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THE LEVEL SET STRUCTURE OF NEARLY ALL REAL CONTINUOUS FUNCTIONS

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Nella presente nota si vede come la nozione di porosità permetta di migliorare alcuni risultati riguardanti proprietà tipiche degli insiemi di livello di una funzione continua di $I=[0,1]$ in $\mathbf{R}$.

Introduction

Let X be a Baire space. A subset of X is

- *nowhere dense* if and only if its closure has empty interior,
- a set *of first category* or *meager* if it is a countable union of nowhere dense subsets,
- a set *of second category* if it is not a set of first category.

Since in such a space the complement of a set of first category is of second category, we can say that *most* elements of X have a certain property if the set of those elements which do not enjoy that property is meager. Such a property is also called *typical* in X. Many results on typical properties have been found in Geometry and Analysis (see e.g. the surveys [4] and [8] and chapter XIII of Bruckner's book [1]).

The notion of a porous set on the real line was introduced by Dolženko ([2]) in 1967 and generalized by Zajíček ([5]) in 1976 to a general metric space. Here we use a slightly stronger notion of porosity ([7]).

DEFINITIONS. A set M in a metric space (X, d) is called *porous* if there is a positive real number α such that for each $x \in X$ and for each positive ε there exists a point y in the open ball $B(x, \varepsilon)$ with center x and radius ε such that

$$B(y, \alpha d(x, y)) \cap M = \emptyset .$$

If the above number α can be chosen as close to 1 as we wish then M is called *strongly porous*.

A countable union of porous sets is said to be *σ-porous*.

Clearly, any porous (σ-porous) set is nowhere dense (of first category). Many examples of nowhere dense but not porous sets can be found on the real line. More generally, it has been proved that σ-porosity is strictly more restrictive than first category in each Banach space ([6], page 322).

Let now $C(I)$ be the space of all continuous functions f from $I = [0, 1]$ into R with the standard metric:

$$d(f, g) = \max_{x \in I} \left| f(x) - g(x) \right| .$$

We shall say (see [7]) that *nearly all* elements of $C(I)$ have a certain property if the set of those elements not enjoying it is σ-porous.

In [3] several classical results involving typical properties of elements in $C(I)$ have been improved by showing that nearly all elements of $C(I)$ have those properties. Here we study the level set structure for nearly all elements of $C(I)$.

1. A result on the level sets

Let m_f and M_f be the minimum and the maximum of an element f of $C(I)$. The following result is known ([1], page 216).

THEOREM A. Let $\mathcal{N}$ be the set of functions f of $C(I)$ to each of which corresponds a dense denumerable subset S_f of the interval (m_f, M_f) such that the level set E_β is:

(i) a nowhere dense perfect set when $\beta \notin S_f \cup \{m_f, M_f\}$,

(ii) a single point when $\beta \in \{m_f, M_f\}$,

(iii) of the form $P_\beta \cup \{x_\beta\}$ where P_β is a nonempty nowhere dense perfect set and x_β is isolated in E_β when $\beta \in S_f$.

Then most elements of $C(I)$ are in $\mathcal{N}$.

In this section we improve Theorem A by showing that nearly all elements of $C(I)$ are in $\mathcal{N}$. In order to do this we need the following lemmas.

LEMMA 1. For nearly all elements $f \in C(I)$, no level set contains more than one point at which f achieves a relative extremum.

Proof. For two disjoint closed intervals J_1 and J_2 of [0,1] with rational endpoints let

$$A_{J_1,J_2} = \{f \in C(I) : \sup_{x\in J_1} f(x) \neq \sup_{x\in J_2} f(x)\}$$

We show that $C(I)\backslash A_{J_1,J_2}$ is porous.

For an arbitrary element f of $C(I)$ we put $y_1 = \sup_{x\in J_1\cup J_2} f(x)$ and choose $x_1 \in J_1 \cup J_2$ such that $f(x_1) = y_1$. Suppose without loss of generality that $x_1 \in J_1$. Let $\varepsilon > 0$. Consider $\delta > 0$ such that

$$[x_1 - \delta, x_1 + \delta] \cap J_2 = \emptyset$$

and

$$f(x_1) - \varepsilon/2 < f(x) \leq f(x_1) \text{ for } x \in [x_1 - \delta, x_1 + \delta].$$

We can define a continuous function g satisfying:

(i) $g(x) = f(x)$ for $x \notin [x_1 - \delta, x_1 + \delta]$,

(ii) $g(x_1) = f(x_1) + \varepsilon/2$,

(iii) g is linear in $[x_1 - \delta, x_1]$ if $x_1 - \delta \in I$ and constant in $[x_1 - \delta, x_1] \cap I$ otherwise,

(iv) g is linear in $[x_1, x_1 + \delta]$ if $x_1 + \delta \in I$ and constant in $[x_1, x_1 + \delta] \cap I$ otherwise.

For $x_1 - \delta \leq x \leq x_1 + \delta$ we have $f(x_1) - \varepsilon/2 \leq g(x) \leq f(x_1) + \varepsilon/2$ and therefore

$$|g(x) - f(x)| < \varepsilon.$$

It follows that

$$(*) \qquad d(f, g) < \varepsilon$$

We show now that $B(g, \varepsilon/4) \subset A_{J_1,J_2}$.

Take $h \in B(g, \varepsilon/4)$. Then $\sup_{x \in J_2} h(x) < y_1 + \varepsilon/4$ and

$$h(x_1) > g(x_1) - \varepsilon/4 = y_1 + \varepsilon/4.$$

It follows that

$$\sup_{x \in J_1} h(x) \geq h(x_1) > y_1 + \varepsilon/4 > \sup_{x \in J_2} h(x),$$

whence $h \in A_{J_1,J_2}$.

This and $(*)$ imply $B(g, \frac{1}{4}d(f, g)) \subseteq A_{J_1,J_2}$. Thus $C(I) \backslash A_{J_1,J_2}$ is porous. Analogously, the complements of

$$A'_{J_1,J_2} = \{f \in C(I) : \inf_{x \in J_1} f(x) \neq \inf_{x \in J_2} f(x)\}$$

and

$$A''_{J_1,J_2} = \{f \in C(I) : \inf_{x \in J_1} f(x) \neq \sup_{x \in J_2} f(x)\}$$

are porous too. (In the proof of the porosity of $C(I) \setminus A''_{J_1,J_2}$, if $f \in C(I)$ is such that $\gamma_f = \inf_{x \in J_1} f(x) - \sup_{x \in J_2} f(x) > 0$, ε should be chosen smaller than γ_f.) Hence $C(I) \setminus \cap_{J_1,J_2} (A_{J_1,J_2} \cap A'_{J_1,J_2} \cap A''_{J_1,J_2})$ is σ-porous, and this is precisely the set of all functions f some level set of which contains at least two relative extrema of f.

LEMMA 2. ([3], Theorem 1.) Nearly all elements of $C(I)$ are of non-monotonic type.

Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of nearly all elements in Lemmas 1 and 2 respectively and $\mathcal{N}$ the residual set from Theorem A. Since $\mathcal{A} \cap \mathcal{B} \subset \mathcal{N}$ (see [1], page 216) we immediately get the following result.

THEOREM 1. Let $\mathcal{N}$ be the set of functions f of $C(I)$ to each of which corresponds a dense denumerable subset S_f of the interval (m_f, M_f) such that the level set E_β is:

(i) a nowhere dense perfect set when $\beta \notin S_f \cup \{m_f, M_f\}$,

(ii) a single point when $\beta \in \{m_f, M_f\}$,

(iii) of the form $P_\beta \cup \{x_\beta\}$ where P_β is a nonempty nowhere dense perfect set and x_β is isolated in E_β when $\beta \in S_f$.

Then nearly all elements of $C(I)$ are in $\mathcal{N}$.

2. A result on the zero-sets

If instead of all level sets we restrict ourselves to only one, say the zero-set $Z(f)$ of an element f of $C(I)$, we have the following result.

THEOREM 2. For nearly all $f \in C(I)$, $Z(f)$ is strongly porous.

Proof. Let $\xi \in (0,1)$ and put

$$C_m = \{f \in C(I) : \forall x \in [0,1], \exists y \in B(x, 1/m) \text{ such that } B(y, \xi|y-x|) \cap Z(f) = \emptyset\}\,.$$

We show that $C(I) \backslash C_m$ is porous. Let $f \in C(I)$, $\eta \in (0,1)$ and $\varepsilon > 0$. There is a number $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \varepsilon(1-\eta)\,.$$

Consider the points $0 = a_0, a_1, \ldots, a_n, a_{n+1} = 1$ such that

$$a_{i+1} - a_i = \delta' \leq \min\{\delta, 1/m\} \quad (i = 0, \ldots, n)\,.$$

Also let

$$c_i = a_i + \frac{\delta'(1-\xi)}{2}\,,\ d_i = a_{i+1} - \frac{\delta'(1-\xi)}{2} \quad (i = 0, \ldots, n)$$

We define a function $g \in C(I)$ linear on each one of the intervals $[a_i, c_i]$, $[c_i, d_i]$, $[d_i, a_{i+1}]$ $(i = 0, \ldots, n)$ such that

$$g(a_i) = f(a_i)\,,$$
$$g(c_i) = g(d_i) = \begin{cases} f(a_i) + \varepsilon\eta & \text{if } f(a_i) \geq 0 \\ f(a_i) - \varepsilon\eta & \text{if } f(a_i) < 0\,. \end{cases}$$

Clearly $d(f,g) < \varepsilon$.

Now we show that $B(g, \eta d(f,g)) \subset C_m$. Let $h \in B(g, \eta d(f,g))$.

First we remark that $h([c_i, d_i]) \not\ni 0$ $(i = 0, \ldots, n)$. Indeed for $x \in [c_i, d_i]$, if $f(a_i) \geq 0$, then

$$h(x) > g(x) - \eta d(f,g) > g(x) - \varepsilon\eta = f(a_i) \geq 0\,,$$

and, if $f(a_i) < 0$, then

$$h(x) < g(x) + \eta d(f,g) < g(x) + \varepsilon\eta = f(a_i) < 0 .$$

Now let $x \in [0,1]$. If $x \in [c_i, d_i]$ for some i then, clearly, we find $y \in [c_i, d_i]$ such that $B(y, \xi|x-y|) \subset [c_i, d_i]$, whence $B(y, \xi|x-y|) \cap Z(h) = \emptyset$. If $x \in [a_i, c_i] \cup [d_i, a_{i+1}]$ for some i, we choose $y = (c_i + d_i)/2$ and obtain $B(y, \xi|x-y|) \subset [c_i, d_i]$ whence again $B(y, \xi|x-y|) \cap Z(h) = \emptyset$.

Hence $\mathcal{C}(I)\backslash \mathcal{C}_m$ is porous and $\mathcal{C}(I)\backslash \cap \mathcal{C}_m$ is σ-porous. Since ξ was chosen arbitrarily in (0,1), the theorem follows.

REMARK. Clearly the previous result also holds for countably many level sets of the function f.

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