Mh. Math. 103, 57–62 (1987)

Monatshefte für
Mathematik

Nearly All Convex Bodies Are Smooth and Strictly Convex

By

Tudor Zamfirescu, Dortmund

(Received 11 July 1985; in revised form 22 November 1985)

Abstract. By an old result of Klee, those convex bodies which are not smooth or not strictly convex form a set of first Baire category. It is proved here that they are "even fewer": they only form a σ-porous set.

In 1959, V. KLEE [4] published probably the first applications of the Baire category theorem to Convex Geometry. His results state that most convex bodies are smooth and strictly convex. 18 years later, P. GRUBER [2] rediscovered these theorems and added some new ones, for instance that most convex surfaces are not of class C^2. Since then several further results on "most" (which means "all, except those in a set of first Baire category") convex bodies or surfaces have been found (see for example [5], [6], [7]). The reader may also consult the survey articles [3], [8].

In this paper we shall use sharper notions than the purely topological notions "nowhere dense" and "of first category". These are the notions "porous" and "σ-porous", respectively.

In a metric space (X, ϱ), a set M is called *porous* if there is an $\alpha > 0$ such that, for every $x \in X$ and any ball $B(x, \varepsilon)$ with centre x and radius ε, there exists a point y in $B(x, \varepsilon)$ such that

$$B(y, \alpha\varrho(x, y)) \cap M = \emptyset .$$

A set is then called *σ-porous* if it is a countable union of porous sets. We say the *nearly all* elements of a Baire metric space have a certain property, if those not enjoying it form a σ-porous set.

Clearly, if nearly all elements of a Baire metric space enjoy some property, so do most elements of that space. But, in contrast to sets of first category, any σ-porous set in $\mathbb{R}^d$ has Lebesgue measure zero. This follows immediately from Lebesgue's density theorem. Thus, sets containing nearly all elements of a Baire subspace of $\mathbb{R}^d$ are necessarily large from both topological and measure-theoretical point of view.

Remark 1. A set in the metric space (X, ϱ) is σ-porous if it is a countable union of sets M_i such that for every $x \in X$ there exists $\alpha > 0$ with the property that in each neighbourhood of x there is a point y verifying

$$B(y, \alpha\varrho(x, y)) \cap M_i = \emptyset .$$

We shall look again at the first two results on most convex bodies and prove that they are actually true not only for most, but even for nearly all of them.

Throughout the paper $\mathscr{C}$ will denote the space of all convex bodies, $\mathscr{A}$ the set of all strictly convex bodies and $\mathscr{B}$ the set of all smooth convex bodies, in $\mathbb{R}^d$. Endowed with the Hausdorff distance δ, $\mathscr{C}$ is a Baire space. Also, let B_1 be the closed unit ball in $\mathbb{R}^d$.

In the proofs we shall make use of the following simple observations.

Remark 2. Consider two concentric circles of radii $r + \gamma$ and $r - \gamma$ $(0 < \gamma < r)$. Then

a) the angle subtended by the smaller circle at any point between or on the circles is more than 2β if and only if

$$\gamma < r(1 - \sin\beta)\,(1 + \sin\beta)^{-1} .$$

b) there is no line segment of length k lying between (possibly touching) the circles if and only if $\gamma < 2^{-4} r^{-1} k^2$.

Smoothness

Theorem 1. *Nearly all convex bodies are smooth.*

Proof. Let $\mathscr{B}_n$ be the set of all convex bodies which admit at a certain point two supporting hyperplanes, the (smaller) angle between which is n^{-1}.

Let $C \in \mathscr{C}$ and $\xi > 0$. Consider the convex body $C_\xi = C + \xi B_1$; clearly, $\delta(C, C_\xi) = \xi$. Let

$$\gamma_\xi = \frac{1 - \cos(2n)^{-1}}{2 + \cos(2n)^{-1}}\xi .$$

The boundary of every convex body T, whose Hausdorff distance from C_ξ is at most γ_ξ, lies between the two parallel surfaces S_1, S_2 (with $S_1 \subset \operatorname{conv} S_2$) at distance γ_ξ from $\operatorname{bd} C_\xi$, possibly touching them. Suppose such a T belongs to $\mathscr{B}_n$. Then, at a certain point $v \in T$, there are two supporting hyperplanes, the (smaller) angle between which is n^{-1}. Suppose now $d \geqslant 3$ (the planar case will then be obvious). Take a 2-dimensional flat Π orthogonal to both these hyperplanes. Let $p: \mathbb{R}^d \to \Pi$ be the orthogonal projection onto Π. Then $\operatorname{bd} p(T)$ is a convex subtending an angle at most $\pi - n^{-1}$ at $p(v)$; also

$$p(S_1) \subset p(T) \subset p(S_2)\,.$$

Let v' be the closest point of $p(S_1)$ to $p(v)$. Since S_1 has at every point an internally tangent sphere of radius $\xi - \gamma_\xi$, $\operatorname{bd} p(S_1)$ has an internally tangent circle of the same radius at v'. Then this circle subtends an angle at most $\pi - n^{-1}$ at the point $p(v)$, which lies on or inside another concentric circle of radius $\xi + \gamma_\xi$. By Remark 2 a), this yields

$$\gamma_\xi \geqslant \frac{1 - \sin\left(\dfrac{\pi - n^{-1}}{2}\right)}{1 + \sin\left(\dfrac{\pi - n^{-1}}{2}\right)}\,\xi\,,$$

which contradicts our choice of γ_ξ.

Hence no $T \in \mathscr{C}$ with $\delta(T, C_\xi) \leqslant \gamma_\xi$ belongs to $\mathscr{B}_n$. Thus the set $\mathscr{B}_n$ is porous. Since

$$\mathscr{C} \setminus \mathscr{B} = \bigcup_n \mathscr{B}_n,$$

the theorem is proved.

Strict Convexity

Lemma. *Let $Q \subset \mathbb{R}^d$ be a finite intersection of balls of radii at most r and let $\gamma > 0$. Consider the following inner and outer parallel sets:*

$$Q_1 = \{x \in Q : \forall\, y \in \operatorname{bd} Q, \|x - y\| > \gamma\}\,,$$

$$Q_2 = Q + \gamma B_1\,.$$

If $\gamma < 2^{-4} r^{-1} k^2$, then $Q_2 \setminus Q_1$ contains no line segment of length k.

Proof. Let $d \geqslant 3$ (the planar case is similar and simpler). Suppose s is a line segment of length k in $Q_2 \setminus Q_1$. Let Ξ be a hyperplane containing s and supporting $\overline{Q_1}$, the closure of Q_1, at some point a. Let O be the ball of radius $r - \gamma$ tangent to Ξ at a and lying in the halfspace with boundary Ξ, which contains Q_1. An easy geometric argument shows that

$$O + \gamma B_1 \supset Q \cap \Xi .$$

If follows that

$$O + 2\gamma B_1 \supset Q_2 \cap \Xi .$$

Hence

$$s \subset \overline{(O + 2\gamma B_1) \setminus O} ,$$

which contradicts Remark 2b).

Theorem 2. *Nearly all convex bodies are strictly convex.*

Proof. Let $\mathscr{A}_n$ be the set of all convex bodies containing a line segment of length n^{-1} in their boundaries. We use Remark 1.

Let $C \in \mathscr{C}$ and $\varepsilon \in (0, 1)$. Choose $C^* \in \mathscr{B}$ and a polytope P whose facets are tangent to C^*, such that $\delta(C, C^*) < \varepsilon/4$, $\delta(C^*, P) < \varepsilon/4$ and $\operatorname{diam} \nu(v) < 1$ for every vertex v of P, ν denoting the "spherical image" (see [1], p. 25).

Now choose for every facet F of P a point $p \in F \cap C^*$ and a point $q_p \in \mathbb{R}^d$ such that:

i) the hyperplane $Y \supset F$ separates q_p from the interior of C^*,

ii) $p - q_p$ and F are orthogonal,

iii) $\xi = \|p - q_p\|$ does not depend on p,

iv) $\delta(\bigcap_p H_p, P) = \varepsilon/2$, where H_p denotes the closed halfspace containing Y with q_p as a boundary point.

Clearly, $\xi < \varepsilon/2 < 1/2$. Consider the smallest ball K_p tangent at q_p to $\operatorname{bd} H_p$ which contains C^*. Let $L = \bigcap_p K_p$. Since $C^* \subset L \subset \bigcap_p H_p$,

$$\delta(C^*, L) \leqslant \delta(C^*, \bigcap_p H_p) \leqslant \delta(C^*, P) + \delta(P, \bigcap_p H_p) < 3\varepsilon/4$$

and

$$\delta(C, L) \leqslant \delta(C, C^*) + \delta(C^*, L) < \varepsilon .$$

Let Δ be the diameter of some ball K containing $C + B_1$. Then $\operatorname{diam}(K \cap Y) \leqslant \Delta$. Any ball tangent to $\operatorname{bd} H_p$ at q_p and containing $K \cap Y$ includes C^*, because K contains C^* and q_p. Since $\delta(\{p\}, K \cap Y) \leqslant \Delta$, the radius r_p of K_p satisfies

$$r_p \leqslant \frac{\xi^2 + \Delta^2}{2\xi} < \frac{2^{-2} + \Delta^2}{2\xi}.$$

We claim that

$$B\left(L, \frac{\cos 1}{2^4 n^2 (2^{-2} + \Delta^2)} \cdot \delta(C, L)\right) \cap \mathscr{A}_n = \emptyset .$$

First observe that

$$\delta\Big(\bigcap_p H_p, P\Big) = \xi / \cos \alpha ,$$

α being the distance between two certain points of $\nu(v)$, for some vertex v of P. Thus, $\xi/\cos\alpha = \varepsilon/2$ and

$$\delta(C, L) < \varepsilon = \frac{2\xi}{\cos\alpha} < \frac{2\xi}{\cos 1} .$$

Therefore,

$$\frac{\cos 1}{2^4 n^2 (2^{-2} + \Delta^2)} \cdot \delta(C, L) < \frac{2\xi}{2^4 n^2 (2^{-2} + \Delta^2)} = 2^{-4} \left(\frac{2^{-2} + \Delta^2}{2\xi}\right)^{-1} n^{-2}$$

and according to the Lemma, no convex body at Hausdorff distance at most

$$\frac{\cos 1}{2^4 n^2 (2^{-2} + \Delta^2)} \cdot \delta(C, L)$$

from L contains a line segment of length n^{-1} in its boundary.

Hence, by Remark 1, $\bigcup_n \mathscr{A}_n$ is σ-porous, which proves Theorem 2.

Acknowledgement: Thanks are due to the referee for his helpful comments.

References

[1] BUSEMANN, H.: Convex Surfaces. New York: Interscience. 1958.

[2] GRUBER, P. M.: Die meisten konvexen Körper sind glatt, aber nicht zu glatt. Math. Ann. **229**, 259—266 (1977).

[3] GRUBER, P. M.: Results of Baire Category Type in Convexity. In: Discrete Geometry and Convexity. Ann. New York Acad. Sci. **440**, 163—169 (1985).

[4] KLEE, V.: Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. **139**, 51—63 (1959).

[5] ZAMFIRESCU, T.: The curvature of most convex surfaces vanishes almost everywhere. Math. Z. **174**, 135—139 (1980).

[6] ZAMFIRESCU, T.: Many endpoints and few interior points of geodesics. Invent. Math. **69**, 253—257 (1982).

[7] ZAMFIRESCU, T.: Points on infinitely many normals to convex surfaces. J. Reine Angew. Math. **350**, 183—187 (1984).

[8] ZAMFIRESCU, T.: Using Baire categories in geometry. Rend. Sem. Mat. Univ. Politecn. Torino **43**, 67—88 (1985).

TUDOR ZAMFIRESCU
Fachbereich Mathematik
Universität Dortmund
D-466 Dortmund, Bundesrepublik Deutschland
and
Department of Mathematics and Computer Science
California State University
Los Angeles, CA 90032, U.S.A.