces do not enjoy it!

Theorem 5 [25]. *For most convex surfaces, at most points x,*

$$\gamma_i^\tau(x) = 0 \quad and \quad \gamma_s^\tau(x) = \infty$$

in every tangent direction τ.

The proof in [25] used a previously published similar result of R. Schneider [21], in which the set of points x, instead of being residual, was only dense. We give here in the last section a different, direct proof.

Do the typical convex surfaces possess only points of the types described in Theorems 4 and 5? No, certainly not: think about the points of a convex surface which lie on its circumscribed sphere.

In $\mathbb{R}^2$ the following holds.

Theorem 6 [34]. *For most convex curves, $\gamma^\tau(x) = \infty$ at uncountably, densely many points x, in both tangent directions τ.*

This generalizes as follows to $\mathbb{R}^d$:

Theorem 7 [34]. *For most convex surfaces S,*

$$\{(x, \tau) : \gamma^\tau(x) = \infty\}$$

is uncountable and dense in the spherical bundle associated to S.

The proofs of Theorems 6 and 7 show that every convex curve or surface satisfying the condition of Theorem 4 automatically enjoys the property of Theorem 6 or 7. Clearly, the set of points x we speak about is small in both senses, that of measure and that of categories.

Problem 1. *Is it true that most convex surfaces in $\mathbb{R}^d$ possess a point x such that $\gamma^\tau(x) = \infty$ in every tangent direction τ?*

This problem, which has, by Theorem 6, a positive answer for $d = 2$, and probably a negative one for $d = 3$, is still open.

Typical convex surfaces also have other types of points than those described utill now.

Theorem 8 [34]. *For most convex surfaces S,*

$$\{(x, \tau) : 0 = \gamma_i^{-\tau}(x) < \gamma_s^{-\tau}(x) = \gamma_i^\tau(x) < \gamma_s^\tau(x) = \infty\}$$

is uncountable and dense in the spherical bundle associated to S.

In Theorem 8, as formulated in [34], a lower or upper bound can be prescribed for the common value of $\gamma_s^{-\tau}(x)$ and $\gamma_i^\tau(x)$.

Problem 2. *Is it possible, in Theorem 8, to precisely prescribe the common value of* $\gamma_s^{-\tau}(x)$ *and* $\gamma_i^{\tau}(x)$?

Normals to convex surfaces

Closely related to the curvature behaviour of typical convex surfaces is the aspect of their normals. While for any usual surface most points of $\mathbb{R}^d$ lie on finitely many normals, we shall see that typical convex surfaces behave completely differently.

Theorem 9 [33]. *For most convex surfaces* S, *most points of* $\mathbb{R}^d$ *lie on infinitely many normals to* S.

The case $d = 2$ is easier than the case $d \geqslant 3$ and was already treated in [30].

It is interesting to remark that all infinite pencils of concurrent normals exhibited in the proof of Theorem 9 realise relative maxima for the distance from the common point. An analog proof can provide pencils realising relative minima. Two questions arise in connection to this.

Problem 3. *Is it true, for most convex surfaces, that for every point lying on infinitely many normals, each of these normals realises a relative maximum or a relative minimum of the distance from that point to the surface?*

Problem 4. *Does there exist, for most convex surfaces, a point in* $\mathbb{R}^d$ *lying on uncountably many normals?*

It is interesting to notice that the existence of convex surfaces with the properties of Theorem 9 was unknown before.

In the same spirit are the following reflection considerations. Let x, $y \in \mathbb{R}^2$, S be a smooth convex curve in $\mathbb{R}^2$ and $M \subset S$ with card $M = \alpha$. If, for each $z \in M$, xz and yz make angles of equal measures with S, then we say that x *sees* α *images* of y. In general it can only be said that x sees 2 images of y. However, this can be drastically strengthened for typical convex curves.

Theorem 10 [30]. *For most convex curves, every point of* $\mathbb{R}^2$ *sees* $\aleph_0$ *images of most other points.*

Regarding pairs of points in $\mathbb{R}^2$, the following in true.

Theorem 11 [30]. *For most convex curves, most pairs of points* $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ *are such that* x *sees* $\aleph_0$ *images of* y.

The generalizations to $\mathbb{R}^d$ are not yet worked out.

The geodesics of convex surfaces.

A shortest path on a convex surface is called a *segment*. A curve which is locally a segment is called a *geodesic* (see for a precise definition [5], p. 77). A point of a segment different from its two endpoints will briefly be called *interior*. Is each point of a surface an interior point? The answer is easy for non-smooth surfaces: no conical points is (for any segment) an interior point ([2], p. 155). Points which are not for any segment interior will be called *endpoints*. They are, of course, endpoints of many geodesics. Smooth surfaces with an endpoint are also known ([2], p. 58-59). But, for each convex surface of class C^2, every point is an interior point of a segment in each tangent direction. More generally, this happens at a point x if the lower indicatrix at every point y in some neighbourhood of x does not contain y as a boundary point, ([5], p. 92). Clearly, the set of all interior points is uncountable and dense, for an arbitrary convex surface.

Thus, it seems that usually convex surfaces must have many interior points. But let us look more closely at a typical convex surface: it is of class C^1, but not of class C^2 and at most points the lower indicatrix reduces to a point (Theorem 5). In fact, we established the following result.

Theorem 12 [31]. *On most convex surfaces in* $\mathbb{R}^d$, *most points are endpoints.*

It is well-known that in a certain tangent direction at a point of a convex surface may not start any segment. Such a tangent direction is called by Aleksandrov *singular*. He shows that there are smooth convex surfaces with a dense set of singular tangent directions at a certain point ([2], p. 59). Also, non-smooth surfaces all points of which are of this kind do exist; take, for example, a convex surface with a dense set of conical points. But, again, at any point of a convex surface of class C^2 or with the above indicatrix condition, a segment starts in each tangent direction. And for an arbitrary convex surface, at each point, the set of singular tangent directions has measure zero, as Aleksandrov proved ([2], p. 213). However we have the following theorem.

Theorem 13 [31]. *On most convex surfaces in* $\mathbb{R}^3$, *at every point, most tangent directions are singular.*

The restriction $d=3$ is imposed by the proof methods, which essentially use Aleksandrov's concepts and results in [2].

Problem 5. *Prove Theorem* 13 *for* $d \geqslant 4$.

We notice that the existence of convex surfaces with the properties described in Theorems 12 and 13 was unknown.

A *circle* on a convex surface in $\mathbb{R}^3$ is the set of all points at intrinsic distance equal to some positive number from a fixed point of the surface. A circle which is a Jordan closed curve will be called a *Jordan circle*. It is known that a Jordan circle may possess *vertices*, i.e. points where the circle is not smooth (see also [2], p. 61).

Theorem 14 [31]. *On most convex surfaces in* $\mathbb{R}^3$, *every Jordan circle has countably, densely many vertices.*

We have raised in [31] the question whether each point with infinite curvature in every tangent direction must be an endpoint.

Consider the number $\alpha > 1$ and the surface Σ_α expressed in cylindrical coordinates (ρ, φ, z) by $z=\rho^\alpha$, with $0 \leqslant \rho \leqslant 1$ and $0 \leqslant \varphi < 2\pi$. Complete Σ_α (add a hemisphere for instance) to a convex surface S_α. Then S_α verifies $\gamma^\tau(0)=\infty$ for every tangent direction τ in $0=(0,\varphi,0)$ if $1<\alpha<2$.

V. Bangert (private communication) gave a negative answer to the preceding question, providing a proof (sketch) of the following: **0** *is an endpoint of* S_α *if and only if* $1<\alpha<3/2$.

Spreads

Let $D \subset \mathbb{R}^2$ be a Jordan domain, i.e. a connected open set with a Jordan closed curve as boundary. Let

$$i : \text{bd } D \to \text{bd } D$$

be a fixed point free, continuous involution. Let $\mathcal{L}$ be a family of Jordan arcs in $\overline{D}$ admitting a surjective continuous function

$$L : \text{bd } D \to \mathcal{L}$$

such that $L(x)$ has x as an endpoint for any $x \in \text{bd } D$ and

i) $L(x) = L(i(x))$ for all $x \in \text{bd } D$,

ii) $L(x) \cap L(y)$ has a single point whenever $y \notin \{x, i(x)\}$.

Then $\mathcal{L}$ was called by B. Grünbaum a *spread*.

The diameters of smooth strictly convex curves, the area bisectors of planar convex bodies, the perimeter bisectors of convex curves, the midpoint curves of smooth strictly convex curves are all examples of spreads. For more information about spreads consult [15], [27].

In the investigation of spreads a special attention was paid to the sets

$$M_\alpha = \{x \in D : \text{card } \{\Gamma \in \mathcal{L} : x \in \Gamma\} \geqslant \alpha\},$$

$$T_\alpha = \{x \in D : \text{card } \{\Gamma \in \mathcal{L} : x \in \Gamma\} = \alpha\}.$$

In 1971 Grünbaum [15] conjectured that $M_\alpha \neq \phi$ implies $T_\alpha \neq \phi$. Since always $M_3 \neq \phi$ [14], the conjecture includes the assertions that T_1, T_2 and T_3 are nonempty. Grünbaum's conjecture was disproved by K. S. Watson [23]. In his example, $M_{\aleph_0} = D$ and the arcs in $\mathcal{L}$ are polygonal, hence not smooth (in general).

Take bd D to be S^1, define $i(x) = -x$ and let $\mathcal{L}$ consist of circular arcs and line segments only. Let the distance between two such spreads $\mathcal{L}_1$ and $\mathcal{L}_2$ be

$$\rho(\mathcal{L}_1, \mathcal{L}_2) = \max_{\|x\|=1} \delta(L_1(x), L_2(x)),$$

where L_j plays in the definition of $\mathcal{L}_j$ the rôle of L $(j=1, 2)$. The space $\mathfrak{L}$ of all such spreads is a Baire space.

Theorem 15 [39]. *For most spreads in* $\mathfrak{L}$, $M_{\aleph_0} = D$.

The above result of A. Zucco and myself shows that simplest smootl curves may serve to exhibit counterexamples to Grünbaum's conjecture anc that, in fact, most spreads in $\mathfrak{L}$ are such counterexamples.

For spreads consisting only of line segments, called ***straight spreads***, th situation is different. For every straight spread, $T_1 \neq \phi$ [27] and $T_2 \neq$ [23], [28]. Thus, a part of Grünbaum's conjecture is valid for straight spreads What about the rest?

Problem 6. *Prove (or disprove) Grünbaum's conjecture for straight spreads*

Returning to arbitrary spreads, under certain additional continuity hy potheses on $\mathcal{L}$, int M_2 is a Jordan domain different from D [28]. Thu $T_1 \neq \phi$. It is also proved in [28] that most points of bd M_2 belong to T_2 The mentioned continuity hypotheses are verified, for instance, by the spread of all area bisectors of a planar convex body and by that of all perimete bisectors of a convex curve.

The spread of all diameters of a convex curve will be examined in th next section.

The diameters of a convex curve

Reducible convex bodies in $\mathbb{R}^2$, introduced by P.C. Hammer [16], ar characterized in the smooth case by

$$\overline{M_2} \subset D\ ,$$

$\overline{D}$ being the convex body and $\mathcal{L}$ the family of all essential diameters of $\overline{D}$ (for a definition of essential diameters see [16]). For strictly convex bodie with smooth boundary, $\mathcal{L}$ coincides with the family of all diameters. Clearly at least for smooth reducible convex bodies, M_2 cannot be dense in D

The situation is different for typical convex bodies.

Theorem 16 [32]. *For most convex curves, most points lie on infinitely man diameters.*

It follows, of course, that typical convex bodies are irreducible.

P.C. Hammer and A. Sobczyk [17] proved that $M_{\aleph_0}$ has measure zero. If a set $A \subset D$ is such that

$$A \subset \bigcup_{x \in X} L(x)$$

for some set $X \subset \mathrm{bd}\, D$ of measure zero, A is said to be ***null-swept*** [32]. Of course, $M_{\aleph_0}$ does not need to be null-swept: take bd D to be a circle. But it is easy to prove that any null-swept set in D has measure zero.

Theorem 17 [32]. *There exist smooth strictly convex curves such that, for the spread of diameters,* $M_{\aleph_0}$ *is residual and null-swept.*

The connectivity properties of M_α are also surprising.

Though it is very easy to see that M_α may be disconnected for $\alpha \geqslant 4$ [27], the following is true for typical convex curves.

Theorem 18 [32]. *For most convex curves,* M_α *is connected for every* $\alpha \leqslant \aleph_0$ *and* T_α *is totally disconnected for every* $\alpha < \aleph_0$.

Notice that T_1 is connected if and only if $\overline{D}$ is reducible. Thus we obtain again the irreducibility of typical convex bodies.

It is not easy to find and investigate higher dimensional analogs of spreads. However, all above results on diameters make a sense in $\mathbb{R}^d$; the proofs await to be done. Not even the case $d = 3$ was settled so far.

Pairs of convex curves

We say that two convex curves in $\mathbb{R}^2$ are ***internally tangent***, if they have a common point, at which there is a common supporting line, and both curves lie on the same side of the line. Let $\mathcal{V}$ be the space of all pairs of internally tangent convex curves. With the topology induced by the product topology of $\mathcal{C} \times \mathcal{C}$, $\mathcal{V}$ is a Baire space.

Theorem 19 [29]. *Most pairs in* $\mathcal{V}$ *intersect each other infinitely many times and are tangent at just one point.*

Another interesting behaviour we find for internally tangent convex curves without any other common points. The space $\mathscr{W}$ of all such pairs is again a Baire space. It can be proved that most pairs in $\mathscr{W}$ behave like wheels in gear: *none of them can be rotated alone, without cutting the other* [35].

All properties of convex curves or surfaces considered until now had more or less a local character. The next two sections present global properties of typical convex surfaces.

Circumscribed spheres and ellipsoids

An ellipsoid (or sphere) is said to be *circumscribed* to a convex surface in $\mathbb{R}^d$, if it has minimal volume and the surface lies in its convex hull. Both the circumscribed ellipsoid, also called *Löwner ellipsoid,* and the circumscribed sphere are unique. The second assertion is obvious, the first was proved by L. Danzer, D. Laugwitz and H. Lenz [7]. If the ellipsoid E and the sphere F are circumscribed to $S \in \mathscr{C}$, then

$$\text{card}\,(E \cap S) \geqslant d + 1$$

and, clearly,

$$\text{card}\,(F \cap S) \geqslant 2\ .$$

How many contact points have typical convex surfaces in common with their circumscribed spheres and ellipsoids? The answer is given by the next theorem.

Theorem 20. *For most convex surfaces* S,

$$\text{card}\,(F \cap S) = d + 1 \quad [26]$$

and

$$\text{card}\,(E \cap S) = \frac{d\,(d+3)}{2} \quad \text{(Gruber [12])}.$$

Notice that the set of surfaces $S \in \mathscr{C}$ such that

$$\text{card}\,(F \cap S) \leqslant d$$

or

$$\text{card}\,(E \cap S) \leqslant (d^2 + 3d - 2)/2$$

is nowhere dense, while the set of those $S \in \mathscr{C}$ such that

$$\text{card}\,(F \cap S) = \alpha$$

or

$$\text{card}\,(E \cap S) = \beta$$

is dense in $\mathscr{C}$, for every $\alpha \geqslant d + 2$ and $\beta \geqslant (d^2 + 3d + 2)/2$ $(\alpha, \beta \in \mathbb{N})$.

Approximation by polytopes

If $\mathscr{P}_n$ denotes the set of all polytopal surfaces in $\mathscr{C}$ with at most n vertices, then, for any surface $S \in \mathscr{C}$ and $n \geqslant d + 1$, there exists $P^* \in \mathscr{P}_n$ such that

$$\delta\,(S\,,P^*) = \nu\,(S\,,n)\,,$$

where

$$\nu\,(S\,,n) = \inf_{P \in \mathscr{P}_n} \delta\,(S\,,P)\,.$$

Such a polytopal surface P^* is called *best approximation* of S. Clearly, the best approximations of a convex surface do not need to be unique. But P. Gruber and P. Kenderov proved the following result.

Theorem 21 [13]. *For* $d = 2$ *and any* $n \geqslant 3$, *most convex curves admit a unique best approximation.*

A refinement of this result was given by V. Zhivkov [40]. R. Schneider and J. Wieacker [22] and, independently, P. Gruber and P. Kenderov [13] studied the asymptotic behaviour for $n \to \infty$ and found that it is - typically - very irregular:

Theorem 22. *Let* $f : \mathbb{N} \to [0\,,\infty\,)$ *be arbitrary and* $g : \mathbb{N} \to [0\,,\infty\,)$ *satisfy* $g(n) = o\,(1/n^{2/(d-1)})$ *as* $n \to \infty$. *Then, for most convex surfaces* S,

$$\nu\,(S\,,n) < f\,(n)$$

for infinitely many n *and*

$$\nu\ (S, n) > g\ (n)$$

for infinitely many n .

Analogous results have also been obtained with respect to other metrics than Hausdorff's [13].

We shall now leave the Baire spaces of convex surfaces and investigate in the next two sections the typical aspect of starshaped compact sets and other related sets.

Starshaped and *n*-starshaped compact sets

A compact set $M \subset \mathbb{R}^d$ is said to be *n-starshaped* if there is a point $z \in M$ such that, for every point $x \in M$, there exists a polygonal line $P \subset M$ joining x to z and formed by n segments. 1-starshaped sets are also called *starshaped*. For any n-starshaped set M, the set of all possible points z is the *kernel* of M: Let $\mathcal{S}_n$ be the space of all n-starshaped sets. Each of these spaces, endowed with Hausdorff's metric, is a Baire space. The typical starshaped sets have a rather strange aspect:

Theorem 23 [36]. *Most starshaped sets are nowhere dense and have a single point as kernel.*

Thus, typical starshaped sets are unions of line segments, each of which has the kernel as one endpoint. Let M be a typical element of $\mathcal{S}_1$, $\Lambda (M)$ be the set of the lengths of the above line segments, $\Delta (M)$ be the set of their directions and

$$l\ (M) = \max \Lambda\ (M)\ .$$

Theorem 24 [36]. *For a typical starshaped set* M , $\Lambda (M)$ *is dense in* $[0, l\ (M)]$ *and* $\Delta (M)$ *is dense in* S^{d-1} .

These results admit analogs for $\mathcal{S}_n$.

Consider now the subspace $\mathcal{S}_n^*$ of all n-starshaped sets. the kernels of which include a given convex body B. Clearly, no set in $\mathcal{S}_n^*$ is nowhere dense

because it includes B. Also, sets in $\mathcal{S}_1^*$ are not even outside B nowhere dense, except B itself. However, for $n \geqslant 2$ the situation changes.

Theorem 25 [36]. *Most sets in* $\mathcal{S}_n^*$ *are nowhere dense outside* B , *for* $n \geqslant 2$.

The case $n = 1$ is interesting from other points of view. We treat it in the next section.

Starshaped surfaces

Clearly, the boundary of any set in $\mathcal{S}_1^*$ is homeomorphic to the $(d-1)$-dimensional sphere. We call it a *starshaped surface*. In polar coordinates it is represented by a Lipschitz function; hence it is differentiable a.e.

Let $\mathcal{S}$ be the Baire space of all starshaped surfaces. We say that a point $x \in S$, where $S = \mathrm{bd}\ M$ and $M \in \mathcal{S}_1^*$, *sees only* B (B is the given convex body which is included in the kernel of every member of $\mathcal{S}_1^*$) , if for each line segment $xy \subset M$, there exists a point $z \in B$ collinear with x and y .

The next two theorems on typical starshaped surfaces put in evidence, once again, curious surfaces, the existence of which was ignored before.

Theorem 26 [37]. *For most surfaces* $S \in \mathcal{S}$, *most points of* S *see only* B .

Thus, most starshaped surfaces are at most points nondifferentiable.

Let P_x be the tangent hyperplane at the smooth point x of $S \in \mathcal{S}$.

Theorem 27 [37]. *For most surfaces* $S \in \mathcal{S}$, *the hyperplane* P_x *exists and supports* B *for almost all points* $x \in S$..

It follows that most sets in $\mathcal{S}_1^*$ have precisely B as kernel. Since the convex body B was choosen arbitrarily, this places in a new light an old question of L. Fejes-Tóth, whether every compact convex body is the kernel of a nonconvex set. Constructive answers to Fejes-Tóth's question were given in the plane by K. Post [19], in Banach spaces by V. Klee [10] and, independently, in Euclidean spaces by M. Breen [4].

If the kernels of starshaped sets (from $\mathcal{S}_1$) are supposed to include a given k-dimensional compact convex set, then we get a new Baire space $\mathcal{S}^{(k)}$ of starshaped sets. In particular, $\mathcal{S}^{(d)} = \mathcal{S}_1^*$.

For a description of typical sets in $\mathcal{S}^{(k)}$ the interested reader should consult [37]. We mention here only the case $k = d-1$. We do so, because in this case the aspect of the typical sets in surprisingly nonpathological! Indeed, while a set in $\mathcal{S}^{(d-1)}$ may be locally disconnected and different from the closure of its interior, the following holds.

Theorem 28 [37]. *Most sets in* $\mathcal{S}^{(d-1)}$ *are homeomorphic to a ball.*

Convex sets of convex sets

Another field, not yet intensively explored, is the convexity in the space $\mathcal{K}$ of all compact convex sets of $\mathbb{R}^d$.

Let $A, B \in \mathcal{K}$. Then

$$\{\lambda A + (1-\lambda)B : \lambda \in [0, 1]\}$$

is called the *segment of endsets* A, B. A set $\mathcal{A} \subset \mathcal{K}$ is said to be *convex* if, for any two sets $A, B \in \mathcal{A}$, the segment of endsets A, B lies in $\mathcal{A}$.

Let $\mathfrak{C}$ be the space of all convex closed bounded sets (not elements!) of $\mathcal{K}$. These notions are considered with respect to the Hausdorff distance in $\mathcal{K}$. We equip $\mathfrak{C}$ with Hausdorff's metric too. Since $\mathcal{K}$ is complete, the space $2^{\mathcal{K}}$ of all closed bounded sets in $\mathcal{K}$ is also complete. Also $\mathfrak{C}$, being closed in $2^{\mathcal{K}}$, is complete and therefore a Baire space, by Baire's theorem.

We look now for a description of typical elements in $\mathfrak{C}$ and present here one result in this direction.

Theorem 29. *Most elements of* $\mathfrak{C}$ *are nowhere dense in* $\mathcal{K}$.

For this theorem, which is new, we give a proof in the next section.

Many questions regarding typical elements of $\mathfrak{C}$ can be raised. We state here only two, related to the extremal structure of elements in $\mathfrak{C}$.

For $\mathcal{A} \in \mathfrak{C}$, $A \in \mathcal{A}$ is called an *extreme element* of $\mathcal{A}$ if A belongs only as an endset to segments in $\mathcal{A}$. Let ext $\mathcal{A}$ be the set of all extreme elements of $\mathcal{A}$.

Problem 7. *Prove (or disprove) that, for most* $\mathcal{A} \in \mathfrak{C}$, *ext* $\mathcal{A}$ *is arcwise connected.*

Conjecture. *Most elements of most* $\mathcal{A} \in \mathfrak{C}$ *lie in ext* $\mathcal{A}$.

Two proofs

Proof of Theorem 5. Let $\mathcal{C}^+$ be the space of all smooth convex surfaces in $\mathbb{R}^d$.

A ball B is called *internally tangent* to $S \in \mathcal{C}^+$ at $x \in S$ if bd B contains x, both S and bd B have a common tangent hyperplane H at x and both lie on the same side of H.

$S \in \mathcal{C}^+$ is called *ε-round* at x if, for some tangent direction τ at x, the normal section $S(\tau)$ of S in direction τ, which is a convex arc, is disjoint from the interior of the ball of radius ϵ internally tangent to S at x. $S \in \mathcal{C}^+$ is called *ε-cornered* at x if, for some tangent direction at x, $S(\tau)$ lies within some ball of radius ϵ internally tangent to S at x.

Let $\mathcal{C}^* \subset \mathcal{C}^+$ be the set of all surfaces in $\mathcal{C}^+$, for which the set of all points x with a tangent direction τ verifying

$$\gamma_i^\tau(x) > 0 \quad \text{or} \quad \gamma_s^\tau(x) < \infty$$

is of second category. Then

$$\mathcal{C}^* = \{S \in \mathcal{C}^+ : \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \text{ is of second category}\},$$

where A_n is the set of all n^{-1}-round points and B_n the set of all n^{-1}-cornered points of S. Putting

$$\mathcal{C}_n^1 = \{S \in \mathcal{C}^+ : A_n \text{ is not nowhere dense}\}$$

$$\mathcal{C}_n^2 = \{S \in \mathcal{C}^+ : B_n \text{ is not nowhere dense}\},$$

we have

$$\mathcal{C}^* \subset \bigcup_{n=1}^{\infty} \mathcal{C}_n^1 \cup \bigcup_{n=1}^{\infty} \mathcal{C}_n^2 .$$

By Theorem 1, most elements of $\mathcal{C}$ lie in $\mathcal{C}^+$. Thus, we have only to show that $\mathcal{C}_n^1$ and $\mathcal{C}_n^2$ are of first category in $\mathcal{C}^+$.

Since we work in $\mathscr{C}^+$, the argument in [24], p. 137 proves that on each $S \in \mathscr{C}^+$, A_n and B_n are closed. Hence, if $S \in \mathscr{C}_n^1$, then there is in S a circular disk (with respect to the intrinsic metric of S) included in A_n. Let $\mathscr{C}_{n,m}^1$ be the subset on $\mathscr{C}_n^1$ consisting of all S for which A_n includes a disk of radius m^{-1}. Then

$$\mathscr{C}_n^1 = \bigcup_{m=1}^{\infty} \mathscr{C}_{n,m}^1 .$$

The same argument from [24], p. 137 provides a proof of the fact that $\mathscr{C}_{n,m}^1$ is closed, for any $n, m \in \mathbb{N}$. Analogously, $\mathscr{C}_n^2$ is a countable union of closed sets $\mathscr{C}_{n,m}^2$, which consist of all $S \in \mathscr{C}_n^2$ for which B_n includes a disk of radius m^{-1}.

To see that $\mathscr{C}^+ - \mathscr{C}_{n,m}^1$ and $\mathscr{C}^+ - \mathscr{C}_{n,m}^2$ are dense, take an arbitrary open set $\mathcal{O} \subset \mathscr{C}$, a polytopal surface $P \in \mathcal{O}$ with faces of diameter smaller than m^{-1}, and the boundary Z of

$$\operatorname{conv} P + \epsilon \operatorname{conv} S^{d-1} ,$$

where $\epsilon < n^{-1}$. Then, for ϵ small enough,

$$Z \in \mathcal{O} \cap \mathscr{C}^+ \text{ and } Z \notin \mathscr{C}_{n,m}^1 \cup \mathscr{C}_{n,m}^2 .$$

Hence $\mathscr{C}_{n,m}^1$ and $\mathscr{C}_{n,m}^2$ are nowhere dense in $\mathscr{C}^+$. Therefore, $\mathscr{C}_n^1$ and $\mathscr{C}_n^2$ are, for every $n \in \mathbb{N}$, of first category and the theorem is proved.

Proof of Theorem 29. Let δ and Δ denote the Hausdorff metrics in $\mathscr{K}$ and $\mathfrak{C}$, respectively.

We show that the set $\mathfrak{F}_\epsilon$ of all elements of $\mathfrak{C}$ including, as sets of $\mathscr{K}$, a ball of radius ϵ is nowhere dense.

Let $\mathfrak{O}$ be open in $\mathfrak{C}$. Take $\mathscr{A} \in \mathfrak{O}$. Consider $A = \cup \mathscr{A}$ and $B \subset \alpha \mathbb{Z}^d$ such that $\delta(A, B) < \alpha$. Clearly, for each $A_i \in \mathscr{A}$, there is a $B_i \subset B$ such that $\delta(A_i, B_i) < \alpha$. Since B is finite, the family of all B_i's obtained in this way is finite. Since each A_i is convex,

$$\delta(A_i, \operatorname{conv} B_i) < \alpha$$

too. Let $\{P_1, \ldots, P_m\}$ be the family of these sets conv B_i. Putting

$$\mathscr{P} = \{ \sum_{i=1}^{m} \lambda_i P_i : 0 \leqslant \lambda_i \leqslant 1, \sum_{i=1}^{m} \lambda_i = 1 \} ,$$

we have

$$\Delta(\mathscr{A}, \mathscr{P}) < \alpha .$$

For α small enough, $\mathscr{P} \in \mathfrak{O}$. For each $P \in \mathscr{P}$, let

$$\beta_P = \inf \{\lambda : \quad Q \in \mathscr{C}^+ \text{ with } \delta(P, Q) = \lambda \text{ and } \gamma_i(\operatorname{bd} Q) \geqslant \epsilon\},$$

where $\gamma_i(\operatorname{bd} Q) \geqslant \epsilon$ means that $\gamma_i^\tau(x) \geqslant \epsilon$ for all points $x \in \operatorname{bd} Q$ and tangent directions τ at x. Clearly,

$$\beta = \inf_{P \in \mathscr{P}} \beta_P \neq 0 .$$

Choose

$$\mathfrak{A} = \{ \mathscr{Q} : \Delta(\mathscr{P}, \mathscr{Q}) < \beta/2 \} .$$

No element of $\mathfrak{A}$ contains the parallel body $K + \epsilon \operatorname{conv} S^{d-1}$ of any $K \in \mathscr{K}$, because $\gamma_i^\tau(x) < \epsilon$ at some point $x \in \operatorname{bd} Q$ and some tangent direction τ at x, for every $Q \in \mathscr{Q}$ and $\mathscr{Q} \in \mathfrak{A}$. Hence no element of $\mathfrak{A}$ includes an entire ball of $\mathscr{K}$ of radius ϵ. Since $\mathfrak{O} \cap \mathfrak{A}$ is open, $\mathfrak{F}_\epsilon$ is nowhere dense.

Every set $\mathfrak{F}_{n^{-1}}$ being nowhere dense,

$$\bigcup_{n=1}^{\infty} \mathfrak{F}_{n^{-1}}$$

is of first category. Hence, most elements of $\mathfrak{C}$ belong to

$$\bigcap_{n=1}^{\infty} \complement \mathfrak{F}_{n^{-1}} ,$$

i.e. have empty interior. Being closed, they are nowhere dense, q.e.d.

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