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HOW MANY SETS ARE POROUS?

TUDOR ZAMFIRESCU

ABSTRACT. The notion of a σ -porous set is often used to sharpen results on sets of first Baire category or of measure zero. It essentially uses the related notion of porosity. We find out in this note that there are quite a few porous sets: In complete convex metric spaces, most totally bounded closed sets are porous! Then we strengthen this result for the case of a Banach space.

Let (X, d) be a complete, convex metric space. The *convexity* means that for any distinct points $x, y \in X$ there exists a third point $z \in X$ such that $d(x, z) + d(z, y) = d(x, y)$. It is known (see, for example, [4, pp. 314, 315]) that the space $(B(X), \delta)$ of all bounded closed subsets of X , endowed with the Hausdorff metric δ , is also complete. In $B(X)$, the subset \mathcal{B} of all totally bounded closed sets M (for which, by definition, $\forall \varepsilon > 0, \exists$ a finite set $F \subset X$ such that $\delta(F, M) < \varepsilon$) is closed, hence (\mathcal{B}, δ) is complete.

For $x \in X$ and $M \in B(X)$ let

$$\begin{aligned}K(x, \rho) &= \{y \in X : d(x, y) \leq \rho\}, \\K^0(x, \rho) &= \{y \in X : d(x, y) < \rho\}, \\K(M, \rho) &= \{N \in B(X) : \delta(M, N) \leq \rho\}.\end{aligned}$$

For $G \subset X$, \bar{G} means the closure of G .

A set $M \subset X$ is called *porous with coefficient* α if $\forall x \in X$ and $\varepsilon > 0, \exists y \in K(x, \varepsilon)$ such that

$$K(y, \alpha d(x, y)) \cap M = \emptyset.$$

A countable union of porous sets (with various coefficients) is called *σ -porous*. A countable union of porous sets with coefficient α is said to be *α -uniformly σ -porous*.

We say that *most* (or *nearly all*, or *α -nearly all*) elements of a Baire metric space have a certain property if those elements not enjoying it form a set of first Baire category (or a σ -porous set, or an α -uniformly σ -porous set).

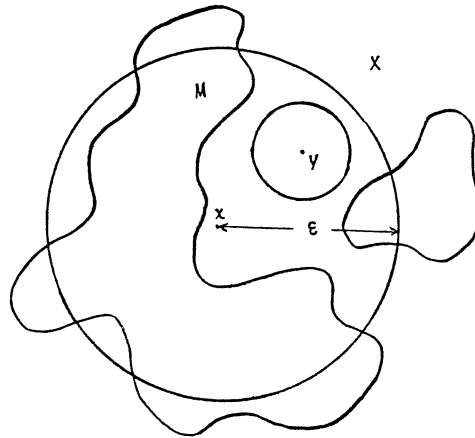
Besides the many old and new results in analysis involving most elements of various spaces, there are recent similar results in the geometry of convex bodies (for a survey see [2 or 6]). While several results on nearly all elements of certain spaces already appeared in functional analysis, [7] contains the first similar results so far in the geometry of convex bodies.

P. Gruber [3] proved in Euclidean spaces that most compact sets have measure zero. The fact that they are also nowhere dense is implicitly contained in C. Kuratowski's book [4]. So, the following natural question arises: Are most compact sets in \mathbb{R}^n porous? The answer is yes, as the following theorems show.

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THEOREM 1. *Let (X, d) be a complete convex metric space. Then, for any $\alpha \in (0, 1)$, most totally bounded closed sets in X are porous with coefficient α .*

PROOF. Let

$$\mathcal{B}_n = \{M \in \mathcal{B} : \exists y \in X \text{ s.t., } \forall x \in K(y, n^{-1}), K(x, \alpha d(x, y)) \cap M \neq \emptyset\}.$$

Obviously, $\bigcup_n \mathcal{B}_n$ is the set of all elements of \mathcal{B} which are not porous with coefficient α . Thus, to prove the theorem it suffices to show that \mathcal{B}_n is nowhere dense for every n .

Let O be an open set in \mathcal{B} . Consider the finite set $F \in O$. Let

$$\xi = \min\{\min\{d(x, y) : x, y \in F, x \neq y\}, n^{-1}\}.$$

Put

$$G = \bigcup_{x \in F} K^0\left(x, \frac{1 - \alpha}{5 - \alpha} \xi\right).$$

We claim that, $\forall y \in X, \exists x \in K(y, n^{-1})$ such that

$$K(x, \alpha d(x, y)) \cap G = \emptyset.$$

This is obvious if $y \notin \bar{G}$. Suppose now $y \in \bar{G}$. Let $x_0, x_1 \in F$ be such that

$$d(y, x_0) = \min_{x \in F} d(y, x), \quad d(y, x_1) = \min_{x \in F \setminus \{x_0\}} d(y, x).$$

Then necessarily $y \in K(x_0, (1 - \alpha)\xi/(5 - \alpha))$.

We have

$$(*) \quad d(y, x_1) \geq d(x_0, x_1) - d(y, x_0) \geq \xi - \frac{1 - \alpha}{5 - \alpha} \xi = \frac{4}{5 - \alpha} \xi.$$

Since X is complete and convex, by Menger's theorem [5], there exists a geodesic path P joining y with x_1 of length $d(y, x_1)$. Let x be the point of P verifying

$$d(x, y) = \min\{\frac{1}{2} d(y, x_1), n^{-1}\}.$$

We have $d(x, x_i) \geq d(x, x_1)$ for every $x_i \in F$ with $i \neq 0$. Suppose, indeed, $d(x, x_2) < d(x, x_1)$ for some $x_2 \in F$. Then

$$d(y, x_2) \leq d(y, x) + d(x, x_2) < d(y, x) + d(x, x_1) = d(y, x_1),$$

which contradicts the choice of x_1 .

Obviously, $x \in K(y, n^{-1})$. Suppose $z \in K(x, \alpha d(x, y)) \cap G$. Then there is an $x_i \in F$ such that

$$d(z, x_i) < \frac{1 - \alpha}{5 - \alpha} \xi.$$

If $i = 0$, then

$$d(x, y) \leq d(x, z) + d(z, x_0) + d(x_0, y) < \alpha d(x, y) + \frac{1 - \alpha}{5 - \alpha} \xi + \frac{1 - \alpha}{5 - \alpha} \xi,$$

whence $d(x, y) < 2\xi/(5 - \alpha)$.

Now, either $d(x, y) = \frac{1}{2} d(y, x_1)$, which implies $d(y, x_1) < 4\xi/(5 - \alpha)$, thus contradicting (*), or $d(x, y) = n^{-1} \geq \xi$, which yields $\xi < 2\xi/(5 - \alpha)$, contradicting the choice of α . If $i \neq 0$, then

$$d(x, y) \leq d(x, x_1) \leq d(x, x_i) \leq d(x, z) + d(z, x_i) < \alpha d(x, y) + \frac{1 - \alpha}{5 - \alpha} \xi,$$

whence $d(x, y) < \xi/(5 - \alpha)$ and either $d(x, y) = \frac{1}{2} d(y, x_1)$, whence $d(y, x_1) < 2\xi/(5 - \alpha)$, in contradiction with (*), or else $d(x, y) = n^{-1} \geq \xi$, whence $\xi < \xi/(5 - \alpha)$, which again contradicts the choice of α .

Consider now any set $M \in \mathcal{O}$ verifying $\delta(M, F) < (1 - \alpha)\xi/(5 - \alpha)$. Clearly, $M \subset G$ and for every $y \in X$, there is an $x \in K(y, n^{-1})$ such that

$$K(x, \alpha d(x, y)) \cap M = \emptyset.$$

Therefore $M \notin \mathcal{B}_n$. Hence \mathcal{B}_n is nowhere dense and the theorem is proved.

THEOREM 2. *In a Banach space X , for any $\alpha, \beta > 0$ with $2\alpha + \beta < 1$, α -nearly all bounded closed sets are porous with coefficient β .*

PROOF. We prove that

$$\mathcal{B}_n = \{M \in B(X) : \exists y \in X \text{ such that } \forall x \in K(y, n^{-1}) : K(x, \beta \|x - y\|) \cap M \neq \emptyset\}$$

is porous with coefficient α .

Let $M \in B(X)$ and $r \in \mathbf{N}$ such that $M \subset K(\mathbf{0}, 2r)$. Then choose a support functional of $K(\mathbf{0}, 2r)$, i.e. a nonidentically zero linear functional f on X such that, letting $\mu = \sup f(K(\mathbf{0}, 2r))$, we can find a point x_μ in $K(\mathbf{0}, 2r) \cap f^{-1}(\mu)$. (By the second Bishop-Phelps theorem, the support functionals of any bounded closed convex set lie densely in X^* , see [1]. In our case a simple corollary of the Hahn-Banach theorem suffices.) Of course,

$$\|x_\mu\| = 2r, \quad f(K(\mathbf{0}, 2r)) = [-\mu, \mu].$$

Take $\nu_0 = -\mu, \nu_1, \dots, \nu_{nr} = 0, \dots, \nu_{2nr} = \mu$ with $\nu_i - \nu_{i-1} = \mu/nr$. Let

$$L = \bigcup_{x \in M} K(x, n^{-1}) \cap \bigcup_{i=0}^{2nr} f^{-1}(\nu_i).$$

We claim that, for any $x \in M$, there exists $x^* \in K(x, n^{-1}) \cap \bigcup_i f^{-1}(\nu_i)$. Indeed, since $-\mu \leq f(x) \leq \mu$, there is an index i satisfying $\nu_{i-1} < f(x) \leq \nu_i$. Suppose $\nu_i - f(x) \leq f(x) - \nu_{i-1}$; the other case is similar. We take

$$x^* = x + \frac{\nu_i - f(x)}{\mu} x_\mu.$$

Then

$$\|x^* - x\| = \frac{\nu_i - f(x)}{\mu} \|x_\mu\| \leq \frac{\nu_i - \nu_{i-1}}{2\mu} \cdot 2r = n^{-1}$$

and

$$f(x^*) = f(x) + \frac{\nu_i - f(x)}{\mu} f(x_\mu) = \nu_i.$$

Hence

$$M \subset \bigcup_{x \in L} K(x, n^{-1}), \quad L \subset \bigcup_{x \in M} K(x, n^{-1}),$$

which yield $\delta(M, L) \leq n^{-1}$.

Consider now the ball $K(L, \alpha\delta(L, M))$. Since $\delta(L, M) \leq n^{-1}$, we have $\alpha\delta(L, M) \leq \alpha/n$. Take $Q \in K(L, \alpha\delta(L, M))$. Let

$$S_i = f^{-1} \left(\left[\nu_i - \frac{\alpha\mu}{2nr}, \nu_i + \frac{\alpha\mu}{2nr} \right] \right)$$

and $S = \bigcup_{i=0}^{2nr} S_i$.

Then $Q \subset S$. Indeed, let $q \in Q$. There exists $x \in L$ such that $\|q - x\| \leq \alpha/n$. Let $\nu_i = f(x)$ and suppose $f(q) > \nu_i$. The “<” case is analogous and “=” is clear.

$$f(q - x) = \frac{\alpha}{2nr} f \left(\frac{2nr}{\alpha} (q - x) \right),$$

where $\|2nr(q - x)/\alpha\| \leq 2r$; whence $f(2nr(q - x)/\alpha) \leq \mu$ and $f(q) \leq \nu_i + \alpha\mu/2nr$.

We now show that $Q \notin \mathcal{B}_n$. Take any $y \in X$. If $y \notin S$, it suffices to remark that $Q \subset S$. If $y \in S$, then $y \in S_i$ for some index i . Assume, for instance, $f(y) \geq \nu_i$; the case $f(y) < \nu_i$ is similar. Consider

$$x = y + \frac{1}{2nr} x_\mu.$$

We have

$$\|x - y\| = \frac{1}{2nr} \|x_\mu\| = n^{-1},$$

hence $x \in K(y, n^{-1})$. We prove that $K(x, \beta\|x - y\|) \cap S = \emptyset$. Let $z \in K(x, \beta\|x - y\|)$. Then $\|z - x\| \leq \beta/n$. Also,

$$f(z - x) = \frac{\beta}{2nr} f \left(\frac{2nr}{\beta} (z - x) \right).$$

Since

$$-\mu \leq f \left(\frac{2nr}{\beta} (z - x) \right) \leq \mu,$$

we have

$$-\frac{\beta\mu}{2nr} \leq f(z - x) \leq \frac{\beta\mu}{2nr}.$$

It follows, using $2\alpha + \beta < 1$, that

$$f(z) \geq f(x) - \frac{\beta\mu}{2nr} = f(y) + \frac{1}{2nr} f(x_\mu) - \frac{\beta\mu}{2nr} \geq \nu_i + \frac{\mu}{2nr} - \frac{\beta\mu}{2nr} > \nu_i + \frac{\alpha\mu}{2nr},$$

and

$$\begin{aligned} f(z) &\leq f(x) + \frac{\beta\mu}{2nr} = f(y) + \frac{1}{2nr} f(x_\mu) + \frac{\beta\mu}{2nr} \leq \nu_i + \frac{\mu}{2nr} (1 + \alpha + \beta) \\ &< \nu_i + \frac{\mu}{2nr} (2 - \alpha) = \nu_i + \frac{\mu}{nr} - \frac{\alpha\mu}{2nr} = \nu_{i+1} - \frac{\alpha\mu}{2nr}, \end{aligned}$$

which shows that $z \notin S_i \cup S_{i+1}$ and therefore $z \notin S$. This completes the proof.

THEOREM 3. *In a Banach space X , for any $\alpha, \beta > 0$ with $2\alpha + \beta < 1$, α -nearly all totally bounded closed sets are porous with coefficient β .*

The proof is identical to that of Theorem 2. We only have to add the obvious remark that, if M is totally bounded, then L is totally bounded too.

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