CONJUGATE POINTS ON CONVEX SURFACES

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On a convex surface $S \subset \mathbb{R}^d$, two points x, y are *conjugate* if there are at least two shortest paths, called *segments*, from x to y. This paper is about the set of points conjugate to some fixed point $x \in S$.

First we make a few simple remarks in $\mathbb{R}^3$. On any convex surface S whose tangent cones have full angle larger than π, for example on any smooth convex surface, for every point $x \in S$ each furthest point of S (in the inner metric) is conjugate to x. However, for such S, there may well be just one point conjugate to a given $x \in S$. If S is a lense (boundary of the intersection of two congruent balls), on the circle of all points of non-differentiability of S any two points are conjugate. However, if Σ is a segment on a convex surface S and $x \in S \backslash \Sigma$, is close enough to an interior point of Σ, then the set of points of Σ conjugate to x is easily seen to be at most countable. This set is dense on Σ, if S is a typical convex surface in the sense of Baire categories, which follows from the existence on such a convex surface of a dense set of *endpoints*, *i.e.*, points not interior to any geodesic (see Theorem 1 in [7]). The same is true for intrinsic circles around x instead of Σ (see the Corollary in [7]). These remarks can make the reader eager to know more about the set of all conjugate points of some given point or a convex surface.

When speaking about a property P shared by several elements of a Baire space, we say that *most* or *typical* elements in the space have P if those without P form a set of first Baire category. For results on typical convex bodies see the survey [10].

P. Gruber [5] showed that on most convex surfaces S, for any point $x \in S$, most points of S are not conjugate to x. Moreover, he proved that, for $d = 3$, on most convex surfaces S, for any point $x \in S$, the set of points conjugate to x is dense in S.

We shall show here that, for $d = 3$, the first result can be extended to all convex surfaces and complemented by the corresponding measure-theoretic statement. Concerning the second result, we shall establish it for an arbitrary dimension d, using a short proof which avoids any reference to the mutual position of segments on arbitrary convex surfaces (which is known only for $d = 3$, see Aleksandrov [1]).

To state our first theorem we need the notion of porosity. In a metric space (X, ρ), a set M is called *porous at* $y \in M$ if there is some $\alpha > 0$ such that, for every $\varepsilon > 0$, there is a point z in the open ball $B(y, \varepsilon)$ of centre y and radius ε such that

$$B(z, \alpha\rho(y, z)) \cap M = \varnothing.$$

A set in X is called *porous* if it is porous at all its points, and is called *σ-porous* if it is a countable union of porous sets [4]. We say that *nearly no*

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elements of a Baire metric space have property P if those points enjoying P form a σ-porous set [**8**]. For results around porosity see the survey [**6**] and for applications to convexity see [**9**].

THEOREM 1. *On any convex surface $S \subset \mathbb{R}^3$ and for any point $x \in S$, nearly no points are conjugate to x.*

To prove this theorem we need the following elementary lemma.

LEMMA. *In $\mathbb{R}^2$, if $a \neq \pm b$, $\|a\| = \|b\|$ and $c = \frac{1}{2}a + b$, then each point of the triangle Θ with vertices $b, 2b, c$ is closer to the line-segment $B = [b, 2b]$ than to the line-segment $A = [\mathbf{0}, a]$.*

Proof. We start with two easy remarks.

(a) The line L through $b/2$ orthogonal to $[\mathbf{0}, b]$ separates Θ from $\mathbf{0}$ since it separates the vertices of Θ from $\mathbf{0}$: b and $2b$ obviously, and c because $\|b - c\| = \|a\|/2 = \|b\|/2 = \|b - \frac{1}{2}b\|$.

(b) The angle-bisector of the angle $a\mathbf{0}b$ separates Θ from a because it cuts $[a, 2b]$ in a point c' such that $\|a - c'\|/\|2b - c'\| = \|a\|/\|2b\| = \frac{1}{2}$.

To prove the lemma we have to consider three cases.

Case I. $\langle a, c\rangle \leq 0$. In this case, for any point $x \in \Theta$, by (a),

$$d(x, B) \leq \|x - b\| \leq \|x\| = d(x, A),$$

where $d(x, X)$ denotes the distance from x to X, that is $\inf\{\|x - y\|: y \in X\}$.

Case II. $\langle a, c\rangle > 0$ and $\langle a, b\rangle < 0$. In this case let $u \in [c, 2b]$ and $v \in [b, c]$ be such that $\langle b - u, b\rangle = \langle b - c, v\rangle = 0$. Also let Ψ be the triangle of vertices b, c, u. For any point $x \in \Psi$,

$$\begin{aligned} d(x, B) &= \|x - b\| < \|c - b\| = \|b\|/2 < \|b\| \cos\frac{\pi}{6} < \|b\| \cos \measuredangle b\mathbf{0}v \\ &= \|v\| \leq d(x, [-a, a]) \leq d(x, A) \end{aligned}$$

and, for any point $x \in \Theta \setminus \Psi$, by (b),

$$d(x, B) < d(x, [-2a, a]) \leq d(x, A).$$

Case III. $\langle a, b\rangle \geq 0$. In this case, for any point $x \in \Theta$, by (b),

$$d(x, B) < d(x, [\mathbf{0}, 2a]) \leq d(x, A).$$

Proof of Theorem 1. Let ρ be the inner metric of S. Denote by S' the set of all points of S at which the full angle of the tangent cone is 2π, *i.e.*, the set of all non-conical points of S.

For any point $y \in S$ different from x, let $\alpha(y)$ be the maximal angle at x between two segments from x to y. Clearly $0 \leq \alpha(y) \leq \pi$ for any y. Let

$$S_n = \{y \in S': 2^{-n}\pi < \alpha(y) \leq 2^{-n+1}\pi\}.$$

Of course,

$$(S \setminus S') \cup \bigcup_{n=1}^{\infty} S_n \supset \{y \in S: x \text{ and } y \text{ are conjugate}\}.$$

We shall prove that S_n is porous for arbitrary n, which will establish the theorem because $S\backslash S'$ is countable.

Let $y \in S_n$ and consider the segments Σ_1, Σ_2 from x to y and the domain $D \subset S$ with boundary $\Sigma_1 \cup \Sigma_2$, such that the angle at x between Σ_1 and Σ_2 towards D equals $\alpha(y)$. (See Fig. 1.) For $\varepsilon < \rho(x, y)$ small enough, the intrinsic circle J of all points in S at distance $\varepsilon/2$ from y is a Jordan curve (see [1], p. 383). It obviously intersects Σ_i in precisely one point σ_i $(i = 1, 2)$, and $J_0 = J \cap \bar{D}$ is a Jordan arc with endpoints σ_1, σ_2.

The continuity of ρ implies the existence of a point $j \in J_0$ with $\rho(j, \sigma_1) = \rho(j, \sigma_2)$. Let Σ_i' be a segment from j to σ_i $(i = 1, 2)$ and Σ a segment from j to x. One of the two angles that Σ makes with Σ_1 and Σ_2 at x is at most $\alpha(y)/2$. Suppose for example that Σ and Σ_1 determine that angle. Let $\sigma_1' \in \Sigma_1$ satisfy $\rho(y, \sigma_1') = \varepsilon$. Consider a segment Σ_1'' from j to σ_1' and a segment Σ'' from σ_1 to the midpoint σ_3 of Σ_1''. Consider also the segment $\Sigma' \subset \Sigma_1$ from σ_1 to σ_1', the segment $\Sigma''' \subset \Sigma_1''$ from σ_3 to σ_1' and the segment $\Sigma_1^- \subset \Sigma_1$ from σ_1 to x. As we shall see later, for ε small enough, for any point z in the interior of the triangle $\Sigma' \cup \Sigma'' \cup \Sigma'''$ and for any segment Σ_z from z to x, $\Sigma_z \backslash \{x\}$ lies in the interior T of the triangle $\Sigma \cup \Sigma_1' \cup \Sigma_1^-$, whence $\alpha(z) < \alpha(y)/2$. This will prove $T \cap S_n = \emptyset$. Subsequently we shall find a disk in T which is large enough to ensure the porosity of S_n at y. The fact that the metric of S is, locally at y, the planar metric of the tangent cone at y is essentially the only tool we use in the rest of the proof.

Choose the origin of $\mathbb{R}^3$ at y. For simplicity, we shall suppose S to be smooth at y. However, the proof is essentially the same if S is not smooth at y, because then its tangent cone is the union of two halfplanes. Let Π be the supporting plane of conv S at y and denote by p the projection from $\mathbb{R}^3$ onto Π.

By Theorem (11.4) in its form (11.6) from [2], for any $\gamma > 0$ there is some $\beta > 0$ such that, for $v, w \in S$,

$$|\rho(v, w) - \| p(v) - p(w) \| | \leq \gamma \max \{\rho(v, y), \rho(w, y)\},$$

if $\max \{\rho(v, y), \rho(w, y)\} < \beta$. It follows that, for $\varepsilon = \beta/3$,

$$\frac{|\rho(v, w) - \| p(v) - p(w) \||}{\varepsilon} \leq 3\gamma, \tag{$*$}$$

for any $v, w \in S$ at distance at most 3ε from y. Remark that all points $j, \sigma_1, \sigma_1', \sigma_2, \sigma_3$ have distance at most 3ε from y.

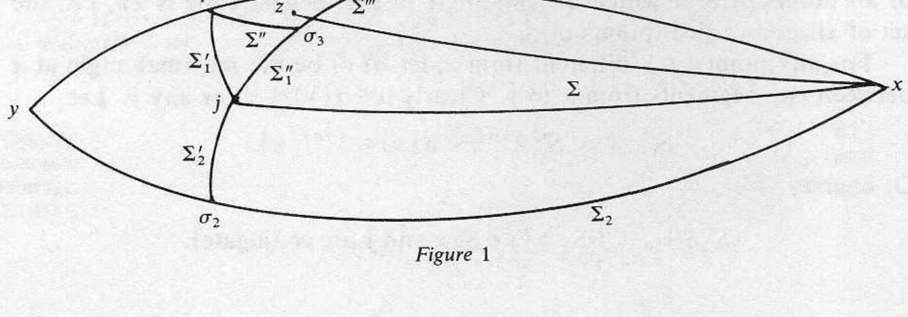

Figure 1

Let σ_1^*, $\sigma_1'^*$ be the points on the halfline $L_1 \subset \Pi$ tangent to Σ_1 at y, with $2\|\sigma_1^*\| = \|\sigma_1'^*\| = 1$. Let j^* be the point on the halfline in Π besecting the angle towards D between the tangents to Σ_1 and Σ_2 at y, with $\|j^*\| = \frac{1}{2}$. Put $\sigma_3^* = \frac{1}{2}(j^* + \sigma_1'^*)$.

Further, let Δ^* be the largest open disk of Π included in the open triangle T^* of vertices σ_1^*, σ_3^*, $\sigma_1'^*$. Denote by $\Sigma_i^+ \subset \Sigma_i$ the segment from y to σ_i on S, by $\Sigma_1'^+ \subset \Sigma_1$ the segment from y to σ_1' on S, by Σ_i^{+*} and $\Sigma_1'^{+*}$ the corresponding line-segments from y to σ_i^* and $\sigma_1'^*$ respectively $(i = 1, 2)$, and by Σ'^* the line-segment from σ_1^* to $\sigma_1'^*$.

Suppose $\varepsilon \to 0+$. Let λA denote the length of A. From [**2**], Corollary (11.8), it follows that $\lambda\Sigma_1^+/\|\sigma_1\| \to 1$, whence $\varepsilon^{-1}\|\sigma_1\| \to \frac{1}{2}$. This, together with the fact that the halfline starting at y and passing through σ_1 tends to L_1, implies $\varepsilon^{-1}\sigma_1 \to \sigma_1^*$. Since $\lambda(\varepsilon^{-1}\Sigma_1^+) = \frac{1}{2}$, $\varepsilon^{-1}\Sigma_1^+ \to \Sigma_1^{+*}$ too. Analogously $\varepsilon^{-1}\sigma_2 \to \sigma_2^*$, $\varepsilon^{-1}\sigma_1' \to \sigma_1'^*$, $\varepsilon^{-1}\Sigma_2^+ \to \Sigma_2^{+*}$ and $\varepsilon^{-1}\Sigma_1'^+ \to \Sigma_1'^{+*}$, whence $\varepsilon^{-1}\Sigma' \to \Sigma'^*$. Let ρ' be the inner metric of $\varepsilon^{-1}S$. $\rho(j, \sigma_1) = \rho(j, \sigma_2)$ implies

$$\rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_1) = \rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_2).$$

By (*),

$$\rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_1) - \|p(\varepsilon^{-1}j) - p(\varepsilon^{-1}\sigma_1)\| \to 0$$

and

$$\rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_2) - \|p(\varepsilon^{-1}j) - p(\varepsilon^{-1}\sigma_2)\| \to 0.$$

Hence

$$\|p(\varepsilon^{-1}j) - p(\varepsilon^{-1}\sigma_1)\| - \|p(\varepsilon^{-1}j) - p(\varepsilon^{-1}\sigma_2)\| \to 0.$$

This, together with $p(\varepsilon^{-1}\sigma_i) \to \sigma_i^*$ and $\varepsilon^{-1}\|\sigma_i - p(\sigma_i)\| \to 0$ $(i = 1, 2)$, implies

$$\|p(\varepsilon^{-1}j) - \sigma_1^*\| - \|p(\varepsilon^{-1}j) - \sigma_2^*\| \to 0. \qquad (**)$$

Moreover

$$\rho'(y, \varepsilon^{-1}j) - \|p(\varepsilon^{-1}j)\| \to 0.$$

This, together with $\rho'(y, \varepsilon^{-1}j) = \varepsilon^{-1}\rho(y, j) = 1$, implies

$$\|p(\varepsilon^{-1}j)\| \to 1,$$

which, together with (**), proves $p(\varepsilon^{-1}j) \to j^*$. Since

$$\varepsilon^{-1}\|j - p(j)\| \to 0, \ \varepsilon^{-1}j \to j^*$$

too. From (*) it follows

$$\rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_1') - \|p(\varepsilon^{-1}j) - p(\varepsilon^{-1}\sigma_1')\| \to 0,$$

which, together with $\varepsilon^{-1}j \to j^*$, $p(\varepsilon^{-1}j) \to j^*$, $\varepsilon^{-1}\sigma_1' \to \sigma_1'^*$ and $p(\varepsilon^{-1}\sigma_1') \to \sigma_1'^*$ implies

$$\lambda(\varepsilon^{-1}\Sigma_1'') = \rho'(\varepsilon^{-1}j, \varepsilon^{-1}\sigma_1') \to \|j^* - \sigma_1'^*\|.$$

This and the fact that $\varepsilon^{-1}\Sigma_1''$ joins $\varepsilon^{-1}j$ to $\varepsilon^{-1}\sigma_1'$ imply that $\varepsilon^{-1}\Sigma_1'$ converges to the line-segment in Π joining j^* to $\sigma_1'^*$. Analogously, $\varepsilon^{-1}\Sigma'$, $\varepsilon^{-1}\Sigma''$ and $\varepsilon^{-1}\Sigma'''$ converge to the three sides of the triangle T^*. Also, if Σ_j is a segment from y to j, then Σ_j clearly converges to the line-segment $[y, j^*]$. Suppose now that $\Sigma_z\backslash\{x\}$ meets $\Sigma \cup \Sigma_1' \cup \Sigma_1^-$ for small $\varepsilon > 0$. Then it must meet Σ_1' and, also, Σ_j. Suppose $\rho(z, s) > \rho(z, t)$, where $s \in \Sigma_z \cap \Sigma_j$ and t is a point of Σ' closest to z. Then, for ε small enough,

$$\rho(z, x) = \rho(z, s) + \rho(s, x) \geq \rho(z, s) + \rho(y, x) - \rho(y, s)$$

$$> \rho(z, t) + \rho(y, x) - \frac{\varepsilon}{2} = \rho(z, t) + \rho(\sigma_1, x) \geq \rho(z, t) + \rho(t, x),$$

which is impossible. Hence $\rho(z, s) \leq \rho(z, t)$. This implies, by taking the limit as $\varepsilon \to 0$, that, for some point $z^* \in \bar{T}^*$, its distance to the line-segment $[y, j^*]$ is not larger than its distance to the line-segment $[\sigma_1^*, \sigma_1'^*]$. This contradicts the lemma. Thus, indeed, $T \cap S_n = \varnothing$ for ε small enough.

Let c^* and δ^* be the centre and the radius of Δ^*. Let c be the point of T with $p(c) = \varepsilon c^*$ and put

$$\Delta = \{u \in T\colon \varepsilon^{-1} p(u) \in \Delta^*\}.$$

For any point $u \in \operatorname{bd} \Delta$,

$$\varepsilon^{-1}\rho(u, c) - \varepsilon^{-1}\| p(u) - p(c)\| \to 0$$

u-uniformly. Since $\varepsilon^{-1}\| p(u) - p(c)\| = \|\varepsilon^{-1}p(u) - c^*\| = \delta^*$, it follows that

$$\varepsilon^{-1}\rho(u, c) \to \delta^*$$

u-uniformly. This in turn, together with $\varepsilon^{-1}c \to c^*$, implies $\varepsilon^{-1}\bar{\Delta} \to \bar{\Delta}^*$ with respect to the Hausdorff metric. Then the open disk

$$\Delta' = \{u \in T\colon \rho(u, c) < \delta^*\varepsilon/2\}$$

lies in Δ for ε small enough. It is easily seen that $\|z^*\| < 1$ for any point $z^* \in T^*$, in particular $\|c^*\| \leq 1$. Since

$$\varepsilon^{-1}c \to c^*, \qquad \varepsilon^{-1}\rho(y, c) \to \|c^*\|,$$

therefore, for ε small enough, $\rho(y, c) < 2\varepsilon$ and the radius of Δ' is larger than $(\delta^*/4)\rho(y, c)$. This proves that S_n is porous at y.

COROLLARY 1. *On any convex surface $S \subset \mathbb{R}^3$ and for any point $x \in S$, most and almost all points are not conjugate to x.*

Proof. A porous set on S is by definition nowhere dense and by Lebesgue's density theorem of measure zero.

THEOREM 2. *On any convex surface $S \subset \mathbb{R}^d$ with a dense set of endpoints, for any point $x \in S$, the set of points conjugate to x is dense too.*

Proof. Let S be a convex surface with a dense set of endpoints, and $x \in S$. Suppose there is an open set $O \subset S$ no point of which is conjugate to x and let $z \in O$ be an endpoint other than x. Since every point $y \in O$ is joined by precisely one segment Σ_y with x, the mapping $y \mapsto \Sigma_y$ is continuous (see [2], p. 81). Let $a \in S\backslash\Sigma_z$. Because z is an extreme point of conv S, we can choose a convex cap (a $(d-1)$-cell on S cut off by a hyperplane) $Y \subset O$ containing z in its interior so small that $a \notin \Sigma_y$ for all $y \in Y$. If $\sigma(y, r)$ denotes the point of Σ_y at distance r from y then σ is continuous in both variables (see [2], (10.5), (10.5′), (11.3)). Thus the set

$$\Omega = \bigcup_{y \in \mathrm{bd}\, Y} \Sigma_y$$

is contractible: take the homotopy $H: \Omega \times [0, 1] \to \Omega$ defined by

$$H(u, t) = \sigma^*(u, t\rho(x, u)),$$

where $\sigma^*(u, r)$ denotes the point of the unique segment joining x and u, at distance r from u (see, for example, [3], p. 362).

On the other hand, Ω neither contains z, because z is an endpoint not belonging to $\{x\} \cup \mathrm{bd}\, Y$, nor a, but Ω includes the topological $(d-2)$-sphere bd Y. Moreover, $z \in \mathrm{int}\, Y$ and $a \notin Y$. Thus Ω is not contractible and this contradiction ends the proof.

COROLLARY 2. *On most convex surfaces $S \subset \mathbb{R}^d$, for any point $x \in S$, the set of points conjugate to x is dense.*

Proof. Combine Theorem 2 with Theorem 1 in [7].

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