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A CHARACTERIZATION THEOREM FOR CERTAIN UNIONS OF TWO STARSHAPED SETS IN R^2

ABSTRACT. Let S be a compact set in R^2. For S simply connected, S is a union of two starshaped sets if and only if for every F finite, $F \subseteq \text{bdry } S$, there exist a set $G \subseteq \text{bdry } S$ arbitrarily close to F and points s, t depending on G such that each point of G is clearly visible via S from one of s, t. In the case where $\sim S$ has at most finitely many components, the necessity of the condition still holds while the sufficiency fails.

1. INTRODUCTION

We begin with some definitions. Let S be a set in R^d. For points x and y in S, we say x *sees* y via S (x is *visible* from y via S) if the corresponding segment $[x, y]$ lies in S. We say x *sees* $Y \subset S$ via S if x sees all points of Y via S; also we say x is *clearly visible* from y via S if there is some neighborhood N of x such that y sees each point of $N \cap S$ via S. The set S is *starshaped* if there is some point p in S such that p sees each point of S via S, and the set of all such points p is called the (convex) *kernel* of S. A compact set in R^d is called *simply connected* if its complement has one component and *finitely connected* if its complement has finitely many components.

A well-known theorem of Krasnosel'skii [2] states that if S is a nonempty compact set in R^d, then S is starshaped if and only if every $d + 1$ points of S are visible via S from a common point. Furthermore, a stronger result may be produced by replacing points of S with boundary points of S. In [1], the concept of clear visibility together with work by Lawrence *et al.* [3] were used to obtain the following Krasnosel'skii-type theorem for unions of two starshaped sets in R^2. Let S be a compact set in R^2, and assume that for each finite set $F \subseteq \text{bdry } S$ there exist points s, t (depending on F) such that every point of F is clearly visible via S from at least one of s, t. Then S is a union of two starshaped sets. By allowing F to vary over S (instead of bdry S), a characterization theorem can be obtained for unions of k compact starshaped sets in a linear topological space. However, a characterization theorem in terms of bdry S has not been produced, even in the plane.

In this paper, the following result will be established. Let S be a compact finitely connected set in R^2. If S is a union of two starshaped sets, then for F finite, $F \subseteq \text{bdry } S$, there exist a finite $G \subseteq \text{bdry } S$ arbitrarily close to F and two points a_G, b_G in S (depending on G) such that every point of G is clearly visible via S from one of a_G, b_G. If, in addition, set S is required to be simply connected, the converse holds as well.

The following terminology will be used throughout the paper: conv S, cl S, int S, and bdry S will denote the convex hull, closure, interior, and

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boundary, respectively, for set S. For distinct points x and y, $L(x, y)$ will be the line through x and y and $R(x, y)$ will be the ray emanating from x through y. For $x \notin S$, cone (x, S) will represent $\bigcup \{R(x, s): s \in S\}$. Finally, $D_\varepsilon(p)$ will denote the set of all points at distance at most ε from p in R^2. The reader is referred to Valentine [6] and to Lay [4] for a discussion of these concepts and to Nadler [5] for information on the Hausdorff metric.

2. Main Result

THEOREM 1. *Let S be a compact, finitely connected set in R^2. Assume that S is a union of two starshaped sets at a, b in R^2. Then for F finite, $F \subseteq$ bdry S, there is a finite $G \subseteq$ bdry S arbitrarily close to F (in the sense of the Hausdorff metric) such that each point of G is clearly visible via S from one of a, b.*

Proof. Let $x \in F$ and let N be any spherical neighborhood of x. We must find some point y in $N \cap$ bdry S with y clearly visible via S from a or b. In the case where there is some point y_0 of $N \cap$ bdry S not visible from one of a or b (say a), then there is a neighborhood N' of y_0, $N' \subseteq N$, such that no point of $N' \cap S$ sees a via S. Then b sees via S each point of $N' \cap S$, and y_0 is clearly visible from b. Thus the theorem is satisfied.

Therefore, for the remainder of the argument we may assume that no such y_0 exists. Then for each y in $N \cap$ bdry S, both a and b see y via S. If $N \cap S \subseteq$ bdry S, then point x itself satisfies the theorem. Hence we assume that N meets int S. Moreover, if a sees $N \cap$ int S via S or if b sees $N \cap$ int S via S, again we are through, so assume that neither one occurs.

For the moment, let us suppose that points a, b, x are distinct. Then without loss of generality we may assume that N has been chosen so that a, $b \notin$ cl N.

The following lemma will be useful.

LEMMA 1. *If $s \in N \cap$ int S and s is not visible via S from b, then there is a neighborhood M_s of s, $M_s \subseteq N \cap$ int S, such that a sees via S all of cone $(b, M_s) \cap N$.*

To prove the lemma, choose a disk $M_s \subseteq N \cap$ int S such that b sees via S no point of M_s. Then b sees via S no point of $V \equiv (\text{cone}(b, M_s) \sim \text{conv}(b \cup M_s)) \cap N$, so no point of V is in bdry S. Likewise, no point of V is in $\sim S$, since this would force a boundary point of S in the convex set $V \cup M_s$, impossible. Hence $V \subseteq$ int S, so $V \cup M_s \subseteq$ int S. Since b sees via S no point of $V \cup M_s$, a must see via S all points of $V \cup M_s$.

Likewise, examine rays $R(t, b)$ for t in M_s. Since b sees via S no point of M_s, each ray $R(t, b)$ meets bdry S at a first point $t' \in (t, b)$. Moreover, since b

sees via S all points of (bdry $S) \cap N$, $t' \notin N$. This implies that $(b, t) \cap N \subseteq$ int S and conv$(b, M_s) \cap N \subseteq$ int S. Combining this with our earlier result, cone$(b, M_s) \cap N \subseteq$ int S. Clearly, b sees via S no point of cone$(b, M_s) \cap N$, so a sees all these points via S. The lemma is proved.

To finish this portion of the proof, there are three cases to examine.

Case 1. Assume that the points a, b, x are noncollinear. Let rays $R(a, x) \sim [a, x)$ and $R(b, x) \sim [b, x)$ meet bdry N at points a_0 and b_0, respectively. Label the open halfplanes determined by lines $L(a, b_0)$, $L(b, a_0)$, $L(b, x)$, $L(a, x)$ so that $x \in L(a, b_0)_+ \cap L(b, a_0)_+$, $a \in L(b, x)_+$, $b \in L(a, x)_+$. Define $N_0 = N \cap L(a, b_0)_+ \cap L(b, a_0)_+$. (See Figure 1.)

We assert that point b sees via S all points of $N_0 \cap$ int $S \cap \text{cl}(L(b, x)_-)$. To show this, assume on the contrary that for some $s \in N_0 \cap \text{int } S \cap \text{cl}(L(b, x)_-)$, $[b, s] \nsubseteq S$. Then by Lemma 1, there is a neighborhood M_s of s, $M_s \subseteq N \cap \text{int } S$, such that a sees via S cone$(b, M_s) \cap N$. Select a point $t \in N_0 \cap M_s \cap L(b, x)_-$. Then $R(b, t) \cap (x, a_0) \neq \emptyset$, forcing x to lie in int conv$(a \cup (R(b, t) \cap N)) \subseteq$ int S, contradicting the fact that $x \in$ bdry S. The assertion is established.

By a symmetric argument, the point a sees via S all points of $N_0 \cap \text{int } S \cap \text{cl}(L(a, x)_-)$. Since both a and b see via S all points of (bdry $S) \cap N$, this implies that b sees via S all of $N_0 \cap S \cap \text{cl}(L(b, x)_-)$ and a sees via S all of $N_0 \cap S \cap \text{cl}(L(a, x)_-)$.

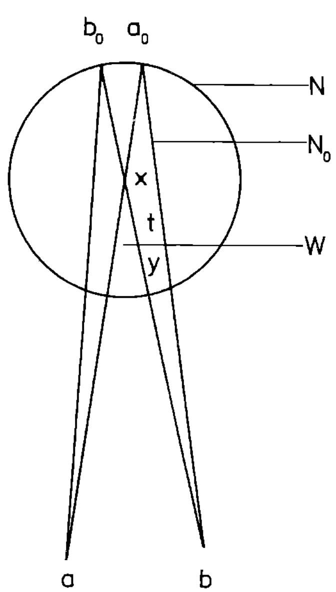

Fig. 1.

Therefore, if $N_0 \cap L(b,x)_-$ meets bdry S at some point y, then y is clearly visible via S from b, and y satisfies the theorem. A parallel statement holds if $N_0 \cap L(a,x)_-$ meets bdry S. Hence we may assume that $N_0 \cap L(b,x)_- \cap \text{bdry}\, S = N_0 \cap L(a,x)_- \cap \text{bdry}\, S = \emptyset$.

This leaves two possibilities: either $(N_0 \sim \{x\}) \cap \text{bdry}\ S = \emptyset$ or $(N_0 \sim \{x\}) \cap \text{bdry}\, S$ is a nonempty subset of $W \equiv \text{cl}(L(a,x)_+) \cap \text{cl}(L(b,x)_+$. If the first occurs, then $([a,x) \cap N_0) \cup ([b,x) \cap N_0) \subseteq \text{int}\, S$, and we shall show that $N_0 \sim S = \emptyset$. Otherwise, for $q \in N_0 \sim S$ and for $a' \in [a,x) \cap N_0$ and $b' \in [b,x) \cap N_0$, at least one of (a',q), (b',q) would meet $(\text{bdry}\, S) \cap N_0 \sim \{x\}$; impossible. Thus $N_0 \subseteq S$, forcing x to belong to int S; a contradiction.

Thus the second possibility must occur. Select $y \in (N_0 \sim \{x\}) \cap \text{bdry}\ S \subseteq W$. Then either x and b are on opposite sides of $L(a,y)$ or x and a are on opposite sides of $L(b,x)$ (or both). Assume $x \in L(a,y)_-$.

Then a sees via S all of $N_0 \cap S \cap \text{cl}(L(a,y)_-)$: Otherwise, by Lemma 1, a fails to see some $r \in N_0 \cap S \cap L(a,y)_-$, and b sees $R(a,r) \cap N$ via S. However, $R(a,r) \cap N$ meets $R(b,y) \sim [b,y]$, so $y \in \text{int conv}(b \cup (R(a,r) \cap N) \subseteq \text{int}\, S$; impossible. Hence a sees $N_0 \cap S \cap \text{cl}(L(a,y)_-)$, x is clearly visible from a via S, and x satisfies the theorem.

Case 2. Assume that a, b, x are distinct collinear points. In the case where there is some point y in $N \cap \text{bdry}\ S \sim L(a,b)$, we select a spherical neighborhood N_y of y, $N_y \subseteq N$. The argument from Case 1 may be applied to y and N_y to complete the proof. In the case where no such y exists, then either $N \cap S \subseteq L(a,b)$ (and x itself satisfies the theorem) or $N \cap S \sim L(a,b)$ is a nonempty subset of int S. Assume that the latter occurs. Then for an appropriate labeling of halfplanes, $N \cap \text{int}\ S \cap L(a,b)_+ \neq \emptyset$. Moreover, since $N \cap \text{bdry}\, S \cap L(a,b)_+ = \emptyset$, it follows that $N \cap L(a,b)_+ \subseteq S$. By similar reasoning, since $x \in \text{bdry}\, S$, $N \cap S \cap L(a,b)_- = \emptyset$.

If a sees via S all points of $N' \cap L(a,b)_+$ for some neighborhood N' of x, $N' \subseteq N$, then the argument is finished. Otherwise, we will show that b has this property. Choose $v_1 \in N \cap L(a,b)_+$ such that $[a, v_1] \nsubseteq S$. Then a sees via S no point of $R(a,v_1) \cap N \subseteq S$, so b necessarily sees all these points via S. If b sees via S all points of N in the open convex region bounded by $R(b,v_1)$ and $R(b,x)$, again the argument is finished. Otherwise, there is some u in this region not seen by b. Then a sees u. However, by our assumption for a, there is some v_2 in N and on the x side of $R(b,u)$ with $[a,v_2] \nsubseteq S$. Then $[b,v_2] \subseteq S$. Moreover, since b sees v_1 and v_2 but not u, there is at least one bounded component K_1 of $\sim S$ in the open convex region bounded by $R(b,v_1)$ and $R(b,v_2)$. (In fact, $K_1 \subseteq \text{int conv}\{b, v_1, v_2\}$.) If b fails to see some point of N in the open convex region bounded by $R(b,v_2)$ and $R(b,x)$, repeating the argument above, we obtain a bounded component K_2 of $\sim S$ in this region, $K_2 \neq K_1$. Since $\sim S$ has finitely many components, by an obvious induction, in finitely many steps we obtain a point v_n in $N \cap L(a,b)_+$ such that b sees via S all points in the open convex region bounded by $R(b, v_n)$ and $R(b, x)$. Hence b has the required property, and the theorem is satisfied.

Case 3. It remains to examine the case in which points a, b, x are not distinct. Certainly if $a = b$, then S is starshaped, so we may assume that $a \neq b$. If either a or b is x, assume $a = x$ and select a spherical neighborhood N of x such that $b \notin \text{cl}\, N$. If N contains a point y in $\text{bdry}\, S \sim L(b, a)$, we may select a spherical neighborhood N_y of y, $N_y \subseteq N$, $a \notin \text{cl}\, N_y$, and apply the argument from Case 1 to y and N_y to complete the proof. If $N \cap \text{bdry}\, S \sim L(a, b) = \emptyset$, then either $N \cap S \subseteq L(a, b)$ (and x satisfies the argument), or $N \cap \text{int}\ S \neq \emptyset$. In the latter case, it is not hard to show that one of $N \cap L(a,b)_+$, $N \cap L(a,b)_-$ is a subset of S, while the other is disjoint from S. Again x satisfies the argument. The theorem is established.

COROLLARY 1. *Let S be a compact, finitely connected set in R^2. Assume that S is a union of two starshaped sets. Then for F finite, $F \subseteq$ bdry S, there exist a finite $G \subseteq$ bdry S arbitrarily close to F and two points a_G, b_G in S (depending on G) such that every point of G is clearly visible via S from one of a_G, b_G.*

It is interesting to observe that both the theorem and its (weaker) corollary fail without the requirement that S be finitely connected. Consider the following example.

EXAMPLE 1. Let U denote the unit square in R^2 having vertices $t_0 = (0,0)$, $t_1 = (1,0)$, $t_2 = (1,1)$, $t_3 = (0,1)$. Let $a = (-2,0)$, $b = (3,0)$, and let V_a and V_b be line segments perpendicular to the x axis at a and b, respectively. Define $T \equiv V_a \cup V_b \cup \text{conv}(U \cup \{a, b\})$. (See Figure 2.)

Define triangular regions in $T \sim U$ as follows. Let u_1 be a segment from b to some point of (a, t_3), and let T_1 be the triangular region with edges on u_1, $(a, t_3]$, and $[t_3, t_0)$. Let $u_1 \cap (t_1, t_2) = \{t'_2\}$, and let w_1 be a segment from a to some point of (b, t'_2). Define T'_1 to be the triangular region with edges on w_1, $(b, t'_2]$, and $[t'_2, t_1)$. By an obvious induction, we obtain sequences of triangular regions $\{T_n\}$, $\{T'_n\}$, $n \geqslant 1$.

Finally, define $S \equiv T \sim \cup \{\text{int}(T_n \cup T'_n): n \geqslant 1\}$. Observe that each point of S sees via S either a or b, so S is a union of two (compact) starshaped sets. However, for $x = (\frac{1}{2}, 0)$, there is no boundary point of S near x which is

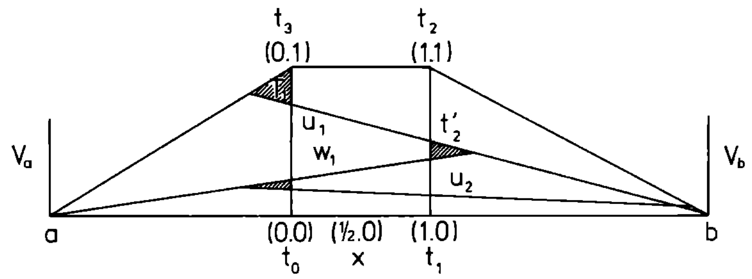

Fig. 2.

clearly visible from a or b. Thus the theorem fails. Moreover, for $x_a \in V_a \sim \{a\}$, $x_b \in V_b \sim \{b\}$, and $F = \{x, x_a, x_b\}$ there are no G, a_G, b_G which satisfy the corollary.

Of course, if we do not require G to lie in bdry S, the finite connectedness of S is not needed. The proof is elementary.

3. Comments on the Main Result

It is interesting to ask how restrictive is the condition of finite connectedness in Theorem 1 and how often the theorem is true without this condition. We can answer the question by considering the Baire space $\mathfrak{U}$ of all unions of two compact starshaped sets, endowed with Hausdorff distance. This $\mathfrak{U}$ has the obvious decomposition $\mathfrak{U} = \mathfrak{U}_1 \cup \mathfrak{U}_2$, where $\mathfrak{U}_1$ is the family of all connected unions of two compact starshaped sets and $\mathfrak{U}_2 = \mathfrak{U} - \mathfrak{U}_1$. Both $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are of second category in $\mathfrak{U}$.

In [7] it is proved that in 'most' (which always means 'all, except those in a set of first Baire category') members of $\mathfrak{U}_1$ the two starshaped sets meet at infinitely many points. By results in [7], most compact starshaped sets have single-point kernels. Similarly, in most members of $\mathfrak{U}_1$, both starshaped sets have single-point kernels. Using a proof in [7], we conclude that for most members of $\mathfrak{U}_1$ the complement has infinitely many components.

While the condition of finite connectedness in Theorem 1 is not fulfilled by most members of $\mathfrak{U}_1$, its conclusion is true in more than most cases.

THEOREM 2. *For all members S of $\mathfrak{U}$ except those in a nowhere dense set, it is true that for any choice of two points in the kernels of the two starshaped sets forming S (one in each) and for each finite set $F \subseteq$ bdry S we may find a finite set $G \subseteq$ bdry S arbitrarily close to F such that each point of G is clearly visible from one of the two points.*

Proof. From the proof of Theorem 1 it is clear that its conclusion is true without the condition of finite connectedness if there are no common boundary points of the two starshaped sets S_1 and S_2 forming S, collinear with a and b (in the notation of Theorem 1).

We shall show that for all $S \in \mathfrak{U}$ except those in a nowhere dense set, for any choice of two points a, b in the kernels of the two starshaped sets S_1, S_2 forming S (one in each) there is no point in bdry $S_1 \cap$ bdry S_2 collinear with a and b.

Let $\mho$ be an open set in $\mathfrak{U}$ and choose $S \in \mho$ with $S = S_1 \cup S_2$, where S_1, S_2 are compact starshaped sets in a, b respectively and $a \neq b$. Consider the disk $D_r(c)$ around the midpoint c of $[a, b]$. Obviously, for given $\varepsilon > 0$, r can be chosen such that

$$\delta(S_1, S_1 \sim \text{cone}(a, D_r(c)) < \varepsilon$$

and

$$\delta(S_2, S_2 \sim \text{cone}(b, D_r(c)) < \varepsilon.$$

Consequently, $\delta(S, A_r) < \varepsilon$, where

$$A_r = (S_1 \sim \text{cone}(a, D_r(c)) \cup (S_2 \sim \text{cone}(b, D_r(c)).$$

Thus, $A_r \in \mho$ for suitable r.

Now choose finite sets Q_1, $Q_2 \subseteq A_r$ such that

$$S_1' = \bigcup_{z \in Q_1} [a, z], S_2' = \bigcup_{z \in Q_2} [b, z]$$

be not line segments and close enough to S_i $(i = 1, 2)$, so that $S_1' \cup S_2' \in \mho$. Evidently, there is a positive $\alpha < r/2$ such that for any compact starshaped set S^* with $\delta(S^*, S_1') < \alpha$, the kernel lies in $D_{r/2}(a)$. Combining this with an analogous argument about S_2', we find a positive $\alpha < r/2$ such that for any $S^+ \in \mathfrak{U}$ with $\delta(S^+, S_1' \cup S_2') < \alpha$, $S^+ \in \mho$ and the two kernels of the starshaped sets S_1^+ and S_2^+ forming S^+ lie in $D_{r/2}(a)$ and $D_{r/2}(b)$ respectively.

Since $S_1' \cup S_2'$ is disjoint from $D_r(c)$, no such S^+ meets $D_{r-\alpha}(c)$. Since $r/2 < r - \alpha$, for simple geometric reasons no three points, one in the kernel of S_1^+, another in the kernel of S_2^+ and the third in $S_1^+ \cap S_2^+$, are collinear. Since this happens for all members of $\mho$ in a ball around $S_1' \cup S_2'$ of radius α, the theorem is proved.

4. Characterization

Observe that the converse of Theorem 1 can be disproved by a minor adaptation of [1, Ex. 1].

EXAMPLE 2. Let S be the compact set in Figure 3, including the broken-line segments and excluding the triangular regions. Only the boundary points x and y fail to be clearly visible from one of a, b, yet S is not a union of two starshaped sets.

THEOREM 3. *Let S be a compact simply connected set in R^2. Assume that for every finite set F in* bdry S *there exist a finite set $G \subseteq$* bdry S *arbitrarily close to F and points s and t (depending on G) such that every point of G is visible via S from at least one of s, t. Then S is a union of two starshaped sets.*

Proof. By comments following [1, Ex. 1], it suffices to prove that for every finite set F in bdry S there exist points s and t (depending on F) such that each point of F is visible via S from s or t. Using our hypothesis, for each n there exist set $G_n \subseteq$ bdry S within $1/n$ of F and points s_n, t_n (depending on G_n) such that each point of G_n is visible via S from s_n or t_n. By standard arguments, we pass to subsequences $\{s_{n(k)}\}$, $\{t_{n(k)}\}$ such that $\{s_{n(k)}\}$ converges to s and $\{t_{n(k)}\}$ converges to t. Then it is easy to show that each point of F sees via S either s or t, and the theorem is proved.

COROLLARY 2. *Let S be a compact, simply connected set in R^2. Then S is a union of two starshaped sets if and only if for F finite, $F \subseteq$* bdry S, *there exist a set $G \subseteq$* bdry S *arbitrarily close to F and points s, t (depending on G) such that each point of G is clearly visible via S from one of s, t.*

The corollary remains true if clear visibility is replaced by visibility.

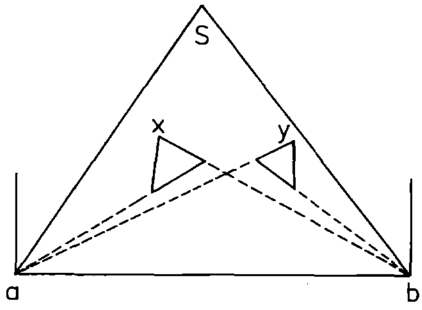

Fig. 3.

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