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Baire Categories in Convexity (**).

0. – Introduction.

It is well-known how useful the Baire category theorem was in Analysis for half a century and it is also known that it keeps being an important tool to distinguish between « small » and « big » sets in the absence of a measure (or in contrast to what a measure provides), or simply to prove existence theorems.

In 1959 V. Klee [35] established the first results of this kind in the geometry of convex bodies. The frame of his work was a Banach space and this is probably the reason why his results seem to have been overlooked by convex geometers. When finally, 18 years later, the interest for this research direction reappeared and P. Gruber published his first paper on this topic (rediscovering, among others, Klee's results mentioned above), the series of results along these lines did not brake any more. We intend to present here an up-to-date survey of results of this kind obtained in geometric Convexity starting with Klee's pioneering paper. Although we shall not try to be exhaustive, the more interesting results will all be mentioned. Compared with the survey article [72] published in 1985 there are here several new chapters reflecting the achievements of the last years. Moreover, some older results have been strengthened in the meantime. For applications of the Baire category theorem not very surprising, we discover sometimes rather strange objects, whose existence was unknown before.

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A set in a topological space is called *nowhere dense*, if its closure has empty interior. Any countable union of nowhere dense sets is said to be of *first category*. If a set is not of first category, then it is of *second category*. A topological space, each open set of which is of second category, is called a *Baire space*. A set in a Baire space is called *residual* if its complement is of first category.

The space $\boldsymbol{R}^d$ with the Euclidean distance, the space $\mathscr{K}$ of all compact sets in $\boldsymbol{R}^d$ with the Hausdorff distance δ, its subspaces $\mathscr{T}$ and $\mathscr{K}^*$ of all starshaped sets (always supposed compact) and of all compact convex sets respectively, the subspaces of $\mathscr{K}$ of all convex bodies and of all convex surfaces (i.e. d-dimensional members of $\mathscr{K}^*$ and their boundaries, respectively), any convex surface with the intrinsic metric, are all examples of Baire spaces.

We say that *most* and *typical* elements of a Baire space enjoy a certain property, if those not enjoying it form a set of first category, i.e. if those enjoying it form a residual set.

In a metric space (X, ϱ) we call a set M *porous at* $x \in X$ if there is a positive number α such that for any positive number ε, there is a point y in the open ball $B(x, \varepsilon)$ of centre x and radius ε such that

$$B\big(y, \alpha\varrho(x, y)\big) \cap M = \emptyset\,.$$

A set which is porous at all points of X is simply called *porous* [13], [74]. If, for some $x \in X$, the above number α can be chosen as close to 1 as we wish, the set M is called *strongly porous at* x. A set which is strongly porous at every point is said to be *strongly porous*.

A countable union of porous sets is called σ-*porous*. We say that *nearly all* elements of a metric Baire space have a certain property, if those which do not enjoy it form a σ-porous set [74].

By Lebesgue's density theorem, any prous set in $\boldsymbol{R}^d$ is of measure zero. Therefore, any σ-porous set in $\boldsymbol{R}^d$ is of first category and of measure zero. Thus, σ-porosity is a convincing smallness attribute and has the advantage to be available in spaces like $\mathscr{K}^*$, in which no geometrically useful Borel measure exists (for a precise formulation see C. Bandt and G. Baraki [5], Theorem 3).

1. – Smoothness and strict convexity of convex surfaces.

The closure of an open convex set in $\boldsymbol{R}^d$ is called a *convex body*, its boundary a *convex surface*. A convex surface needs not be smooth, but must be smooth almost everywhere, with respect to the $(d-1)$-

dimensional Hausdorff measure, as K. Reidemeister [52] proved in 1921 for $d = 3$. For a more precise result and higher dimension, see [3].

Klee's first result mentioned in the Introduction asserts that most convex surfaces are smooth and strictly convex. We have the following strengthening via porosity:

THEOREM 1 [73]. *Nearly all convex surfaces are smooth and strictly convex.*

A generalization of Klee's result in the planar case, for convex curves on arbitrary 2-dimensional convex surfaces, is reported in Section 3.

2. – The curvature of convex surfaces.

Nearly all convex surfaces are smooth, hence of class C^1.

The question whether a typical convex surface also belongs to C^2 was answered by Gruber [20] in the negative.

Let x be a point of the convex surface S at which S is smooth. At x we consider the tangent direction τ, the normal section of S in direction τ and the lower and upper radii of curvature $\varrho_i^\tau(x)$ and $\varrho_s^\tau(x)$ of the normal section (see [5], p. 14). The numbers

$$\gamma_i^\tau(x) = \varrho_s^\tau(x)^{-1}, \quad \gamma_s^\tau(x) = \varrho_i^\tau(x)^{-1}$$

are the *lower* and *upper curvatures* of S at x in direction τ. If they are equal, the common value $\gamma^\tau(x)$ is the *curvature* of S at x in direction τ.

By a theorem of A. D. Aleksandrov [1] (H. Busemann and W. Feller [11] for $d = 3$), on every convex surface there exists a finite curvature a.e. in every tangent direction. How behaves the curvature of typical convex surfaces?

G. de Rham studied more than 30 years ago the following remarkable kind of convex curves in $\boldsymbol{R}^2$: Take a convex polygon, then the two points on every side dividing it into three equal parts. Consider the convex hull of all these points and its boundary polygon. Repeat this procedure. The intersection of all these infinitely many convex sets is a convex set the frontier of which is a smooth convex curve with vanishing curvature a.e., as de Rham has shown [53].

There is another simple way of producing convex curves with vanishing curvature a.e.: Consider any *singular function* (also called

Vitali function) i.e. a strictly increasing continuous function $f\colon [0,1] \to \boldsymbol{R}$ with $f' = 0$ a.e. Then $\int f$ is a convex function, the graph of which has a vanishing curvature wherever $(\int f)'' = 0$, namely a.e. Unexpectedly, it turns out that this curvature behaviour is typical for convex curves.

R. Schneider [55] proved that on most convex surfaces there is a dense set of points x with $\gamma_i^\tau(x) = 0$ and $\gamma_s^\tau(x) = \infty$ in every tangent direction τ. It can be shown that the above set is even residual.

The following theorem describes the curvature behaviour of typical convex surfaces:

THEOREM 2 [65], [66], [76]. *For most convex surfaces* S,

(i) *at each point* $x \in S$,

$$\gamma_i^\tau(x) = 0 \quad \textit{or} \quad \gamma_s^\tau(x) = \infty$$

for any tangent direction τ *in* x,

(ii) $$\gamma^\tau(x) = 0 \ \textit{a.e.}$$

for any tangent direction τ *in* x,

(iii) *at most points* $x \in S$,

$$\gamma_i^\tau(x) = 0 \quad \textit{and} \quad \gamma_s^\tau(x) = \infty$$

for any tangent direction τ *in* x,

(iv) $$\{(x, \tau)\colon \gamma^\tau(x) = \infty\}$$

is uncountable and dense in the sphere bundle associated to S,

(v) $$\{(x, \tau)\colon 0 = \gamma_i^{-\tau}(x) < \gamma_s^{-\tau}(x) = \gamma_i^\tau(x) < \gamma_s^\tau(x) = \infty\}$$

is uncountable and dense in the sphere bundle associated to S.

Since the Dupin indicatrix at any point is convex if it exists, from Theorem 2, (iv) it follows that at many points x on a typical convex surface there is a $(d-2)$-dimensional open halfsphere H in the unit sphere of all tangent directions in x, such that $\gamma^\tau(x) = \infty$ for all $\tau \in H (d \geqslant 3)$. The following question is still unanswered.

PROBLEM 1. Do most convex surfaces in $\boldsymbol{R}^d$ $(d \geqslant 3)$ possess a point with existing and infinite curvature in every tangent direction?

3. – Geodesics and convexity on convex surfaces.

For any two points of a convex surface in $\boldsymbol{R}^d$ there exists at least one shortest path joining them. Such a path is called a *segment*. A curve which is locally a segment is called a *geodesic* (see for a precise definition [10], p. 77). By a *simple closed geodesic* we understand a geodesic which is a closed Jordan curve.

A set M in a convex surface which, for any pair of points $x, y \in M$, contains a segment joining x to y, is called *convex*. A connected set which is the boundary of a convex set in a convex surface $S \subset \boldsymbol{R}^3$ is called *convex curve* on S. A convex curve on a convex surface in $\boldsymbol{R}^3$ is called *strictly convex* if it contains no segment.

Two points of a convex surface which are joined by more than one segment are called *conjugate*. A point of a convex surface which is not for any segment interior (that is different from its endpoints) is called an *endpoint* of the surface. It is well-known that in a certain tangent direction at a point of a convex surface may not start any segment. Such a tangent direction is called by Aleksandrov *singular*.

Every conical point of a convex surface is an endpoint, but there also exist smooth endpoints (see [2], p. 58-59). However, a convex surface of class C^2 has no endpoints.

Concerning the singular directions, Aleksandrov proved that there are smooth convex surfaces with a dense set of singular directions at a certain point ([2], p. 59). But he also showed that, for any convex surface in $\boldsymbol{R}^3$, at each point, the set of singular directions has measure zero in the associated circle S^1 of all tangent direction ([2], p. 213). Also, at no point of a convex surface of class C^2 there is any singular direction.

The following theorem describes the typical convex surfaces with respect to their endpoints and singular directions. It shows that there are unexpectedly many endpoints and illustrates once again the possible contrast between the measure-theoretical and the topological behaviour.

THEOREM 3 [69]. *On most convex surfaces* $S \subset \boldsymbol{R}^d$,

(i) *most points of* S *are endpoints,*

(ii) *for* $d = 3$, *at every point of* S, *most tangent directions are singular.*

Concerning the conjugate points on a convex surface we have the following result.

THEOREM 4 [84]. *Let S be any convex surface in $\boldsymbol{R}^3$ and $x \in S$. Then nearly all points of S are not conjugate to x.*

In higher dimensions the situation is not yet clarified for all convex surfaces. However, Gruber [25] could settle the typical case.

THEOREM 5. *If $S \subset \boldsymbol{R}^d$ is a typical convex surface or $S \in C^3$ and if $x \in S$ then most points of S are not conjugate to x.*

The proof in case $S \in C^3$ (similar to the proofs in [25] and [84]) goes as follows:

Let $S \subset \boldsymbol{R}^d$ be a convex surface of class C^3, $x \in S$ and

$$S_m = \{y \in S \colon \exists \text{ segments } F, G \text{ from } x \text{ to } y \text{ with } \delta(F, G) \geqslant m^{-1}\}.$$

We shall prove that S_m is nowhere dense. This finishes the proof, because then the set $\bigcup\limits_{m=1} S_m$ of all points conjugate to x is of first category.

First, S_m is closed. Indeed, if $y_n \in S_m$, F_n and G_n are segments from x to y_n, $\delta(F_n, G_n) \geqslant m^{-1}$ and $y_n \to y$ then, for a subsequence of indices $\{n_i\}_{i=1}^{\infty}$, $\{F_{n_i}\}_{i=1}^{\infty}$ and $\{G_{n_i}\}_{i=1}^{\infty}$ converge, the limit curves are segments (see [10], p. 81), say F and G, and moreover $\delta(F, G) \geqslant m^{-1}$.

Now let O be open in S. Take $y_0 \in O \setminus \{x\}$ and consider a segment F_0 from x to y_0. Let $y_1 \in O \cap F_0$ be different from x and y_0. If y_1 and x were conjugate then a segment $G_0 \not\subset F_0$ from y_1 to x would exist. Let H_0 be the subsegment of F_0 from y_0 to y_1. Then the geodesics $H_0 \cup G_0$ and F_0 would have the arc H_0 in common. But the surface S is of class C^3, hence the geodesic H_0 can be extended in a unique way, being the unique solution of a differential equation. This contradiction shows that y_1 and x are not conjugate, which yields $O \setminus S_m \neq \emptyset$, whence S_m is nowhere dense.

PROBLEM 2. Prove that for any convex surface S in $\boldsymbol{R}^d$ and any point $x \in S$, most points of S are not conjugate to x ($d \geqslant 4$).

Thus, in many cases, possibly always, the set of points conjugate to some point on a convex surface is small. The following result first proved by Gruber [25] for $d = 3$ shows that typically the set of points conjugate to some point on a convex surface cannot be too small, however.

THEOREM 6 [84]. *For most convex surfaces $S \subset \boldsymbol{R}^d$ and any point $x \in S$, the set of points conjugate to x is dense in S.*

We next pass to a global problem, which attracted for a long time a lot of attention: the number of closed geodesics (simple or not) on a closed surface, in our case a convex one.

Let $d = 3$. Arguments of Aleksandrov ([2], p. 377, 378) show that, with the obvious identification between the space $\mathscr{P}_n$ of all polytopes in $\boldsymbol{R}^3$ with n vertices and a set in $\boldsymbol{R}^{3n}$, and with the naturally induced measure on $\mathscr{P}_n$, the boundaries of almost all polytopes in $\mathscr{P}_n$ admit no simple closed geodesic. On the other hand, by a well-known result of L. Lusternik and L. Schnirelman [45], there are at least 3 distinct simple closed geodesics on any sufficiently smooth convex surface. The typical case was settled by Gruber. He recently proved the following very beautiful theorem.

THEOREM 7 [26]. *Most convex surfaces in $\boldsymbol{R}^3$ admit no closed geodesics.*

This result is considerably harder to prove than the corresponding result restricted to simple closed geodesics, see [25].

It should be mentioned here that, by a result of A. Pogorelov [50], on any convex surface in $\boldsymbol{R}^3$ there are at least three quasigeodesics.

About the length of geodesics on typical convex surfaces, inspite of the lack of knowledge about the situation for arbitrary convex surfaces, we have the following surprising result.

THEOREM 8 [83]. *On most convex surfaces in $\boldsymbol{R}^3$ there are arbitrarily long geodesics (of finite length) without self-intersections.*

The space $\mathscr{C}$ of all convex curves on a convex surface in $\boldsymbol{R}^3$ with the Hausdorff metric derived from its intrinsic metric is a Baire space. It makes sense to ask whether the typical behaviour of such convex curves parallels the typical behaviour of convex curves in $\boldsymbol{R}^2$. Are they, for example, smooth and strictly convex? It is rather obvious that smoothness cannot be a generic property for all convex surfaces S: Take, for instance, S to be polytopal. But in smooth S most convex curves are indeed smooth. By Theorem 1 this is true for most S. Also the strict convexity is not for all S a generic property of convex curves in S. It is, however, a generic property in case the set $\mathscr{G}$ of all convex closed geodesics has empty interior in the space $\mathscr{C}$ of all convex curves in S. This is a rather weak requirement. It is, by Theorem 7, trivially fulfilled by most convex surfaces $S \subset \mathbf{R}^3$. If int $\mathscr{G} \neq \emptyset$ then $\mathscr{C} \setminus$ int $\mathscr{G}$ is also a Baire space.

THEOREM 9 [75]. *If S is a smooth convex surface in $\mathbf{R}^3$ then most convex curves in S are smooth. If S is a convex surface in $\mathbf{R}^3$ then most elements of $\mathscr{C} \setminus$ int $\mathscr{G}$ are strictly convex.*

PROBLEM 3. Generalize Theorem 9 to arbitrary dimension $d \geqslant 3$.

4. – The shadow boundaries of convex bodies.

Let K be a convex body in $\mathbf{R}^d$ and $M \subset \mathbf{R}^d$ a set viewed as a light source. The *shadow boundary* $\Gamma(K, M)$ of K with respect to M is the set of all points $y \in \text{bd}\, K$ such that every line through y which meets M misses int K.

The cases when the light source M is a point in $\mathbf{P}^d$ or a flat of dimension at most $d-3$ in the hyperplane at infinity have been repeatedly considered in the literature. We present here some generic properties of the shadow boundaries.

We shall speak below about typical $M \subset \mathbf{P}^d \setminus \mathbf{R}^d$, where M is a k-dimensional flat. This will always mean that the $(k+1)$-dimensional linear subspace of $\mathbf{R}^d$ determined by M is typical in the Grassmannian manifold $\mathscr{G}_{k+1}$ of all $(k+1)$-dimensional linear subspaces of $\mathbf{R}^d$.

Let F be a $(d-3)$-dimensional flat in the hyperplane at infinity. Consider the 2-dimensional subspace P of $\mathbf{R}^d$ orthogonal to the $(d-2)$-dimensional subspace of $\mathbf{R}^d$ determined by F. If K is a convex body in $\mathbf{R}^d$, let $p_F\colon K \to P$ be the orthogonal projection. If K is strictly convex, $p_F^{-1}|_{\text{rlbd}\, p_F(K)}$ is single-valued and continuous (rlbd means relative boundary); the Jordan curve $p_F^{-1}(\text{rlbd}\, p_F(K))$ is precisely the shadow boundary $\Gamma(K, F)$ of K with respect to F. $\Gamma(K, F)$ is called *singular* if all its tangent lines (if any) meet F. The shadow boundary $\Gamma(K, F)$ also determines a real function $f\colon \text{rlbd}\, p_F(K) \to \mathbf{R}$ defined as the distance from a point $x \in \text{rlbd}\, p_F(K)$ to the single point of $p_F^{-1}(x)$. Obviously, $\Gamma(K, F)$ is singular if and only if f is nowhere differentiable. Since such a function is not of bounded variation, a singular $\Gamma(K, F)$ is nonrectifiable.

THEOREM 10 [77]. *Let $F \subset \mathbf{P}^d$ be a fixed or a typical $(d-3)$-dimensional flat at infinity. For most convex bodies K, $\Gamma(K, F)$ is singular, hence nonrectifiable.*

P. Gruber and H. Sorger considered the case when M contains a single point $x \in \boldsymbol{P}^d$ and proved the following result which parallels the assertion of Theorem 10 regarding the generic infinite length of shadow boundaries, and confirms their (naturally expected) Hausdorff dimension $d-2$.

THEOREM 11 (Gruber and Sorger [29]). *Let x be a fixed point in $\boldsymbol{P}^d$ or a typical point of $\boldsymbol{R}^d$ or of $\boldsymbol{P}^d \setminus \boldsymbol{R}^d$. For most convex bodies $K \subset \boldsymbol{R}^d$, if $x \notin K$ then $\Gamma(K, \{x\})$ has Hausdorff dimension $d-2$ and infinite $(d-2)$-dimensional Hausdorff measure.*

The case when x is a given point in $\boldsymbol{P}^d$ was not explicitly stated in [29] but follows from the same arguments. The case of a single point light source is more complicated because the shadow boundary is then higher dimensional. In the proof, an integral geometric surface area measure was used since Hausdorff measures lack a certain semi-continuity property. Still open is the following problem, first formulated in [29].

PROBLEM 4. Is the $(d-2)$-dimensional Hausdorff measure of $\Gamma(K, \{x\})$ for most convex bodies $K \subset \boldsymbol{R}^d$ and most points $x \notin K$ non-σ-finite?

There is a remarkable contrast between the above theorems and measure-theoretical results of P. Steenaerts [59], D. G. Larman and P. Mani-Levitska [39]. Denote by $F(\alpha)$ the k-dimensional flat in the hyperplane at infinity corresponding to $\alpha \in \mathscr{G}_{k+1}$ and let μ_k be the k-dimensional Hausdorff measure. By results in [59] and [39], for any convex body $K \subset \boldsymbol{R}^d$,

$$\mu_k\big(\Gamma(K, F(\alpha))\big) < \infty$$

for almost all $\alpha \in \mathscr{G}_{d-k-1}$ with respect to the Haar measure. Thus, for typical convex bodies the contrast is perfect!

Since the boundary of a $(d-1)$-dimensional convex body has finite $(d-2)$-dimensional Hausdorff measure, it follows from Theorem 11 that for most convex bodies $K \subset \boldsymbol{R}^d$ and most points $x \in \boldsymbol{R}^d$ or $x \in \boldsymbol{P}^d \setminus \boldsymbol{R}^d$, $\Gamma(K, \{x\})$ is not contained in a hyperplane.

In 1986 F. Hering conjectured that there are convex bodies $K \subset \boldsymbol{R}^d$ such that, for no point $x \in \boldsymbol{R}^d \setminus K$, $\Gamma(K, \{x\})$ is contained in a hyperplane. Indeed, the following is true.

THEOREM 12 [80]. *For most convex bodies $K \subset \boldsymbol{R}^d$, the shadow boundary $\Gamma(K, \{x\})$ is contained in a hyperplane for no point $x \in \boldsymbol{P}^d \setminus K$.*

Mani-Levitska [46] proved that—in the case of parallel « light rays »—there are classes of convex bodies K which can cover all their shadows (i.e. there is for any $x \in \boldsymbol{R}^d \setminus \{0\}$ a rigid motion c such that $c(p_{x^\perp} K) \subset K$), and classes of convex bodies which cannot cover all their shadows. He also observes (private communication) that every convex body which can cover all its shadows must have a planar shadow boundary. This together with Theorem 12 shows that most convex bodies cannot cover all their shadows. In fact more can be shown: Those convex bodies which can cover all their shadows form a nowhere dense set.

Results, generic or not, on shadow boundaries with respect to flats of positive dimension lying in $\boldsymbol{R}^d$ await to be discovered. Also, light sources M which are not flats might be of interest.

5. – Normals to convex surfaces.

We shall now consider normal lines to convex surfaces. E. Heil [34] proved that any convex surface in $\boldsymbol{R}^d$ admits a point lying on at least 6 normals to the surface ($d \geqslant 3$). In the planar case, for any convex curve there is a point belonging to 4 normals to the curve.

For any usual surface, the points lying on infinitely many normals are exceptional. However, this is not true for typical convex surfaces, as the following surprising result shows.

THEOREM 13 [68], [71]. *For most convex surfaces, most points of $\boldsymbol{R}^d$ lie on infinitely many normals.*

In Problem 4 from [72] we asked whether for most convex surfaces there is any point lying on uncountably many points. I. Bárány and the author proved that this is indeed the case. In fact the following more detailed description was obtained. Let $\psi(x)$ be the set of directions (unit vectors) of all normals passing through $x \in \boldsymbol{R}^d$.

THEOREM 14 (Bárány and Zamfirescu [6]). *Let $Z \subset \boldsymbol{R}^d$ be a countable set. For most convex suefaces the following is true: For any point $x \in Z$ the set $\psi(x)$ is perfect and for any point $x \in \boldsymbol{R}^d$ the set $\psi(x)$ is porous in S^{d-1}.*

Recently, M. Laczkovich [38] succeeded to show in the planar case that, for most convex curves, most points in $\boldsymbol{R}^2$ lie in uncountably many normals. Unfortunately, his proof is not extendable to higher dimensions.

Problem 5. Do most points of $\boldsymbol{R}^d$ lie on uncountably many normals to most convex surfaces ($d \geqslant 3$)?

In the spirit of Theorem 13 is the following reflection result. Let C be a smooth convex curve in $\boldsymbol{R}^2$ and consider $x, y \in \boldsymbol{R}^2$ and $M \subset C$ with card $M = \alpha$. If, for each $z \in M$, xz and yz make equal angles with C, we say that *x sees α images of y*. In general it can only be said that a point sees two images of another point. For typical convex curves a stronger assertion is valid.

Theorem 15 [68]. *For most convex curves in $\boldsymbol{R}^2$, every point of $\boldsymbol{R}^2$ sees $\aleph_0$ images of most other points, and for most pairs $(x, y) \in \boldsymbol{R}^2 \times \boldsymbol{R}^2$, x sees $\aleph_0$ images of y.*

Also this result was not yet generalized to $\boldsymbol{R}^d$.

6. – Diameters of convex bodies.

A *diameter* of a convex body $K \subset \boldsymbol{R}^d$ is a chord of K such that K admits parallel supporting hyperplanes at its endpoints.

It follows from a result of A. Kosiński [37] that in every convex body there is a point lying on at least three diameters. This statement is possibly improvable for $d \geqslant 3$, but nothing better seems to be known at present.

In the typical case many more diameters must meet together. We have the following result, first proved in the planar case by a different method in [70].

Theorem 16 (Bárány and Zamfirescu [6]). *For most convex bodies $K \subset \boldsymbol{R}^d$, most points of K lie on infinitely many diameters.*

Let x be a point of the convex body K. We denote by $\varphi(x)$ the set of all directions of diameters of K passing through x. The generic aspect of $\varphi(x)$ is described by the following theorem.

Theorem 17 (Bárány and Zamfirescu [6]). *Let $Z \subset \boldsymbol{R}^2$ be countable. For most convex bodies $K \subset \boldsymbol{R}^d$, the following is true: At each point $x \in Z \cap K$ the set $\varphi(x)$ is perfect and at each point $x \in K$ the set $\varphi(s)$ is porous in S^{d-1}.*

It is remarkable how Theorems 16 and 17 parallel Theorems 13 and 14 respectively.

In 1965 P. C. Hammer [33] raised the question whether there exist a convex body K and a point $z \in \operatorname{int} K$ such that the set $R(z)$ of all ratios into which z divides the various diameters through z is uncountable. A. S. Besicovitch and T. Zamfirescu [7] answered the question by providing such a convex body and such an interior point. In fact this is a generic property of convex bodies:

THEOREM 18 (Bárány and Zamfirescu [6]). *Let $Z \subset \boldsymbol{R}^d$ be countable. For most convex bodies $K \subset \boldsymbol{R}^d$, at each point $x \in Z \cap K$, the set $R(x)$ is uncountable.*

Let M_α (respectively T_α) be the set of all interior points of the convex body $K \subset \boldsymbol{R}^d$ lying on at least (respectively precisely) α diameters. Generic connectivity properties of M_α have been investigated, but only in the planar case:

THEOREM 19 [70]. *For most convex bodies in $\boldsymbol{R}^2$, the set M_α is connected for every $\alpha \leqslant \aleph_0$ and the set T_α is totally disconnected for every $\alpha < \aleph_0$.*

7. – Circumscribed balls, shells and ellipsoids.

The union of two concentric spheres is called a *shell.*

A sphere $\mathcal{S}(S)$ is said to be *circumscribed* to $M \subset \boldsymbol{R}^d$ if $M \subset \operatorname{conv} \mathcal{S}(M)$ and $\mathcal{S}(M)$ has minimal radius.

A shell $\mathcal{H}(S)$ is said to be *circumscribed* to $M \subset \boldsymbol{R}^d$ if M lies between (and on) the two spheres forming $\mathcal{H}(S)$ and the difference between their radii is minimal.

An ellipsoid $\mathcal{E}(S)$ is said to be *circumscribed* to $M \subset \boldsymbol{R}^d$ if $M \subset \operatorname{conv} \mathcal{E}(S)$ and $\operatorname{conv} \mathcal{E}(S)$ has minimal volume.

It is easily checked that, for any convex surface $S \subset \boldsymbol{R}^d$, the sets of contact points satisfy

$$2 \leqslant \operatorname{card}(S \cap \mathcal{S}(S)) \leqslant \mathfrak{c}\,,$$
$$4 \leqslant \operatorname{card}(S \cap \mathcal{H}(S)) \leqslant \mathfrak{c}\,,$$
$$d + 1 \leqslant \operatorname{card}(S \cap \mathcal{E}(S)) \leqslant \mathfrak{c}\,.$$

and that any cardinal number between the given lower and upper bounds is realizable.

Can the number of contact points be more precisely estimated in the case of typical convex surfaces? Yes, indeed, the above number can be exactly determined in the typical case!

THEOREM 20. *For most convex surfaces $S \subset \mathbf{R}^d$ the following holds*:

$$\operatorname{card}(S \cap \mathcal{S}(S)) = d + 1 \ [67]\,.$$

$$\operatorname{card}(S \cap \mathcal{H}(S)) = d + 2 \ \text{(A. Zucco [86], [87])}\,.$$

$$\operatorname{card}(S \cap \mathcal{E}(S)) = d(d+3)/2 \ \text{(Gruber [22])}\,.$$

The third result in Theorem 20 had been jointly conjectured by Gruber and the author. Its proof in [22] involves ingenious, not easy arguments.

Notice that the set of all surfaces S, such that the contact points are fewer than one of the above numbers, is nowhere dense, while the set of those surfaces S, such that, for any given α larger than one of the above numbers, the corresponding number of contact points equals α, is dense.

8. – Approximation by polytopes.

The set $\mathcal{P}_n$ of all polytopal surfaces in $\mathbf{R}^d$ with at most n vertices is closed in $\mathcal{K}$. By Blaschke's selection theorem, for any convex surface S and $n > d$, there exists $P^* \in \mathcal{P}_n$ such that

$$\delta(S, P^*) = \nu(S, n)\,,$$

where

$$\nu(S, n) = \inf_{P \in \mathcal{P}_n} \delta(S, P).$$

Such a polytopal surface P^* is called *best approximation* of S. Clearly, the best approximations of a convex surface need not be unique. But P. Gruber and P. Kenderov proved the following result.

THEOREM 21 [28]. *For $d = 2$ and any $n > 3$, most convex curves admit a unique best approximation.*

A refinement of this result was given by V. Zhivkov [85]. R. Schneider and J. Wieacker [51] and, independently, Gruber and Kenderov [28] studied the asymptotic behaviour for $n \to \infty$ and found that it is typically very irregular:

THEOREM 22. *Let $f\colon \mathbf{N} \to [0, \infty)$ be arbitrary and $g\colon \mathbf{N} \to [0, \infty)$ satisfy $g(n) = o(1/n^{2/(d-1)})$ as $n \to \infty$. Then, for most convex surfaces S,*

$$\nu(S, n) < f(n)$$

for infinitely many n and

$$\nu(S, n) > g(n)$$

for infinitely many n.

Analogous results have also been obtained with respect to other metrics than Hausdorff's [28].

Seen more abstractly, a best approximation in a given set of some element is a point (from the given set) closest to that element. This explains the special interest in the nearest point mapping, for which we describe some generic properties in the following section.

9. – Nearest point mapping.

We shall consider here the nearest point mapping p_K defined on $\boldsymbol{R}^d$ as the multi-valued function:

$$p_K(x) = \left\{y \in K:\ \|x-y\| = \min_{z\in K} \|x-z\|\right\},$$

where K is a compact set in $\boldsymbol{R}^d$. In particular the set K can be convex. In that case (and only then) the function p_K is single-valued everywhere. A natural question to ask is about the proportion between the set on which p_K is single-valued and the set on which P_K is properly multi-valued. The answer is known and the same from both points of view, measure-theoretical and topological (via Baire categories): For any compact set K, the nearest point mapping p_K is not single valued on a set of measure zero and first category [58]. Here we can use again the notion of porosity and strengthen the above statement in the following way:

THEOREM 23 [82]. *The nearest point mapping is single-valued at nearly all points of* $\boldsymbol{R}^d$.

A question which then comes into one's mind is whether

$$K^+ = \{x \in \boldsymbol{R}^d:\ \mathrm{card}\, p_K(x) > 1\}$$

must even be nowhere dense or not. That this need not be the case is shown by the next result.

THEOREM 24 [82]. *For most compact sets $K \subset \mathbf{R}^d$, the nearest point mapping p_K is not single-valued at a dense set of points.*

If the boundary of K is smooth enough K^+ can no longer be dense. This is shown by the next theorem.

THEOREM 25 [82]. *If $K \subset \mathbf{R}^2$ is closed and* $\mathrm{bd}\, K$ *is an analytic Jordan curve then p_K is not single-valued on a nowhere dense set.*

While extending this to higher dimensions is only a technical problem, more interesting and possibly difficult is to reduce the degree of smoothness of $\mathrm{bd}\, K$ until a further reduction would lead to a wrong statement. We formulate this as a problem.

PROBLEM 6. Prove, for a minimal integer k, that for any continuum $K \subset \mathbf{R}^d$ with a boundary of class C^k the function p_K is not single-valued on a nowhere dense set.

Of a special interest are also the differentiability properties of the nearest point mapping p_K. As E. Asplund proved in [4], p_K is not only single-valued almost everywhere, but also Fréchet differentiable almost everywhere. From the viewpoint of Baire categories the situation now changes. We shall see that, for many sets K, p_K is not differentiable at most points outside K.

Let now K be convex. By $p'_K(y)$ we denote the Fréchet derivative of p_K at $y \in \mathbf{R}^d$ and by $P_K(y)$ the orthogonal projection of $\mathbf{R}^d$ onto the hyperplane

$$H(y) = \{z \in \mathbf{R}^d \colon \langle y - p_K(y), z\rangle = 0\}\,.$$

It is known [16] that the operators $p'_K(y)$ and $P_K(y)$ satisfy:

$$p'_K(y) \circ P_K(y) = p'_K(y) = P_K(y) \circ p'_K(y)\,.$$

The generic aspect of the Fréchet derivative is described by the next theorem.

THEOREM 26 [81]. *For most convex bodies $K \subset \mathbf{R}^d$,*

(i) $p'_K(y) = P_K(y)$ *for any* $y \in p_K^{-1}(x)$ *and almost every* $x \in \mathrm{bd}\, K$,

(ii) $p'_K(y)$ *does not exist at most points* $y \notin K$.

The existence of convex bodies K for which the requirement (ii) is satisfied has been previously verified for $d = 2$ by L. Zajíček [64].

In the planar case additional information on the generic differentiability properties of p_K is provided by the following result.

THEOREM 27 [81]. *For most convex bodies $K \subset \mathbf{R}^2$,*

(i) *p_K has no directional derivative in any nonnormal direction at most points outside K,*

(ii) *$p'_K = \mathbf{0}$ at a set of points dense in $\mathbf{R}^2 \setminus K$.*

10. – Billiards.

We regard any smooth convex body K in $\mathbf{R}^d$ as a billiard table. A point playing the rôle of a billiard ball will move along straight lines in the interior of K and reflect obeying the light reflection law at the boundary points of K. The broken line with infinitely many (not necessarily distinct) vertices described by such a point is called *trajectory*. A compact convex set $C \subset \operatorname{int} K$ is called a *caustic* if any trajectory which touches K once touches it again after the next reflection. A strange phenomenon, that of a trajectory with finite length, was studied by B. Halpern [31] and Gruber [24]. The *phase space* $\operatorname{ph} K$ associated to K is, by definition, the set

$$\{(p, v) : p \in \operatorname{bd} k, v \in S_{d-1}, \langle v, n(p)\rangle < 0\} ,$$

where $n(p)$ denotes the exterior normal unit vector of $\operatorname{bd} K$ at p. For $(p, v) \in \operatorname{ph} K$, let $T(p, v)$ denote the trajectory starting at p in direction v.

It is known that there are convex bodies admitting no caustic, so are for example all convex bodies which are not strictly convex. In $\mathbf{R}^2$ all convex bodies whose boundaries are sufficiently smooth (of class C^7 [14]; the smoothness degree could gradually be decreased from class C^{553} to class C^7, see [40], [41], [42], [48], [54], [9]) and have positive curvature admit caustics. For $d \geqslant 3$ no convex body different from an ellipsoid and having a caustic seems to be known. Gruber [24] conjectures that they do not exists.

Concerning the trajectories of finite length, it is known that convex bodies of class C^3 have no such trajectory. This was proved by Halpern [31] in $\mathbf{R}^2$ and by Gruber [24] for $d \geqslant 3$.

Let β be the Borel measure defined by

$$\beta(B) = -\int_B \langle v, n(p)\rangle \, d\sigma \times \tau$$

for any Borel set $B \subset \mathrm{ph}\, K$, where σ and τ denote the surface area measures on bd K and S^{d-1} respectively. The next result shows that, in any convex body, a trajectory of finite length is rather exceptional.

THEOREM 28 (Gruber [24]). *Let $K \subset \boldsymbol{R}^d$ be a convex body. For most and β-almost all pairs $(p, v) \in \mathrm{ph}\, K$, the trajectory $T(p, v)$ has infinite length.*

The generic situation is also clarified. The following result of Gruber shows that having caustics or trajectories of finite length is untypical.

THEOREM 29 [24]. *Most convex bodies in $\boldsymbol{R}^d$ contain no caustic and no trajectory of finite length.*

The vertices of a trajectory $\mathcal{T}$ in a convex body K and the directions of its line segments determine a certain infinite subset $S(\mathcal{T}) \subset \subset \mathrm{ph}\, K$. Theorem 29 is complemented in the planar case by the following:

THEOREM 30 (Gruber [24]). *For most convex bodies $K \subset \boldsymbol{R}^2$ and most pairs $(p, v) \in \mathrm{ph}\, K$, the set $S(T(p, v))$ is dense in* ph K.

It follows immediately from Theorem 30 that, for most convex bodies $K \subset \boldsymbol{R}^2$ and most pairs $(p, v) \in \mathrm{ph}\, K$, the trajectory $T(p, v)$ is dense in K.

For more details on this topic the reader should consult [24].

Another type of trajectory will appear in the next section.

11. – Tomography.

An X-ray picture of a convex body taken in a certain direction may be identified with its Steiner symmetral in that direction. In recent years P. C. Hammer's problem [32] of determining a convex body from its X-ray pictures was investigated by R. Gardner and P. McMuller [18], Gardner [17], K. Falconer [15] and A. Volčič [61]. An earlier result is due to O. Giering [19]. Gardner and McMullen proved that there are four directions such that the corresponding X-ray pictures distinguish between all convex bodies, and that no three directions can do this. Giering proved that, given a plane convex body K, there exist three directions depending on K, such that the corresponding X-ray pictures distinguish K from any other convex body. He has also shown that two directions are in general not enough.

Convex bodies with the same X-ray pictures in two fixed directions as a given one were called *ghosts* in [60], in analogy with the ghost densities from computerized tomography [44].

The characterization of those convex bodies which are uniquely determined by two X-ray pictures is still open. We mention that the analogous problem for measurable sets has been solved by G. G. Lorentz [43].

The existence of ghosts is very much related to the existence of inscribed broken lines whose line-segments are parallel to the axes. The importance of these broken lines was first pointed out by Giering.

Let $V(K)$ be the union of the vertex sets of all these broken lines inscribed in K. The following result of Volčič and the author suggests that there might be many ghosts in the space of all convex bodies.

THEOREM 31 [62]. *For most convex bodies* $K \subset \boldsymbol{R}^2$, *the set* $V(K)$ *is nonempty and nowhere dense in* bd K.

However, the next results of the same authors shows that in fact for most convex bodies X-ray pictures in two given directions are enough to distinguish them from all the others.

THEOREM 32 [62]. *Most convex bodies in* $\boldsymbol{R}^d$ *are not ghosts.*

This settles previous independent conjectures of Volčič and Gru ber [23].

12. – Compact sets.

The typical compact set is rather thin. This is not very surprising. In fact rather standard arguments show that it is both of measure zero and of first category. This suggests, of course, the question whether it is also porous. In answering this question we shall exceptionally enlarge our usual frame $\boldsymbol{R}^d$.

THEOREM 33 [74]. *In a complete convex metric space, most compact sets are strongly porous. In a Banach space, nearly all compact sets and nearly all closed bounded sets are strongly porous.*

Theorem 33 has not been stated this way in [74], but is an immediate corollary of Theorems 1, 2 and 3 in [74].

For recent refinements of Theorem 33 see Gruber's Theorems 2 and 3 in [27]. The Hausdorff dimension of typical compact sets is computed by the next result which follows in the separable case from A. Ostaszewski's paper [49]. This is also proved by Gruber in [21], where the separability assumption can be dropped, as he remarks in [27].

THEOREM 34. *In a complete metric space, most compact sets have Hausdorff dimension zero.*

This result implies that most compact sets in a complete metric space have topological dimension zero and are therefore totally disconnected. This together with the easy remark that most compact sets in a connected complete metric space are perfect shows that, in such a space, they are in fact Cantor sets.

Now we come back to the Euclidean space $\boldsymbol{R}^d$ and consider the Baire space $\mathscr{K}$ of all compact sets in $\boldsymbol{R}^d$. J. A. Wieacker studied the generic properties of the convex hull of an element in $\mathscr{K}$. In analogy to a generic property of convex bodies, we have the following result.

THEOREM 35 (Wieacker [63]). *For most* $C \in \mathscr{K}$, bd conv C *is of class* C^1, *but not of class* C^2.

Note that strict convexity is not a generic property of the convex hull of compact sets.

Also different from the generic aspect of convex bodies is the extremal aspect of the convex hull of typical compact sets. First let us recall two notions important in Convexity. A point x in a convex body $K \subset \boldsymbol{R}^d$ is called

(i) *extreme* if $K \setminus \{x\}$ is convex,
(ii) *exposed* if $\{x\} = K \cap H$ for some hyperplane H.

THEOREM 36 (Wieacker [63]). *For most* $C \in \mathscr{K}$, *the extreme points of* conv C *form a Cantor set and the exposed points of* conv C *form a set homeomorphic to the space of all irrational numbers.*

13. – Starshaped sets.

In $\boldsymbol{R}^d$, the space $\mathscr{T}$ of all starshaped sets, always considered compact here, being closed in the complete space of all compact sets, is a Baire space too. As in the case of compact sets typical starshaped set are rather thin, as shown by the next theorem.

THEOREM 37 [78]. *For most starshaped sets, their orthogonal projection on any 2-dimensional flat is nowhere dense.*

It follows from Theorem 37 that a typical member T of $\mathscr{T}$ is nowhere dense and has a kernel consisting of a single point $k(T)$. Using porosity, the first assertion was strengthened in the following manner.

THEOREM 38 (Gruber and Zamfirescu [30]). *Most starshaped sets $T \in \mathscr{T}$ are not porous at $k(T)$, but strongly porous anywhere else.*

The exceptional rôle of $k(T)$ can also be seen in the next result on local connectedness.

THEOREM 39 [78]. *Most starshaped sets $T \in \mathscr{T}$ are not locally connected at any point different from $k(T)$.*

Since, typically, the kernel of $T \in \mathscr{T}$ consists of the single point $k(T)$, generic properties of the following sets are of interest:

$$V(T) = \left\{ \frac{x - k(T)}{\|x - k(T)\|} : x \in T \setminus \{k(T)\} \right\}$$

and

$$U(T) = \{ \|x - k(T)\| : x \in T, \forall y \in T \setminus \{x\}, x \notin yk(T) \} .$$

Also, set $I(T) = [0, \max U(t)]$. The following result is due to Gruber and the author.

THEOREM 40 [30], [78]. *For most starshaped sets $T \in \mathscr{T}$,*

(i) *$V(T)$ is dense, uncountable and of first category in S^{d-1},*
(ii) *$U(t)$ is dense in $I(T)$.*

PROBLEM 7. Describe more precisely the generic topological aspect of $U(T)$.

From Theorem 38 it follows that most starshaped sets have Lebesgue measure zero. This suggests the possibility that their Hausdorff dimension is less than d. That this is indeed the case we learn from the following result of Gruber and the author.

THEOREM 41 [30]. *Most starshaped sets belonging to $\mathcal{T}$ have Hausdorff dimension* 1 *and non-σ-finite* 1*-dimensional Hausdorff measure.*

14. – Starshaped surfaces.

Let K be a compact convex set in $\boldsymbol{R}^d$ and let $\mathcal{T}_K$ be the subspace of $\mathcal{T}$ consisting of all compact starshaped sets whose kernels include K. In case $\dim K = d$, we call the boundary of any $T \in \mathcal{T}_K$ a *starshaped surface.* In polar coordinates it is represented by a Lipschitz function; hence it is differentiable a.e. Clearly every starshaped surface is homeomorphic to S^{d-1}. Being closed in the complete space $\mathcal{K}$ of all compact sets, $\mathcal{T}_K$ and the space of all starshaped surfaces are Baire spaces.

Suppose $\dim K = d$. We say that a point $x \in \operatorname{bd} T$, where $T \in \mathcal{T}_K$, *sees only* K if $xy \subset T$ implies the existence of a point $z \in K$ collinear with x and y. The strange aspect of the typical starshaped surfaces is pointed out in the next theorem.

THEOREM 42 [79]. *Assume* $\dim K = d$. *For most starshaped surfaces* S,

(i) *most point of* S *see only* K,

(ii) *for almost all points* $x \in S$, *there is a supporting hyperplane of* K *which is tangent to* S *at* x.

It follows that most members of $\mathcal{T}_K$ have precisely K as kernel. Since the convex body K was chosen arbitrarily, this places in a new light an old question of L. Fejes Tóth, whether every compact convex body is the kernel of a nonconvex set. Constructive answers to Fejes Tóth's question were given in the plane by K. Post [51], in Banach spaces by Klee [36] and, independently, in Euclidean spaces by M. Breen [8]. I am indebted to Klee for having suggested the connection between Theorem 42 and Fejes Tóth's question.

New let $\dim K = d - 1$. While a member of $\mathcal{T}_K$ may be locally disconnected and different from the closure of its interior, the following theorem reveals a surprisingly nonpathological typical aspect.

THEOREM 43 [79]. *Assume* $\dim K = d - 1$. *Most members of* $\mathcal{T}_K$ *are homeomorphic to a ball.*

For generic properties of members of $\mathcal{T}_K$ in the case of a lower-dimensional convex set K, see [79].

15. – Convex sets of convex sets.

The convexity in the space $\mathscr{K}^*$ of all compact convex sets in $\boldsymbol{R}^d$ was not yet intensively investigated. Let $A, B \in \mathscr{K}^*$. Then

$$\{\lambda A + (1-\lambda)B \colon \lambda \in [0,1]\}$$

is called the *segment of endsets* A, B. A set $\mathscr{A} \subset \mathscr{K}^*$ is said to be *convex* if, for any two sets $A, B \in \mathscr{A}$, the segment of endsets A, B lies in $\mathscr{A}$. For a convex set $\mathscr{A} \subset \mathscr{K}^*$, $A \in \mathscr{A}$ is called an *extreme element* of $\mathscr{A}$ if A belongs only as an endset to segments in $\mathscr{A}$. Let $\operatorname{ext} \mathscr{A}$ be the set of all extreme elements of $\mathscr{A}$.

Let $\mathfrak{C}$ be the space of all convex closed bounded sets (not elements!) in $\mathscr{K}^*$. These notions are considered with respect to the Hausdorff distance δ. We equip $\mathfrak{C}$ with Hausdorff's metric derived from δ, too. Since $\mathscr{K}^*$ is complete, the space $2^{\mathscr{K}^*}$ of all closed bounded sets in $\mathscr{K}^*$ is also complete. Also $\mathfrak{C}$, being closed in $2^{\mathscr{K}^*}$, is complete and therefore a Baire space.

The following result presents generic properties of members of $\mathfrak{C}$. Its second part, due to T. Schwarz and the author, confirms a conjecture in [72].

THEOREM 44. *For most members* $\mathscr{A} \in \mathfrak{C}$,

(i) $\mathscr{A}$ *is nowhere dense in* $\mathscr{K}^*$ [72],

(ii) *most elements of* $\mathscr{A}$ *are extreme* [57].

Still without an answer remained the following problem from [72].

PROBLEM 8. Prove (or disprove) that, for most $\mathscr{A} \in \mathfrak{C}$, the extreme elements of $\mathscr{A}$ form an arcwise connected set.

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REFERENCES

[1] A. D. ALEKSANDROV, *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it* (Russian), Uchenye Zapiski Leningrad. Gos. Univ., Math. Ser., **6** (1939), pp. 3-35.

[2] A. D. Aleksandrov, *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin, 1955.

[3] R. D. Anderson - V. L. Klee, *Convex functions and upper semi-continuous collections*, Duke Math. J., **49** (1952), pp. 349-357.

[4] E. Asplund, *Differentiability of the metric projection in finite-dimensional Euclidean space*, Proc. Amer. Math. Soc., **38** (1973), pp. 218-219.

[5] C. Bandt - G. Baraki, *Metrically invariant measures on locally homogeneous spaces and hyperspaces*, Pacific J. Math., **121** (1986), pp. 13-28.

[6] I. Bàràny - T. Zamfirescu, *Diameters in typical convex bodies*, Canadian J. Math., **42** (1990), pp. 50-61.

[7] A. S. Besicovitch - T. Zamfirescu, *On pencils of diameters in convex bodies*, Rev. Roum. Math. Pures Appl., **11** (1966), pp. 637-639.

[8] M. Breen, *Admissible kernels for starshaped sets*, Proc. Amer. Math. Soc., **82** (1981), pp. 622-628.

[9] L. A. Buminovich, *On the ergodic property of nowhere dispersing billiards*, Comm. Math. Phys., **65** (1979), pp. 295-312.

[10] H. Busemann, *Convex surfaces*, Interscience Publishers, New York, 1958.

[11] H. Busemann - W. Feller, *Krümmungseigenschaften konvexer Flächen*, Ann. Math., **66** (1936), pp. 1-47.

[12] L. Danzer - D. Laugwitz - H. Lenz, *Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden*, Arch. Math., **8** (1957), pp. 214-219.

[13] E. Dolženko, *Boundary properties of arbitrary function*, Izv. Akad. Nauk SSSR Ser. Mat., **31** (1967), pp. 3-14.

[14] R. Douady, *Applications du théorème des tores invariants*, Thèse 3ième cycle, U. Paris VII, 1982.

[15] K. J. Falconer, *X-ray problems for point sources*, Proc. London Math. Soc., (3) **46** (1983), pp. 241-262.

[16] S. Fitzpatrick - R. Phelps, *Differentiability of the metric projection in Hilbert space*, to appear.

[17] R. J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math., **41** (1983), pp. 183-189.

[18] R. J. Gardner - P. McMullen, *On Hammer's X-ray problem*, J. London Math. Soc., **21** (1980), pp. 171-175.

[19] O. Giering, *Bestimmung von Eibereichen und Eikörpern durch Steiner-Symmetrisierungen*, S.-Ber. Bayer. Akad. Wiss. München, Math.-Naturw. Kl. (1962), pp. 225-253.

[20] P. Gruber, *Die meisten konvexen Körper sind glatt, aber nicht zu glatt*, Math. Ann., **229** (1977), pp. 259-266.

[21] P. Gruber, *In most cases approximation is irregular*, Rend. Sem. Mat. Univ. Politecn. Torino, **41** (1983), pp. 19-33.

[22] P. Gruber, *Minimal ellipsoids and their duals*, Rend. Circ. Mat. Palermo, **37** (1988), pp. 35-64.

[23] P. Gruber, *Radon's contribution to convexity*, in: J. Radon, *Gesammelte Abhandlungen*, Kap. 4, *Konvexität/Convexity*.

[24] P. Gruber, *Convex billiards*, Geom. Dedicata, to appear.

[25] P. Gruber, *Geodesics on typical convex surfaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur (8), to appear.

[26] P. Gruber, *A typical convex surface contains no closed geodesics*, to appear.

[27] P. Gruber, *Dimension and structure of typical compact sets, continua and curves*, Mh. Math., **108** (1989), pp. 149-164.

[28] P. Gruber - P. Kenderov, *Approximation of convex bodies by polytopes*, Rend. Circ. Mat. Palermo, **31** (1982), pp. 195-225.

[29] P. Gruber - H. Sorger, *Shadow boundaries of typical convex bodies. Measure properties*, Mathematika, **36** (1989), pp. 142-153.

[30] P. Gruber - T. Zamfirescu, *Generic properties of compact starshaped sets*, Proc. Amer. Math. Soc., **108** (1990), pp. 207-214.

[31] B. Halpern, *Strange billiard tables*, Trans. Amer. Math. Soc., **232** (1977), pp. 297-305.

[32] P. C. Hammer, *Problem 2, Proc. Symp. Pure Math.*, Vol. VII: *Convexity*, Amer. Math. Soc., 1963.

[33] P. C. Hammer, *Problem 14* in: *Colloquium on Convexity*, Copenhagen, 1965.

[34] E. Heil, *Concurrent normals and critical points under weak smoothness assumptions*, Ann. New York Acad. Sci., **440** (1985), pp. 170-178.

[35] V. L. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann., **139** (1959), pp. 51-63.

[36] V. L. Klee, *A theorem on convex kernels*, Mathematika, **12** (1965), pp. 89-93.

[37] A. Kosiński, *On a problem of Steinhaus*, Fund. Math., **46** (1958), pp. 47-59.

[38] M. Laczkovich, private communication.

[39] G. D. Larman - P. Mani-Levitska, *Almost all shadow boundaries are almost smooth*, to appear.

[40] V. F. Lazutkin, *On the asymptotics of the eigenfunctions of the Laplacian*. Dokl. Akad. Nauk SSSR, **200** (1971), pp. 1277-1279 or Soviet Math. Dokl., **12** (1971), pp. 1569-1572.

[41] V. F. Lazutkin, *The existence of caustics for billiards in a convex domain*, Izv. Akad. Nauk SSSR, **37** (1979), pp. 186-216.

[42] V. F. Lazutkin, *Convex billiards and eigenfunctions of the Laplace operator* (Russian), Leningrad Univ., Leningrad, 1981.

[43] G. G. Lorentz, *A problem of plane measure*, Amer. J. Math., **71** (1949), pp. 417-426.

[44] A. K. Louis, *Ghosts in Tomography—The null space of the Radon transform*, Math. Meth. Appl. Sci., **3** (1981), pp. 1-10.

[45] L. A. Lusternik - L. G. Schnirelman, *Sur le problème des trois géodésiques fermées sur les surfaces de genre 0*, C. R. Acad. Sci. Paris, **189** (1929), pp. 269-271.

[46] P. Mani-Levitska, *Can you cover your shadows?*, Discrete Comp. Geom., **1** (1986), pp. 45-58.

[47] P. McMullen, *Problem 54*, in: *Contributions to Geometry*, Ed. by J. Tölke, J. Wills, Birkhäuser Verlag, Basel, 1979.

[48] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II (1962), pp. 1-20.

[49] A. J. Ostaszewski, *Families of compact sets and their universals*, Mathematika, **21** (1974), pp. 116-127.

[50] A. V. Pogorelov, *Quasigeodesics on convex surfaces*, Mat. Sb., **25** (1949), pp. 275-307.

[51] K. Post, *Star extension of plane convex sets*, Indag. Math., **26** (1964), pp. 330-338.

[52] K. Reidemeister, *Über die singulären Randpunkte eines konvexen Körpers*, Math. Ann., **83** (1921), pp. 116-118.

[53] G. de Rham, *Sur quelques courbes définies par des équations fonctionelles*, Rend. Sem. Mat. Univ. Politecn. Torino, **16** (1956/57), pp. 101-113.

[54] H. Rüssmann, *Kleine Nenner I. Über invariante Kurven differenzierbarer Abbildungen eines Kreisringes*, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II (1970), pp. 65-105.

[55] R. Schneider, *On the curvatures of convex bodies*, Math. Ann., **240** (1979), pp. 177-181.

[56] R. Schneider - J. Wieacker, *Approximation of convex bodies by polytopes*, Bull. London Math. Soc., **13** (1981), pp. 149-156.

[57] T. Schwarz - T. Zamfirescu, *Typical convex sets of convex sets*, J. Austral. Math. Soc. (A), **43** (1987), pp. 287-290.

[58] S. Stechkin, *Approximative properties of subsets of Banach spaces* (Russian), Rev. Roum. Math. Pures Appl., **8** (1963), pp. 5-8.

[59] P. Steenaerts, *Mittlere Schattengrenzenlänge konvexer Körper*, Results in Math., **8** (1985), pp. 54-77.

[60] A. Volčič, *Ghost convex bodies*, Boll. U.M.I., **4**-A (1985), pp. 287-292.

[61] A. Volčič, *A three point solution to Hammer's X-ray problem*, J. London Math. Soc., **34** (1986), pp. 349-359.

[62] A. Volčič - T. Zamfirescu, *Ghosts are scarce*, J. London Math. Soc., **40** (1989), pp. 171-178.

[63] J. A. Wieacker, *The convex hull of a typical compact set*, Math. Ann., **282** (1988), pp. 637-644.

[64] L. Zajíček, *On differentiation of metric projections in finite dimensional Banach spaces*, Czech. Math. J., **33** (1983), pp. 325-336.

[65] T. Zamfirescu, *The curvature of most convex surfaces vanishes almost everywhere*, Math., Z., **174** (1980), pp. 135-139.

[66] T. Zamfirescu, *Nonexistence of curvature in most points of most convex surfaces*, Math. Ann., **252** (1980), pp. 217-219.

[67] T. Zamfirescu, *Inscribed and circumscribed circles to convex curves*, Proc. Amer. Math. Soc., **80** (1980), pp. 455-457.

[68] T. Zamfirescu, *Most convex mirrors are magic*, Topology, **21** (1982), pp. 65-69.

[69] T. Zamfirescu, *Many endpoints and few interior points of geodesics*, Invent. Math., **69** (1982), pp. 253-257.

[70] T. Zamfirescu, *Intersecting diameters in convex bodies*, Annals of Discrete Math., **20** (1983), pp. 311-316.

[71] T. Zamfirescu, *Points on infinitely many normals to convex surfaces*, J. Reine Angewandte Math., **350** (1984), pp. 183-187.

[72] T. Zamfirescu, *Using Baire categories in geometry*, Rend. Sem. Mat. Univ. Politecn. Torino, **43** (1984), pp. 67-88.

[73] T. Zamfirescu, *Nearly all convex surfaces are smooth and strictly convex*, Mh. Math., **103** (1987), pp. 57-62.

[74] T. Zamfirescu, *How many sets are porous?* Proc. Amer. Math. Soc., **100** (1987), pp. 383-387.

[75] T. Zamfirescu, *Typical convex curves on convex surfaces*, Mh. Math., **103** (1987), pp. 241-247.
[76] T. Zamfirescu, *Curvature properties of typical convex surfaces*, Pacific J. Math., **131** (1988), pp. 191-207.
[77] T. Zamfirescu, *Too long shadow boundaries*, Proc. Amer. Math. Soc., **103** (1988), pp. 587-590.
[78] T. Zamfirescu, *Typical starshaped sets*, Aequationes Math., **36** (1988), pp. 188-200.
[79] T. Zamfirescu, *Description of most starshaped surfaces*, Math. Proc. Cambridge Phil. Soc., **106** (1989), pp. 245-251.
[80] T. Zamfirescu, *On two conjectures of Franz Hering about convex surfaces*, Discrete Comp. Geom., to appear.
[81] T. Zamfirescu, *Nondifferentiability properties of the nearest point mapping*, J. Analyse Math., **54** 1990, pp. 90-98.
[82] T. Zamfirescu, *The nearest point mapping is single-valued nearly everywhere*, Arch. Math., to appear.
[83] T. Zamfirescu, *Long geodesics on convex surfaces*, to appear.
[84] T. Zamfirescu, *Conjugate points on convex surfaces*, Mathematika, to appear.
[85] V. Zhivkov, *Plane polygonal approximation of bounded convex sets*, C. R. Acad. Bulgare Sci., to appear.
[86] A. Zucco, *Minimal annulus of a convex body*, Arch. Math., **52** (1989), pp. 92-94.
[87] A. Zucco, *The minimal shell of a typical convex body*, Proc. Amer. Math. Soc., to appear.