Math. Ann. 293, 109–114 (1992)

Mathematische Annalen
© Springer-Verlag 1992

Long geodesics on convex surfaces

Tudor Zamfirescu

Fachbereich Mathematik, Universität Dortmund,
W-4600 Dortmund 50, Federal Republic of Germany

Received November 23, 1990; in revised form May 6, 1991; in final form September 9, 1991

Mathematics Subject Classification (1991): 52A15

1 Introduction

We are in $\mathbb{R}^3$. The following is an open problem in the geometry of convex surfaces rather than a joke: Does there exist a convex surface on which no geodesic is closed and all geodesics have length less than 1?

If the surface is smooth enough, then a well-known theorem of Liusternik and Schnirelman asserts that there are at least three simple closed geodesics. However this is not true in general. On many polytopal surfaces there is no simple closed geodesic, as observed already by Aleksandrov [1, p. 377]. But, at least, these surfaces have many infinitely long geodesics, as we easily see. For arbitrary surfaces of class C^1 this is no longer clear and, in fact, neither proved nor disproved. *Most* convex surfaces in the sense of Baire categories (i.e. all except those in a set of first category) are of class C^1 but have no closed geodesic (even with self-intersections), as Gruber [5] showed. At the same time, on most convex surfaces most segments (a *segment* is a shortest path between two points of the surface [4]) are not extendable (use Theorem 1 in [9]), in the sense that no geodesic will contain such a segment as a proper subset. However, we shall prove here that most convex surfaces have many arbitrarily long geodesics. Moreover, we shall even show that most convex surfaces have arbitrarily long geodesics without self-intersections. While the first property does not surprise the differential geometer, the latter surely does.

Pogorelov [7] provided a version of Liusternik-Schnirelman's theorem which applies for any convex surface. He introduced the notion of a quasigeodesic and proved that on any convex surface there exist at least three closed quasigeodesics. However, a quasigeodesic may fail to be locally a segment.

For the reader's convenience we shall recall in the next section Aleksandrov's gluing theorem, a few related notions, as well as existence and uniqueness results concerning the realizability of convex surfaces with given intrinsic metrics, mainly due to Aleksandrov and Pogorelov.

2 Notation and basic facts

We denote by λC the length of the curve C. For $r \in \mathbb{R}$, $x \in \mathbb{R}^3$, $A \subset \mathbb{R}^3$, let

$$B(x,r) = \{y \in \mathbb{R}^3 : \|x-y\| < r\}$$

and

$$\Delta(x,A) = \inf_{y \in A} \|x-y\| .$$

Let $\mathscr{S}$ be the space of all convex surfaces T, i.e. boundaries of bounded open convex sets. Equipped with the usual Hausdorff distance δ, $\mathscr{S}$ is a Baire space. For any convex surface S, let d_S denote its intrinsic metric.

We shall make use of the following fundamental generic result of Klee [6].

Klee's generic theorem. *Most convex surfaces are smooth and strictly convex.*

For other results on most convex surfaces see the survey article [11].

Let S be a convex surface. A *geodesic* is the image of an interval $I \subset \mathbb{R}$ through a continuous mapping $c: I \to S$, such that every point in I has a neighbourhood N in I for which $c(N)$ is a segment. If $I = \mathbb{R}$ and c is periodic then $c(I)$ is called a *closed geodesic*. If I is compact, we call $c(I)$ a *geodesic arc*. This is said to be *closed* (a closed geodesic segment in Klingenberg's terminology [7]) if the images of the endpoints of I through c coincide.

For any Jordan arc J with definite directions at its endpoints the notions of a right and a left swerve can be introduced (see, for example, [4, p. 108–110]). They correspond to the integral of the geodesic curvature along the arc in the differential case. A *quasigeodesic arc* is a Jordan arc which has definite directions at each point and every subarc of which has non-negative right and left swerves. If we take a Jordan (closed) curve instead of a Jordan arc in the above definition we obtain the notion of a *closed quasigeodesic*. If the full angle at the endpoints of a quasigeodesic arc is at most π, we have a degenerate case of a closed quasigeodesic called *quasigeodesic arc traversed back and forth*. All these are *quasigeodesics*.

Pogorelov's theorem [7]. *Any convex surface possesses three distinct quasigeodesics each of which is either a closed quasigeodesic or a quasigeodesic arc traversed back and forth.*

Let $M_1, \dots, M_n$ be two-dimensional manifolds, each with its own intrinsic metric. For every i consider an open set D_i with $\bar{D}_i \subset M_i$ and whose boundary is the union of pairwise disjoint rectifiable Jordan curves $C_1^i, \dots, C_{n_i}^i$. We say that the manifold M is obtained by *gluing* together $\bar{D}_1, \dots, \bar{D}_n$ if all C_l^i are decomposed into Jordan arcs which are pairwise identified in such a way that any two identified subarcs of these identified Jordan arcs have the same length, while $\bar{D}_1 \cup \dots \cup \bar{D}_n = M$.

Aleksandrov's gluing theorem [1, p. 362; 4, p. 154]. *Let $M_1, \dots, M_n$ have non-negative curvature and let the swerve have bounded variation on any subarc of any C_l^i. The manifold M obtained by gluing together $\bar{D}_1, \dots, \bar{D}_n$ has non-negative curvature if and only if for any identified subarcs $A^i \subset C_l^i$ and $A^j \subset C_k^j$ the sum of the swerve of A^i in M_i towards $\bar{D}_i$ and the swerve of A^j in M_j towards $\bar{D}_j$ is non-negative and for any point p belonging to more than two sets $\bar{D}_i$ the sum of the angles of these $\bar{D}_i$ at p is at most 2π.*

The proof of the next result can be found in [1, Chap. VI].

Existence theorem. *Each polyhedral metric with non-negative curvature on the sphere can be realized as a polytopal surface.*

The following theorem is due to Pogorelov [8]. It was proved earlier by Olovianishnikov in the case S is a polytopal surface [1, p. 336].

Uniqueness theorem [2, p. 158]. *Any convex surface isometric to a convex surface S is congruent to S.*

Let again S be a convex surface. We define the *sphere bundle* T_1S associated with S as the set of all pairs (x, τ), where x is a smooth point of S and τ a tangent direction at x. The topology is that induced by $\mathbb{R}^3 \times \mathbb{S}^2$. For $(x, \tau) \in T_1S$, the union of all geodesics starting at x in direction τ (if any) is itself a geodesic and we denote it by $G(x, \tau)$. If there is no such geodesic, set $G(x, \tau) = \{x\}$. Also, put

$$\mu(x,\tau) = \begin{cases} \min\{\lambda G(x,\tau), \lambda G(x,-\tau)\} & \text{if } G(x,\tau) \cup G(x,-\tau) \text{ is a geodesic}, \\ 0 & \text{otherwise}. \end{cases}$$

3 Two lemmas

We shall make use of the following lemma.

Lemma 1 [1, p. 106]. *Let S, S_1, $S_2, \ldots$ be convex surfaces, on which the points x, $y \in S$, x_n, $y_n \in S_n$ are chosen ($n \in \mathbb{N}$). If $S_n \to S$, $x_n \to x$ and $y_n \to y$, then $d_{S_n}(x_n, y_n) \to d_S(x, y)$.*

The next lemma is an essential tool for the proof of our results.

Lemma 2. *Let P be a polytopal surface and G a geodesic arc on P. If $S \in \mathscr{S}$ and $S \to P$ then there are geodesic arcs $G_S \subset S$ such that $G_S \to G$.*

Proof. Let z be a point in the interior of P and $p : \mathbb{R}^3 \to P$ the central projection with centre z. If S is close enough to P, $p_S = p|_S$ is a homeomorphism according to which, by Lemma 1, d_S converges to d_P. First suppose the endpoints p, q of G are not vertices of P. Then there is a line-segment $L \subset \mathbb{R}^2$, an open rectangle $R \supset L$ and a locally isometric mapping $f : \bar{R} \to P$ such that $f(L) = G$. For any points u, $v \in \bar{R}$ joined by arcs $J \subset \bar{R}$, let

$$d'_S(u,v) = \inf_J \lambda p_S^{-1}(f(J)).$$

It is easily checked that d'_S is a metric in $\bar{R}$ and $p_S^{-1} \circ f$ is a local isometry between $(\bar{R}, d'_S)$ and (S, d_S). Let G'_S be a shortest arc from p to q in $(\bar{R}, d'_S)$ and put $G_S = p_S^{-1}(f(G'_S))$. Then $G'_S \to L$ and $G_S \to G$. Also, for S close enough to P, $G'_S \subset R$ and therefore G_S is a geodesic arc in S.

The conclusion can be immediately extended to any geodesic arc $G \subset P$ since G can be approximated by geodesic subarcs with endpoints different from vertices of P.

4 Densely many long geodesics

Theorem 1. *For most convex surfaces S the following holds: for any positive number r there is a set T dense in T_1S such that, for any $(x, \tau) \in T$, there is a geodesic of length r, with midpoint x and with directions τ and $-\tau$ at x.*

Proof. We prove the theorem by showing that most surfaces in $\mathscr{S}^+$ have the required property, the space $\mathscr{S}^+$ of all smooth strictly convex surfaces being residual in $\mathscr{S}$ by Klee's generic theorem from Sect. 2.

Let $K \subset \mathbb{R}^3$ and $\Sigma \subset \mathbb{S}^2$ be open balls. For any surface $S \in \mathscr{S}$, let

$$I(S) = T_1 S \cap (K \times \Sigma).$$

Define

$$\mathscr{S}_n(K, \Sigma) = \{S \in \mathscr{S}^+ : I(S) \neq \emptyset \text{ and } \forall (x, \tau) \in I(S), \mu(x, \tau) \leq n\}.$$

We first show that $\mathscr{S}_n(K, \Sigma)$ is nowhere dense in $\mathscr{S}^+$.

Let $\mathscr{O} \subset \mathscr{S}$ be open. Consider $S_0 \in \mathscr{O} \cap \mathscr{S}^+$ and suppose $I(S_0) = \emptyset$. Then either

(i) $S_0 \cap K = \emptyset$

or

(ii) $S_0 \cap K \neq \emptyset$, but for any point x in $S_0 \cap K$ the set $T_1 x \cap \Sigma$ is empty, where

$$T_1 x = \{\tau \in \mathbb{S}^2 : (x, \tau) \in T_1 S_0\}.$$

If (i) holds then we can easily find a surface in $\mathscr{O} \cap \mathscr{S}^+$ disjoint from $\bar{K}$. If (ii) holds then, for any $x \in S_0 \cap K$, the set $T_1 x \cap \bar{\Sigma}$ is empty; indeed, if $T_1 x_0 \cap \bar{\Sigma} \neq \emptyset$ for some point $x_0 \in S_0 \cap K$ then, since S_0 is of class C^1 and strictly convex, for a suitably chosen point $x \in S_0 \cap K$ we would have $T_1 x \cap \Sigma \neq \emptyset$, which is false. However $T_1 x_0 \cap \bar{\Sigma}$ may be nonempty for some $x_0 \in \operatorname{bd} K$. To avoid this, consider a homothety h having the centre in $S_0 \cap K$ and the ratio $\varrho > 1$. Then

$$h(S_0 \cap K) \supset h(S_0) \cap K, \qquad h(S_0 \cap \operatorname{bd} K) \cap \bar{K} = \emptyset$$

and therefore, since $S_0 \in \mathscr{S}^+$,

$$T_1 h(S_0) \cap \bar{K} \times \bar{\Sigma} = \emptyset.$$

For $\varrho - 1$ small enough, $h(S_0) \in \mathscr{O}$. Now, for a whole neighbourhood $\mathscr{N}$ of $h(S_0)$ in $\mathscr{S}^+$, the sphere bundle $T_1 S$ misses $\bar{K} \times \bar{\Sigma}$ for any $S \in \mathscr{N}$, whence $\mathscr{N} \cap \mathscr{S}_n(K, \Sigma) = \emptyset$. It remains to consider the case $I(S_0) \neq \emptyset$.

Let $P \in \mathscr{O}$ be a polytopal surface such that $I(P) \neq \emptyset$. Consider now $(x_0, \tau_0) \in I(P)$. Since in all tangent directions τ at x_0 except for at most countably many $\lambda G(x_0, \tau) = \infty$, we can find a tangent direction τ_1 at x_0 such that $\tau_1 \in \Sigma$ and $\mu(x_0, \tau_1) = \infty$. Let Γ be the geodesic arc with midpoint x_0, with directions $\pm\tau_1$ at x_0 and of length $2n + 1$. Choose $\alpha \in (0, 1)$ such that $B(x_0, \alpha) \subset K \backslash E$, where E is the union of all edges of P.

According to Lemma 2, on each surface $S \in \mathscr{S}^+$ there is a geodesic arc G_S such that $S \to P$ yields $G_S \to \Gamma$. Hence, the midpoint x_S of G_S converges to x_0 and there is a number $\beta \in (0, \alpha/2)$ such that $\delta(S, P) < \beta$ implies $\delta(G_S, \Gamma) < \alpha^2$. Let u, v be the endpoints of the longest subarc $A \subset G_S$ containing x_S and included in $\overline{B(x_0, \alpha)}$.

Denote by A', u', v' the orthogonal projections of A, u, v on P, respectively. For $\alpha \to 0$, the tangent plane Π_x at x to S converges uniformly for $x \in S \cap B(x_0, \alpha)$ to the plane Π of the facet of P at x_0. Thus, not only A, as a geodesic on a smooth convex surface, is differentiable, but also A', for S close enough to P. Then, there is a point $w' \in A'$ such that the tangent direction τ' of A' at w' (from u' to v') equals $\|v' - u'\|^{-1}(v' - u')$. If $\alpha \to 0$, then $\|v' - u'\|^{-1}(v' - u') \to \tau_1$ because $\Delta(u', \Gamma) < \alpha^2$, $\Delta(v', \Gamma) < \alpha^2$ and $\|v' - u'\| > \alpha$.

Let w be the point of A (unique for α small enough) whose orthogonal projection on P is w' and let τ_w be the tangent direction of A at w (from u to v). Since $w \in B(x_0, \alpha)$, if $\alpha \to 0$ then $w \to x_0$ and $\Pi_w \to \Pi$, whence $\|\tau_w - \tau'\| \to 0$. This together with $\tau' \to \tau_1$ proves that $\tau_w \to \tau_1$. For α small enough, $\tau_w \in \Sigma$. Since $w \in B(x_0, \alpha)$, we have $(w, \tau_w) \in K \times \Sigma$ and, for $\alpha \to 0$, not only $x_S \to x_0$, but also $w \to x_0$. Hence the distance from x_S to w on G_S converges to zero. This together with $\lambda G_S \to 2n+1$ implies $\mu(w, \tau_w) > n$ for α small enough.

This shows that no surface $S \in \mathscr{S}^+$ at Hausdorff distance less than β from P belongs to $\mathscr{S}_n(K, \Sigma)$. Therefore $\mathscr{S}_n(K, \Sigma)$ is nowhere dense in $\mathscr{S}^+$.

Now let $\{x_n : n \in \mathbb{N}\}$ be dense in $\mathbb{R}^3$ and $\{\sigma_n : n \in \mathbb{N}\}$ dense in $\mathbb{S}^2$. Put

$$D(\sigma_n, \varrho) = \{\sigma \in \mathbb{S}^2 : d_{\mathbb{S}^2}(\sigma, \sigma_n) < \varrho\}$$

and

$$\mathscr{S}_{n,m,p,q} = \mathscr{S}_n(B(x_m, q^{-1}), D(\sigma_p, q^{-1}))\,.$$

Each set $\mathscr{S}_{n,m,p,q}$ is nowhere dense in $\mathscr{S}^+$. Thus most surfaces $S \in \mathscr{S}^+$ belong to

$$\mathscr{S}^+ \setminus \bigcup_{n,m,p,q} \mathscr{S}_{n,m,p,q}\,.$$

We verify for such a surface S the property of the statement. Let $n > r/2$. Take any open set $O \subset T_1 S$. Consider the open balls K and Σ in $\mathbb{R}^3$ and $\mathbb{S}^2$ respectively, such that $O \cap (K \times \Sigma) \neq \emptyset$. Take (x^*, τ^*) in this intersection. Choose m, p, and q such that

$$x^* \in B(x_m, q^{-1}) \subset K\,, \qquad \tau^* \in D(\sigma_p, q^{-1}) \subset \Sigma\,.$$

Since $S \notin \mathscr{S}_{n,m,p,q}$ but

$$T_1 S \cap (B(x_m, q^{-1}) \times D(\sigma_p, q^{-1})) \neq \emptyset\,,$$

for some (x, τ) in the above set $\mu(x, \tau) > n > r/2$.

The proof is finished.

5 Long geodesics without self-intersections and many closed geodesic arcs

Theorem 2. *On most convex surfaces there are non-self-intersecting geodesics of arbitrary finite lengths .*

Proof. It suffices to show that the set $\mathscr{S}_n$ of all $S \in \mathscr{S}$ admitting only non-self-intersecting geodesics of length at most n is nowhere dense in $\mathscr{S}$.

Let $\mathscr{O} \subset \mathscr{S}$ be open and choose a smooth surface $S \in \mathscr{O}$. Close to S, in $\mathscr{O}$, we find a polytopal surface P such that each vertex v of P has a spherical image which is small enough to ensure that the full angle of P at v is more than π. By Pogorelov's theorem, P has at least three closed quasigeodesics. We only use the existence of one of them (which is much easier to prove, see [1, p. 378]). Let G be this closed quasigeodesic. The construction which follows was independently found and used by Aleksandrov and Burago in [3]. As a Jordan curve, G decomposes P into two pieces P_1 and P_2, each of which has an inner metric with non-negative curvature. Using Aleksandrov's gluing theorem or only the gluing theorem for polygonal domains (see [1, p. 317] or [4, p. 150]), we obtain a manifold with non-negative curvature by gluing together P_1, P_2 and two rectangles R_1, R_2, of lengths ℓ_1, ℓ_2 and widths w_1, w_2 respectively, such that $\ell_1 + \ell_2 = \lambda G$ and $w_1 = w_2 = \varepsilon$, for small ε. By the Existence theorem and the Uniqueness theorem (from Sect. 2), there is a unique (up to a rigid motion of $\mathbb{R}^3$) polytopal surface P' isometric to the above manifold. By letting $\varepsilon \to 0$ and choosing the corresponding surface P'_ε in some big ball, a sequence of surfaces P'_{ε_k} with $\varepsilon_k \to 0$ converges by Blaschke's selection theorem to a surface which, by the Uniqueness theorem combined with Lemma 1, must be congruent to P. A sequence of suitable congruent copies P_{ε_k} of P'_{ε_k} converges then to P and we therefore find, for some k, a surface $P_{\varepsilon_k} \in \mathscr{O}$. Put $\varepsilon = \varepsilon_k$ and let R be the portion of P_ε corresponding to $R_1 \cup R_2$. Choose $m > n(\lambda G)^{-1} + 1$ and let $x_1, \dots x_m \in R$ correspond to the m points of a small edge of R_1 which divide it into $m + 1$ line-segments of equal lengths (the small edges of R_1 and R_2 are pairwise glued together). Thus the rectangle $R_1 \cup R_2$, now considered in $\mathbb{R}^2$ with just one pair of edges identified, has copies x'_i and x''_i (identified in R) of x_i on its opposite short edges. Let L_i be the line-segment from x'_i to x''_{i+1}. Then $\bigcup_{i=1}^{m-1} L_i$ corresponds to a geodesic Γ on P_ε, with endpoints x_1, x_m and passing through $x_2, \dots, x_{m-1}$. Since $\lambda L_i > \lambda G$ $(i = 1, \dots, m-1)$,

$$\lambda\Gamma > (m-1)\lambda G > n\,.$$

Consider

$$T = \{x \in \mathbb{R}^3 : \Delta(x, \Gamma) < \varepsilon(6m)^{-1}\}\,.$$

By Lemma 2, for $\beta > 0$ small enough and for any convex surface S at Hausdorff distance at most β from P_ε, there is a geodesic arc $A \subset T$ on S joining a point close to x_1 and a point close to x_m. Thus A is a geodesic on S of length close to $\lambda\Gamma$ and without self-intersections. This yields $S \notin \mathscr{S}_n$, and the theorem is proved.

It is unknown whether an arbitrary convex surface contains closed geodesic arcs.

Theorem 3. *On most convex surfaces there are infinitely many pairwise disjoint closed geodesics arcs.*

Proof. We see as in the preceding proof that the family of all convex surfaces admitting at most n pairwise disjoint closed geodesic arcs is nowhere dense in $\mathscr{S}$: we take $m = n + 1$ and consider the line-segments from x'_i to x''_i instead of L_i.

References

1. Aleksandrov, A.D.: Die innere Geometrie der konvexen Flächen. Berlin: Akademie-Verlag 1955
2. Aleksandrov, A.D.: Existence of a convex polyhedron and of a convex surface with a given metric (in Russian). Mat. Sb. **11**, 15–61 (1941)
3. Aleksandrov, A.D., Burago, J.D.: Quasigeodesics. Proc. Steklov Inst. Math. **76**, 58–76 (1967)
4. Busemann, H.: Convex surfaces. New York: Interscience 1958
5. Gruber, P.: A typical convex surface contains no closed geodesic. J. Reine Ang. Math. **416**, 195–205 (1991)
6. Klee, V.: Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. **139**, 51–63 (1959)
7. Klingenberg, W.: Contributions to Riemannian Geometry in the large. Ann. Math. **69**, 654–666 (1959)
8. Pogorelov, A.V.: Quasigeodesics on convex surfaces (in Russian). Mat. Sb. **25**, 275–307 (1949)
9. Pogorelov, A.V.: Die eindeutige Bestimmtheit allgemeiner konvexer Flächen. Schriftr. Forsch. Inst. Math. **3** (1956)
10. Zamfirescu, T.: Many endpoints and few interior points of geodesics. Invent. Math. **69**, 253–257 (1982)
11. Zamfirescu, T.: Baire categories in convexity. Atti Semin. Mat. Fis. Univ. Modena **39**, 139–164 (1991)