GHOSTS ARE SCARCE

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In recent years P. C. Hammer's problem [**8**] of determining a convex body from its 'X-ray pictures' was investigated by Gardner and McMullen [**4**], Gardner [**3**], Falconer [**2**] and Volčič [**15**]. An earlier result is due to Giering [**5**].

An X-ray picture of a convex body in a direction may be identified with its Steiner symmetral in that direction. Some of these papers consider X-ray pictures taken from points not on the line at infinity, but here we are not concerned with that situation.

Gardner and McMullen proved that there exist four directions such that the corresponding X-ray pictures distinguish between all convex bodies, and that no three directions can do this. Giering proved that, given a plane convex body K, there exist three directions depending on K, such that the corresponding X-ray pictures distinguish K from any other convex body. He has also shown that two directions are in general not enough.

Convex bodies with the same X-ray pictures as a given one were called 'ghosts' in [**14**], in analogy with the ghost densities from computerized tomography [**12**].

It should be remembered that in the fundamental case of parallel rays from two orthogonal directions, besides a few triangular or quadrangular examples by Giering [**6**] and a rather obvious construction which basically interchanges two diagonally opposite, symmetrical pieces with two other diagonally opposite congruent pieces (diagonals of a rectangle), no deeper insight into the soul of a ghost of a convex body has been won.

We are—as a consequence—far away from being able to characterize convex
are not ghosts! (Note the equivalence between *having* and *being* a ghost!) Thus we

In this situation, the question about the generic behaviour of convex bodies with regard to their ghosts appears interesting, but looks at a first glance, in view of the lack of knowledge described above, rather hopeless. However, in this paper we establish the validity of the (more comfortable?) assertion that most convex bodies are not ghosts! (Note the equivalence between *having* and *being* a ghost!) Thus we confirm a conjecture of the first author, motivated by the symmetries described above and also present in his examples from [**14**].

It is clear that the orthogonality of the two considered directions is unessential, because of the affine character of our problem. When we state it we do so just to fix the ideas.

As a main open problem there remains the characterization of those convex bodies which are uniquely determined by two X-ray pictures. The analogous problem for measurable sets has been solved by Lorentz [**11**].

The space $\mathscr{C}$ of all convex curves in $\mathbb{R}^2$, like the space $\mathscr{B}$ of all convex bodies in $\mathbb{R}^d$, equipped with the Hausdorff distance δ is a Baire space. 'Most' means 'all, except those in a set of first category'. For a survey on properties of most convex bodies, see [**16**].

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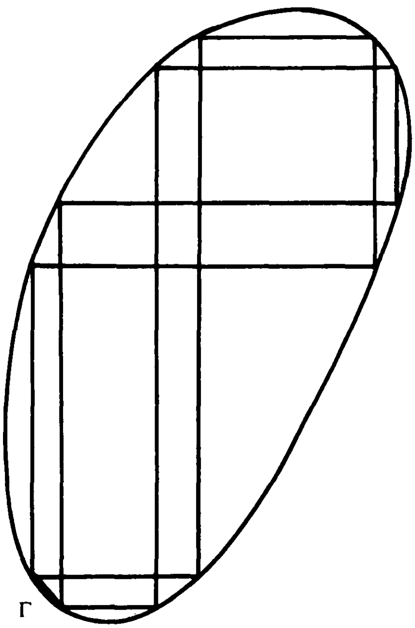

FIG. 1

The existence of a ghost is very much related to the existence of inscribed closed broken lines whose line-segments are parallel to the axes. The importance of these broken lines was first noted by Giering.

Indeed, if a convex body B has a ghost B' such that the component Γ of $B\setminus B'$ lies as in Figure 1, then $B\setminus B'$ and $B'\setminus B$ have finitely many components of the same area as Γ (possibly among many, even infinitely many other components) and B must have the depicted inscribed closed broken lines. We say that a convex body with such an inscribed closed broken line is a *ghost-candidate*.

This is not the only reason why the investigation of convex bodies with closed broken lines of the type described above is of interest. Consider the Dirichlet problem for the hyperbolic differential equation

$$u_{xy} = 0,$$

that is, the problem of determining its solutions from given values on a closed curve C (see Hadamard [7]), that we shall suppose strictly convex. For any $x \in C$, let $Sx \in C$ be the (other, if possible) point with the same abscissa as x and $Tx \in C$ be the (other, if possible) point with the same ordinate as Sx. The transformation T is a homeomorphism and there is a close connection between the Dirichlet problem for C and the topological properties of T, as John has shown [9].

It is easily seen that T is an even homeomorphism (that is, it is orientation-preserving on C). A point $x \in C$ is called *periodic* if it is a fixed point of T^n for some n (the smallest such n is the *period* of x). The set $V(x) = \{T^n x : n = 0, 1, 2, \ldots\}$ is called *orbit* of $x = T^0 x$. There are the following possibilities.

I All points of C are periodic (T is *periodic*).
II C contains periodic and non-periodic points (T is *semiperiodic*).
III No point of C is periodic and no orbit is dense in C (T is *intransitive*).
IV No point of C is periodic and some orbit is dense in C (T is *transitive*).

Thus, the existence of periodic points makes out of conv C a ghost-candidate, to use again the previous wording.

Many ghost-candidates

We shall see here that in most cases we have just one of the four types of homeomorphisms T, namely T must be semiperiodic. The generic situation is even more precisely described by the following theorem.

THEOREM 1. *On most convex curves C, there is a non-empty, nowhere dense set of periodic points.*

Proof. First of all, it suffices to prove the theorem in the space $\mathscr{C}^*$ of all smooth and strictly convex curves, because $\mathscr{C}^*$ is residual in $\mathscr{C}$, as first proved by Klee [**10**]. Let $\mathscr{A} \subset \mathscr{C}^*$ be the set of all smooth and strictly convex curves without any periodic points. We show that $\mathscr{A}$ is nowhere dense in $\mathscr{C}^*$. Let $\mathscr{O} \subset \mathscr{C}^*$ be open and choose

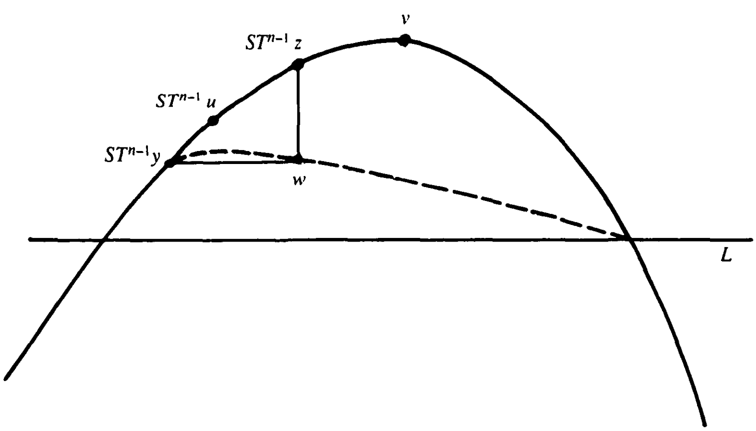

FIG. 2

$C \in \mathscr{O}$ of class C^2. Then T is not intransitive (see Denjoy [**1**, p. 372] and John [**9**, p. 147]). If $C \in \mathscr{A}$, then T is transitive and it is well known that all points of C have dense orbits. Let u and v be the points of C of smallest, respectively largest ordinate. Consider the horizontal line L meeting C, at distance $\varepsilon > 0$ from v. For ε small enough, no modification of C above L preserving smoothness and strict convexity throws C out of $\mathscr{O}$. Let $T^n u$ be the first point of the orbit of u which lies above L. Also, let $y, z \in C$ be points of equal ordinates, chosen so close to u that $T^n y$ and $T^n z$ both lie above L. Let w be the point having the same abscissa as $ST^{n-1}z$ and the same ordinate as $T^n y$. By interchanging y and z if necessary we can arrange that $w \in \operatorname{conv} C$. If $w \notin C$, modify C above L so as to contain $ST^{n-1}y$ and w. Then y becomes a periodic point.

Thus, in any case, there is a curve $C \in \mathscr{O}$ with a periodic point $y \in C$ of period, say, m. We may also suppose that y is close to the point of C with minimal ordinate and $V(y)$ does not contain any one of the four points which have one coordinate maximal or minimal.

Let $D \in \mathscr{C}$ and $x \in D$. Consider the point $f(x, D)$ of the same abscissa as x and of the same ordinate as $T^m x$, where T is taken with respect to D, of course. It is easily seen that f is continuous.

Now consider again our curve C and point y. Clearly, $y = f(y, C)$. Choose a point $y' \in C \setminus (V(y) \cup SV(y))$. There are three possible cases:

(1) $f(y', C) \notin \operatorname{conv} C$,
(2) $f(y', C) \in C$,
(3) $f(y', C) \in \operatorname{int}\operatorname{conv} C$.

We consider here the first case only; the remaining two can be treated in a similar way. Let $Y \subset \mathbb{R}^2$ be a neighbourhood of y such that $T^i y' \notin Y$, $ST^i y' \notin Y$ $(i = 0, 1, \ldots, m)$, $V(y) \cap Y = \{y\}$ and $SV(y) \cap Y = \emptyset$. (Observe that, since $V(y)$ does not contain the four points of C with a coordinate maximal or minimal, $V(y) \cap SV(y) = \emptyset$.) Now consider $C' \in \mathcal{O}$ such that C and C' coincide outside of Y and $y \in \operatorname{int}\operatorname{conv} C'$. Let y'' be the point of C' with the same abscissa as y, but smaller ordinate. Clearly, $f(y'', C') = y$. Thus $f(y', C') \notin \operatorname{conv} C'$ and $f(y'', C') \in \operatorname{int}\operatorname{conv} C'$.

There exists a neighbourhood $\mathcal{N} \subset \mathcal{O}$ of C' and neighbourhoods Y' of y' and Y'' of y'' such that for every curve $D \in \mathcal{N}$, $D \cap Y'$ and $D \cap Y''$ are non-empty and, for arbitrary $t' \in D \cap Y'$ and $t'' \in D \cap Y''$, $f(t', D) \notin \operatorname{conv} D$ and $f(t'', D) \in \operatorname{int}\operatorname{conv} D$. This follows from the continuity of f, which also yields the existence of a third point t such that $f(t, D) \in D$. Thus, every curve in $\mathcal{N}$ has a periodic point. Hence $\mathcal{A}$ is nowhere dense.

Now let $\mathcal{C}_n$ be the set of all curves in $\mathcal{C}^*$ containing an arc A of length n^{-1} whose points are all periodic and whose supporting lines at the points of relint A are neither vertical nor horizontal. We show that $\mathcal{C}_n$ is nowhere dense. Clearly, $\mathcal{C}_n$ is closed in $\mathcal{C}^*$. Let $C \in \mathcal{C}_n$ and $A \subset C$ be an arc as mentioned above. It is well known that all periodic points have the same period, say m. Let $\mathcal{U}$ be a neighbourhood of C in $\mathcal{C}^*$. Also, let y be close to the midpoint of A and such that $V(y) \cap SV(y) = \emptyset$. Let $Y \subset \mathbb{R}^2$ be a neighbourhood of y disjoint from $(V(y) \cup SV(y)) \setminus \{y\}$. As before, we can modify C inside Y and obtain a curve $C' \in \mathcal{U}$ such that some point $y' \in C' \cap Y$ does not have period m, hence y' is not periodic. By repeating this procedure (finitely, but sufficiently many, times) with other arcs of length n^{-1}, we eventually find a curve which belongs to $\mathcal{U} \setminus \mathcal{C}_n$. Thus, the complement of $\mathcal{C}_n$ is dense in $\mathcal{C}^*$, whence on most curves in $\mathcal{C}^*$ the set of periodic points is nowhere dense. Since $\mathcal{A}$ is nowhere dense, the theorem is completely proved.

We remark that all convex curves of types I (T periodic), III (T intransitive) and IV (T transitive) form a nowhere dense subset of $\mathcal{C}$. The argument for the curves of type I parallels that about $\mathcal{C}_n$ in the above proof, while the curves of type III or IV constitute the set $\mathcal{A}$.

This theorem has as a consequence that the number τ associated to T, introduced by van Kampen in **[13]**, is rational for most C.

As another consequence, we see that we are faced with many ghost-candidates. Are there in fact many ghosts?

A lemma

We shall make use of the following simple lemma. Recall that $\mathcal{B}$ is the space of all convex bodies, that is, compact convex sets with interior points, in $\mathbb{R}^d$.

LEMMA. *If $\mathcal{T}$ is a topological space, $\mathcal{B}'$ is a closed subspace of $\mathcal{B}$ and $f: \mathcal{B}' \to \mathcal{T}$ is continuous, then the set*

$$\{B \in \mathcal{B}' : \exists B' \in f^{-1}(f(B)) \text{ with } B' \neq B\}$$

is an F_σ.

Proof. Since the above set is equal to $\bigcup_{n=1}^{\infty}\mathscr{B}_n$, where

$$\mathscr{B}_n = \{B\in\mathscr{B}' : \exists B'\in f^{-1}(f(B)) \text{ with inradius at least } n^{-1} \text{ and } n^{-1}\leqslant \delta(B,B')\leqslant n\},$$

it suffices to show that every $\mathscr{B}_n$ is closed. If $B_i \to B$ with $B_i\in\mathscr{B}_n$ and $B\in\mathscr{B}$, then there is some $B'_i\in f^{-1}(f(B_i))$ with inradius at least n^{-1}, satisfying

$$n^{-1}\leqslant \delta(B_i,B'_i)\leqslant n$$

for every index i. Clearly there is some ball $K\subset\mathbb{R}^d$ including $\bigcup_{i=1}^{\infty}B_i$. Then the concentric ball obtained by adding n to the radius of K includes $\bigcup_{i=1}^{\infty}B'_i$. By Blaschke's selection theorem, some subsequence $\{B'_{i_j}\}_{j=1}^{\infty}$ of $\{B'_i\}_{i=1}^{\infty}$ converges, say to a compact convex set B'. It follows immediately that B' has inradius at least n^{-1},

$$f(B') = \lim_{j\to\infty} f(B'_{i_j}) = \lim_{j\to\infty} f(B_{i_j}) = f(B)$$

and

$$n^{-1}\leqslant \delta(B,B')\leqslant n,$$

whence $B\in\mathscr{B}_n$.

Scarce ghosts

We start with a planar result.

Let $S_h(K)$ and $S_v(K)$ be the Steiner symmetrals of K in the horizontal and—respectively—vertical direction.

PROPOSITION. *Most planar convex bodies are not ghosts.*

Proof. The function which associates to each convex body in $\mathscr{B}$ its Steiner symmetral is known to be continuous. Thus, by the lemma, the set of all ghosts in $\mathscr{B}$ is an F_σ. It remains to show that the family of all convex bodies without ghosts is dense in $\mathscr{B}$. Let $K\in\mathscr{B}$. Suppose without loss of generality that some point $s(K)$ of K of smallest abscissa has an ordinate not less than that of some point $l(K)$ of largest abscissa. Then K can arbitrarily well be approximated by a convex body K' with unique $s(K')$ and $l(K')$, such that the ordinate of $s(K')$ is larger than that of $l(K')$, the portion P of bdK' above the horizontal line L_1 through $s(K')$ is smooth and strictly convex, and $(\mathrm{bd}K')\backslash P$ is a polygonal line.

We claim that K' has no ghost. Suppose indeed that K'' has the same Steiner symmetrals as K'.

Let $H = \mathrm{bd}S_h(K') = \mathrm{bd}S_h(K'')$ and $V = \mathrm{bd}S_v(K') = \mathrm{bd}S_v(K'')$. Since H is polygonal below L_1 and V is strictly convex on the left-hand side of the vertical line L_2 through $L_1\cap(\mathrm{bd}K')\backslash\{s(K')\}$, $s(K'')$ does not lie below $s(K')$.

Suppose that $s(K'')$ lies above $s(K')$. Then the smoothness of H above L_1 implies the smoothness of bdK'' at $s(K'')$, but this contradicts the non-smoothness of V at $s(S_v)$, which follows from the non-smoothness of bdK' at $s(K')$. Hence $s(K') = s(K'')$. It follows that $L_1\cap\mathrm{bd}K' = L_1\cap\mathrm{bd}K''$. If $P\not\subset\mathrm{bd}K''$, then there is an arc $A'\subset P$ and an arc $A''\subset\mathrm{bd}K''$ with the same endpoints x,y and no other points in common.

Since $S_v(K') = S_v(K'')$, there are two points $x^*, y^*\in(\mathrm{bd}K')\cap(\mathrm{bd}K'')$ below x,y respectively. The arcs $B'\subset\mathrm{bd}K'$ and $B''\subset\mathrm{bd}K''$ between x^* and y^* below L must be polygonal, H being polygonal below L_1.

If $f_{K'}, f_{K''}: I\to\mathbb{R}$ are two concave functions whose graphs are A', A'' respectively and $g_{K'}, g_{K''}: I\to\mathbb{R}$ are two convex functions whose graphs are B', B'' respectively, then $f_{K'}-f_{K''}$ has no root in the interior of I,

$$f_{K'}-f_{K''} = g_{K'}-g_{K''}$$

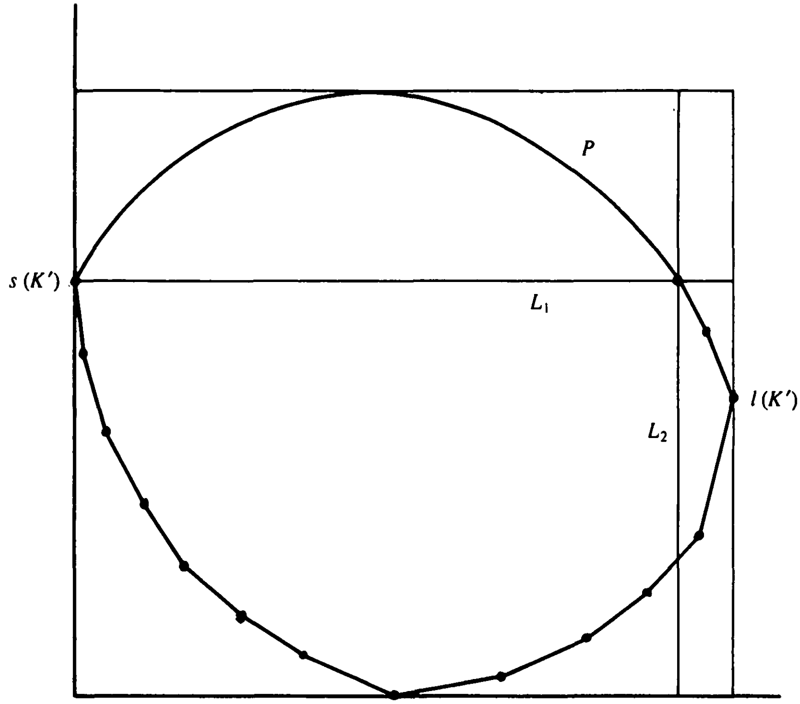

FIG. 3

and the common values at the endpoints of I are both zero. But $f_{K'}-f_{K''}$ is smooth and $g_{K'}-g_{K''}$ is piecewise linear; this is impossible for non-identical $f_{K'}, f_{K''}$.

Hence $P \subset \mathrm{bd}K''$. From $S_v(K') = S_v(K'')$, it follows that K' and K'' coincide on the left-hand side of L_2; $S_h(K') = S_h(K'')$ eventually implies that $K' = K''$. The proposition is proved.

This result will now be generalized to higher dimensions. Thereby we do not modify the number of directions along which Steiner symmetrals are considered: let S_h and S_v continue to mean the Steiner symmetrizations in two fixed orthogonal directions, called horizontal and vertical.

THEOREM 2. *In* $\mathbb{R}^d$ $(d \geqslant 2)$, *most convex bodies are not ghosts.*

Proof. For $d = 2$, the theorem coincides with the previous proposition. Thus let $d \geqslant 3$ and consider a 2-dimensional flat $\Pi \subset \mathbb{R}^d$ parallel to the horizontal and vertical directions.

Let $\mathscr{K}$ be the space of all compact sets in $\mathbb{R}^d$ endowed with the Hausdorff metric. Also, let $\mathscr{A} = \{B \in \mathscr{B} : B \cap \Pi = \varnothing\}$ and $\mathscr{D} = \{B \in \mathscr{B} \setminus \mathscr{A} : B \cap \Pi$ is not a ghost$\}$. We prove that $\mathscr{B}_\Pi = \mathscr{A} \cup \mathscr{D}$ is residual in $\mathscr{B}$.

Since $B \cap \Pi$ depends continuously on B in $\mathscr{B} \setminus \mathscr{A}$ and the Steiner symmetrizations are continuous, the function $f: \mathscr{B} \setminus \mathscr{A} \to \mathscr{K}^2$ associating $(S_h(B \cap \Pi), S_v(B \cap \Pi))$ to B is also continuous. Since $\mathscr{B} \setminus \mathscr{A}$ is closed in $\mathscr{B}$, we may apply the lemma and obtain that $\mathscr{B} \setminus \mathscr{B}_\Pi$ is an F_σ-set.

Thus, it remains to prove that $\mathscr{B}_\Pi$ is dense in $\mathscr{B}$. To this effect, let $\mathscr{O} \subset \mathscr{B}$ be open. If $\mathscr{O} \cap \mathscr{A} \neq \varnothing$, then we already have $\mathscr{O} \cap \mathscr{B}_\Pi \neq \varnothing$. If $\mathscr{O} \cap \mathscr{A} = \varnothing$, then we may choose a smooth convex body $B \in \mathscr{O}$ (the family of all smooth convex bodies is more than only dense: see [10, 17]). It is easily seen that Π cuts B and therefore $B \cap \Pi$ is smooth (in Π). Take $\varepsilon > 0$ such that $\delta(B, B') < \varepsilon$ implies that $B' \in \mathcal{O}$. Consider in Π an open convex set $C \supset B \cap \Pi$ such that bd$C$ is a polygon without any vertical edge and

$$\delta(B \cap \Pi, \bar{C}) < \varepsilon.$$

We round up several corners of $\bar{C}$ in order to get a planar convex body $C' \supset B \cap \Pi$ of the type described in the proof of the preceding proposition, hence uniquely determined by $S_h(C')$ and $S_v(C')$. Then $B' = \operatorname{conv}(B \cup C')$ belongs to $\mathcal{D}$, satisfies $\delta(B, B') < \varepsilon$, and therefore also belongs to $\mathcal{O}$.

Hence $\mathcal{B}_\Pi$ is residual in $\mathcal{B}$. Let $\{\Pi_i\}_{i=1}^{\infty}$ be a sequence of 2-flats parallel to Π, whose union is dense in $\mathbb{R}^d$. Then $\bigcap_{i=1}^{\infty} \mathcal{B}_{\Pi_i}$ is residual in $\mathcal{B}$, which means that for most $B \in \mathcal{B}$, $B \cap \Pi_i$ is not a ghost for any $i \in \mathbb{N}$ for which $B \cap \Pi_i \neq \emptyset$. Now, it easily follows that most B are not ghosts, which proves the theorem.

On the frequency of ghosts

Besides being of first category we would like to know more about the set of all ghosts. In this section we contribute a little to this problem; however, the orthogonal directions of symmetrization will not be fixed in advance any more. For simplicity we work in $\mathbb{R}^2$. With a more complicated but still elementary proof, the theorem can be extended to higher dimensions.

THEOREM 3. *There is a dense family $\mathcal{F}$ of convex bodies, each of which has uncountably many ghosts in $\mathcal{F}$ with respect to each one of two distinct pairs of orthogonal directions.*

Proof. Let $\mathcal{O} \subset \mathcal{B}$ be open and consider $K_1 \in \mathcal{O}$ with polygonal boundary. Let $\omega_M(K_1)$ and $\omega_m(K_1)$ be the directions in which the width of K_1 is maximal, respectively minimal. By slightly modifying K_1 if necessary, we obtain a convex body $K_2 \in \mathcal{O}$ with polygonal boundary, such that $\omega_M(K_2)$ and $\omega_m(K_2)$ are not orthogonal. Let $a_M b_M$ be the chord of K_2 realizing the maximal width and let $a'_m b'_m$ be a chord realizing the minimal width. The two chords are not orthogonal. Clearly, a_M and b_M are vertices of the boundary of K_2 and no edge is orthogonal to $\omega_M(K_2)$. Choose $a_m, b_m \notin K_2$ such that $a'_m b'_m \subset a_m b_m$ and $K_3 = \operatorname{conv}(K_2 \cup \{a_m, b_m\})$ still belongs to $\mathcal{O}$. Consider the two circles having $a_M b_M$ and $a_m b_m$ as diameters. Choose on them four arcs A_M, B_M, A_m, B_m centred at a_M, b_M, a_m, b_m respectively, all of length $\varepsilon > 0$. For ε small enough, all four arcs lie on the boundary of $K_4 = \operatorname{conv}(K_3 \cup A_M \cup B_M \cup A_m \cup B_m)$ and $K_4 \in \mathcal{O}$. For such an ε, consider the sequences $\{a_{mi}\}_{i=1}^{\infty}$, $\{a'_{mi}\}_{i=1}^{\infty}$, $\{b_{mi}\}_{i=1}^{\infty}$, $\{b'_{mi}\}_{i=1}^{\infty}$ such that $a_{mi} \in A_m$, $a_{m(i+1)}$ lies between a_{mi} and a_m, a_{mi} and a'_{mi} are symmetric with respect to the line Λ through a_m and b_m, b_{mi} and b'_{mi} are symmetric with respect to Λ, a_{mi} and b_{mi} are symmetric with respect to the midpoint of $a_m b_m$.

Consider the circle segments α_i, α'_i, β_i, β'_i, defined as the convex hulls of the open arcs $a_{mi} a_{m(i+1)}$, $a'_{mi} a'_{m(i+1)}$, $b_{mi} b_{m(i+1)}$, $b'_{mi} b'_{m(i+1)}$ respectively. For every $i \in \mathbb{N}$ we choose either the pair α_i, β_i or the pair α'_i, β'_i and subtract them from K_4. We do the same with A_M and B_M. All these convex bodies lie in $\mathcal{O}$. We obtain in this way, as in [14], for some fixed choice of circle segments on A_M and B_M uncountably many convex bodies with the same Steiner symmetrals in directions $\omega_m(K_2)$ and $\omega_m(K_2)^\perp$. For some fixed choice on A_m and B_m (and all possible choices on A_M and B_M) we obtain uncountably many convex bodies with the same Steiner symmetrals in directions $\omega_M(K_2)$ and $\omega_M(K_2)^\perp$. This completes the proof.

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