Mh. Math. 103, 241–247 (1987)

Monatshefte für
Mathematik

Typical Convex Curves on Convex Surfaces

By

Tudor Zamfirescu*, Dortmund

(Received 26 September 1985; in revised form 23 April 1986)

Abstract. In the sense of Baire categories, most convex curves on a smooth two-dimensional closed convex surface are smooth. Moreover, if the set of all closed geodesics has empty interior in the space of all convex curves, then most convex curves are strictly convex.

In Geometric Convexity the underlying space is usually $\mathbb{R}^d$. One possible natural generalization which entirely keeps the geometric character consists in replacing $\mathbb{R}^d$ by an arbitrary convex surface $\mathbb{S}$ in $\mathbb{R}^{d+1}$, which is by definition the boundary of an open convex set, different from the union of two parallel hyperplanes [2]. A shortest path joining two points of $\mathbb{S}$ is called a *segment*, and a set in $\mathbb{S}$ is said to be *convex* if, for any two of its points, it also contains a segment joining those points.

For $d = 2$, a closed Jordan curve in $\mathbb{S}$ is called

i) *convex* if it is the boundary of a convex set in $\mathbb{S}$,

ii) a *closed geodesic* if each of its points has a neighbourhood in $\mathbb{S}$ whose intersection with the curve is a segment.

Obviously, a closed Jordan curve is convex if and only if it is the boundary of a "konvexer Bereich" in Aleksandrov's terminology [1].

In a Baire space, we say that "most" elements enjoy a certain property if all those which do not enjoy it form a set of first Baire category. Then we also say that a "typical" element of the space enjoys that property.

Klee [4] initiated the study of typical convex sets. In several other papers, for example in [3], [5]—[9] properties of typical convex surfaces in $\mathbb{R}^d$ are described (see the recent survey article [10]).

* This paper was written during the author's visit at Western Washington University, whose substantial support is acknowledged.

General problem: Extend the description of typical convex surfaces, as far as possible, to the case of an arbitrary d-dimensional convex surface $\mathbb{S}$ as underlying space instead of $\mathbb{R}^d$.

While the definition of a convex surface in $\mathbb{S}$ can be formulated in a straightforward way for any dimension d of $\mathbb{S}$, a (not differential-geometric) theory does not seem to exist except for $d = 2$. Thus, we restrict our study here to typical convex curves on an arbitrary convex surface $\mathbb{S} \subset \mathbb{R}^3$.

The space $\mathscr{C}$ of all convex curves on $\mathbb{S}$ is a Baire space. Indeed, if we add to $\mathscr{C}$ the set $\mathscr{S}$ of all segments on $\mathbb{S}$ and the set $\mathscr{P}$ of all singletons of $\mathbb{S}$, we get a set which is closed in the space of all compact subsets of $\mathbb{S}$. This nontrivial statement essentially follows from the closedness of the space of all compact convex sets on $\mathbb{S}$, which in turn follows from Theorems 10.5, 10.5′ and 11.3 in [2]. Thus $\mathscr{C} \cup \mathscr{S} \cup \mathscr{P}$ is a complete metric space S, hence a Baire space. Since $\mathscr{S} \cup \mathscr{P}$ is clearly nowhere dense in S, $\mathscr{C}$ is a Baire space too.

We shall assume the knowledge of a few very simple facts about segments and geodesics, all of which can be found for instance in [1].

A Lemma. Let ϱ be the intrinsic metric on $\mathbb{S}$ and δ Hausdorff's metric for compact subsets of $\mathbb{S}$ (derived from ϱ).

A finite union of segments

$$u_0 u_1 \cup u_1 u_2 \cup \ldots \cup u_{m-1} u_m \cup u_m u_0$$

is called a *polygon*, $u_0, u_1, \ldots, u_m$ are its *vertices* and the above segments are its *edges*. This polygon is denoted by $u_0 u_1 \ldots u_m u_0$.

A curve $C \subset \mathbb{S}$ is said to *possess an angle* in $x \in C$ if, letting $y \to x$ from one side, the halfline $[x, y) \subset \mathbb{R}^3$ having an endpoint at x and passing through y converges to a halfline H_x^- and, letting $y \to x$ from the other side, $[x, y)$ converges to a halfline H_x^+. Then $H_x^- \cup H_x^+$ is called the *angle* of C at x.

We also recall that the *tangent cone* of $\mathbb{S}$ at $x \in \mathbb{S}$ is the union of all halflines obtained as limits of $[x, y)$ when $y \to x$ on $\mathbb{S}$ in all possible ways.

The angle of any curve on $\mathbb{S}$ at a certain point x will always be measured on the tangent cone of $\mathbb{S}$ at x.

It is well known that the measure of the angle of any geodesic of $\mathbb{S}$ at every point different from an endpoint is π.

Lemma. *If C is a closed Jordan curve on $\mathbb{S}$ and $\varepsilon > 0$, then there exists a polygon P with vertices on C such that*

$$\delta(P, C) < \varepsilon .$$

If C is not a closed geodesic, then P may be chosen so as not to be a closed geodesic. If $C \in \mathscr{C}$, then we may choose $P \in \mathscr{C}$ too.

Proof. Let O_x be the open disk of radius $\varepsilon/2$ around any $x \in C$ (always with respect to ϱ). Consider a homeomorphism h between $\mathbb{S}$ and the unit sphere S_2 in $\mathbb{R}^3$, carrying C onto the great circle S_1 of S_2. In each $h(O_x)$ choose an open disk D_x centered in $h(x)$. Then $\{h^{-1}(D_x) : x \in C\}$ is an open covering of C. Choose a finite subcovering $\{h^{-1}(D_{x_i})\}_{i=1}^{k}$. Suppose this one possesses no proper subcovering. Also, suppose the points $h(x_1), \ldots, h(x_K)$ lie in this order on S_1. Choose

$$x_i' \in S_1 \cap D_{x_i} \cap D_{x_{i+1}} \quad (i = 1, \ldots, k-1) ,$$

$$x_k' \in S_1 \cap D_{x_k} \cap D_{x_1} .$$

Consider the points

$$y_i = h^{-1}(x_i') \quad (i = 1, \ldots, k) .$$

Clearly,

$$\varrho(y_i, y_{i+1}) < \varepsilon \quad (i = 1, \ldots, k-1) ,$$

$$\varrho(y_k, y_1) < \varepsilon,$$

because

$$y_i, y_{i+1} \in h^{-1}(D_{x_i+1}) \subset O_{x_{i+1}} \quad (i = 1, \ldots, k-1) ,$$

$$y_k, y_1 \in h^{-1}(D_{x_1}) \subset O_{x_1} .$$

Hence each point of the polygon $P = y_1 y_2 \ldots y_k y_1$ is at distance less than $\varepsilon/2$ from some vertex of P and therefore at distance less than $\varepsilon/2$ from C. Each point of C lies in some O_{x_i} and therefore has distance less than ε from y_i, hence its distance from P is also less than ε. Consequently, $\delta(C, P) < \varepsilon$.

Suppose C is not a closed geodesic, but the polygon P happens to be one. Then there are two vertices v_1, v_2 of P such that the segment $v_1 v_2$ on P is different from the corresponding arc a on C. Consider the point $v \in v_1 v_2$ which is nearest to v_2 such that $v v_1 \subset a$ (thus v may well be v_1). Now consider a point $v' \in a - v_1 v_2$. If v' is chosen sufficiently close to v, then the polygon P' obtained from P by replacing $v_1 v_2$ with

$v_1 v' \cup v' v_2$ still has Hausdorff distance less than ε from C and has at v_1 and v_2 angles of measures different from π.

We show now that $P = y_1 y_2 \dots y_k y_1$ can be chosen in $\mathscr{C}$ if $C \in \mathscr{C}$. Until now no restriction was imposed on the choice of the segments $y_1 y_2, \dots, y_k y_1$. Thus, they may be chosen in the closed convex set M the boundary of which is C. We claim that the closure N of the open connected subset of M with boundary P is convex. If x, y are consecutive vertices of P, then that edge of P lies in N. We shall prove that for all other $x, y \in N$, any segment xy chosen to lie in M must also lie in N. Choose in this way the segment xy and suppose $xy \not\subset N$. Let $v, w \in xy$ be such that for the subsegment vw of xy,

$$vw \cap N = \{v, w\} .$$

If v, w belong to the same edge of P, then vw lies on that edge, which is in N; this is impossible. If v, w lie on different edges of P and are boundary points of $M - N$, then they obviously belong to different components of $(M - N) \cup \{v, w\}$. Hence $vw - \{v, w\} \not\subset M - N$, again a contradiction.

Thus N is convex and the polygon P is a convex curve.

Smoothness of Most Convex Curves. Results of KLEE [4] and GRUBER [3] show that most convex surfaces in $\mathbb{R}^d$ are smooth and strictly convex. The proper analogue of smoothness for curves on $\mathbb{S}$ is the following:

The closed Jordan curve $C \subset \mathbb{S}$ is *smooth* at $x \in C$ if C possesses an angle in x and this angle, measured on the tangent cone of $\mathbb{S}$ at x, equals π.

That a convex curve $C \subset \mathbb{S}$ possesses an angle at each of its points is an immediate consequence of its convexity. On the other hand this smoothness of C at some point does not automatically imply its differentiability there.

Theorem 1. *Most convex curves in $\mathbb{S}$ are smooth.*

Proof. Let $\mathscr{A} \subset \mathscr{C}$ be the set of all convex curves which are not smooth and $\mathscr{A}_n \subset \mathscr{A}$ the set of all curves possessing a point x where the angle is at most $\pi - n^{-1}$. Clearly,

$$\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n$$

and it suffices to show that $\mathscr{A}_n$ is nowhere dense in $\mathscr{C}$ for each $n \in \mathbb{N}$.

$\mathscr{A}_n$ is a closed set. Consider, indeed, a convergent sequence $\{A_m\}_{m=1}^{\infty}$ of curves from $\mathscr{A}_n$, each of which has a point x_m where the tangent halflines τ_m and σ_m form an angle of measure at most $\pi - n^{-1}$. We may suppose (otherwise we find an appropriate subsequence) that $\{x_m\}_{m=1}^{\infty}$ converges to some point x and $\{\tau_m\}_{m=1}^{\infty}$, $\{\sigma_m\}_{m=1}^{\infty}$ to two halflines τ and σ starting at x. It is easily seen that, if $A_m \to A$, then $x \in A$, the angle $\tau \cup \sigma$ contains (in an obvious sense) the tangent angle of A at x and the measure of $\tau \cup \sigma$ does not exceed $\pi - n^{-1}$. Hence $A \in \mathscr{A}_n$.

Now we show that $\mathscr{C} - \mathscr{A}_n$ is dense. Let $\mathcal{O}$ be open in $\mathscr{C}$ and $C \in \mathcal{O}$. By the Lemma, there exists a polygon $P \in \mathcal{O}$ with vertices on C.

Since the set of all conical points of $\mathbb{S}$ (i.e. points where the tangent cones contain no line) is countable, the vertices of P could be chosen not to be conical points.

Let ν_i be (the measure of) the angle of P at y_i and suppose $\nu_i \leqslant \pi - n^{-1}$. Consider two points $z_i \in y_{i-1} y_i$, $z_i' \in y_i y_{i+1}$ such that

$$\varrho(y_i, z_i) = \varrho(y_i, z_i') = \eta\ .$$

If η is small enough, the angles at z_i and z_i' in the triangle $y_i z_i z_i'$ are arbitrarily close to $(\pi - \nu_i)/2$. By replacing the edges $y_{i-1} y_i$ and $y_i y_{i+1}$ of$P$ by $y_{i-1} z_i$, $z_i z_i'$ and $z_i' y_{i+1}$, we get a new convex polygon P', which still belongs to $\mathcal{O}$ if η was small enough. It is clear that, repeating this procedure an appropriate number of times, we eventually obtain a polygon $Q \in \mathcal{O}$ with all its angles larger than $\pi - n^{-1}$. Thus $Q \notin \mathscr{A}_n$, which proves that $\mathscr{C} - \mathscr{A}_n$ is dense.

Being closed and having a dense complement, $\mathscr{A}_n$ is nowhere dense, which achieves the proof.

Strict Convexity. A convex curve is called *strictly convex*, if it contains no segments. Contrary to the Euclidean case, it can not be said, for arbitrary $\mathbb{S}$, that most convex curves are strictly convex! Let $\mathbb{S}$ be the boundary of the convex hull of the union of two congruent distinct balls in $\mathbb{R}^3$. $\mathbb{S}$ consists of two closed half-spheres S and S^* and an (open) ring R. Every circle C in R is a convex curve on $\mathbb{S}$; and the entire ball $\mathscr{B}$ of radius ε around C in $\mathscr{C}$ consists (for ε small enough) of translates of C. Thus the set of second category $\mathscr{B}$ consists of convex curves none of which is strictly convex.

On the other hand, it can not be said that most convex curves on $\mathbb{S}$ are not strictly convex, since for each convex curve $C \subset S$, and every neighbourhood $\mathscr{N}$ of C in $\mathscr{C}$, the set of those curves of $\mathscr{N}$ which are strictly convex is of second category. Moreover, if C lies in the interior of S, then there is a neighbourhood $\mathscr{N}$ of C in $\mathscr{C}$ such that most curves of $\mathscr{N}$ are strictly convex.

Let $\mathscr{G}$ be the set of all closed geodesics of $\mathbb{S}$ belonging to $\mathscr{C}$. As observed above, the interior $\operatorname{int}\mathscr{G}$ of $\mathscr{G}$ in $\mathscr{C}$ may well be nonvoid. In any case, $\mathscr{C} - \operatorname{int}\mathscr{G}$ is a Baire space.

Theorem 2. *Most curves of* $\mathscr{C} - \operatorname{int}\mathscr{G}$ *are strictly convex.*

Proof. Let $\mathscr{D}$ be the set of all curves in $\mathscr{C} - \operatorname{int}\mathscr{G}$ which are not strictly convex and $\mathscr{D}_n \subset \mathscr{D}$ the set of all curves including a segment of length n^{-1}. Since

$$\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}_n ,$$

it suffices to prove that $\mathscr{D}_n$ is nowhere dense in $\mathscr{C} - \operatorname{int}\mathscr{G}$ for each $n \in \mathbb{N}$.

First, it is clear by Theorems 10.5, 10.5′ and 11.3 in [2] that $\mathscr{D}_n$ is closed in $\mathscr{C}$. Hence $\mathscr{D}_n$ is closed in $\mathscr{C} - \operatorname{int}\mathscr{G}$ too.

Let $\mathscr{O} \subset \mathscr{C} - \operatorname{int}\mathscr{G}$ be open and $C \in \mathscr{O} - \mathscr{G}$. Our Lemma provides a polygon $P \in \mathscr{O}$ with vertices on C, which is not a closed geodesic. There is a vertex v of P where P is locally not a segment. Denote by m_1, m_2 the midpoints of the two edges s_1, s_2 meeting at v. Take a point $w \in v m_2$ close to v and a segment $w m_1$. The polygon derived from P, with vertices m_1 and w instead of v, still lies in $\mathscr{O}$, has instead of the edge s_1 two edges of lengths close to half the length of s_1 and is not smooth at m_1 and w. By repeating this procedure sufficiently many times we get a polygon $Q \in \mathscr{O}$ with all its edge lengths less than n^{-1} and nonsmooth at all vertices. Thus Q includes no segment of length n^{-1}, hence $Q \notin \mathscr{D}_n$. It follows that $\mathscr{D}_n$ is nowhere dense and $\mathscr{D}$ of first category in $\mathscr{C} - \operatorname{int}\mathscr{G}$.

References

[1] ALEKSANDROV, A. D.: Die Innere Geometrie der Konvexen Flächen. Berlin: Akademie-Verlag. 1955.

[2] BUSEMANN, H.: Convex Surfaces. New York: Interscience Publ. 1958.

[3] GRUBER, P. M.: Die meisten konvexen Körper sind glatt, aber nicht zu glatt. Math. Ann. **229**, 259—266 (1977).

[4] KLEE, V.: Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. **139**, 51—63 (1959).

[5] ZAMFIRESCU, T.: The curvature of most convex surfaces vanishes almost everywhere. Math. Z. **174**, 135—139 (1980).

[6] ZAMFIRESCU, T.: Nonexistence of curvature in most points of most convex surfaces. Math. Ann. **252**, 217—219 (1980).

[7] ZAMFIRESCU, T.: Inscribed and circumscribed circles to convex curves. Proc. Amer. Math. Soc. **80**, 455—457 (1980).

[8] ZAMFIRESCU, T.: Many endpoints and few interior points of geodesics. Invent. Math. **69**, 253—257 (1982).

[9] ZAMFIRESCU, T.: Points on infinitely many normals to convex surfaces. J. Reine Angew. Math. **350**, 183—187 (1984).

[10] ZAMFIRESCU, T.: Using Baire categories in geometry. Rend. Sem. Mat. Univ. Politec. Torino **43**, 67—88 (1985).

TUDOR ZAMFIRESCU
Fachbereich Mathematik
Universität Dortmund
D-4600 Dortmund
Bundesrepublik Deutschland