There is a Universal Topological Plane

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Abstract

We show that every arrangement of pseudolines in the plane can be extended to a topological projective plane, a projective geometry whose points and topology agree with the real projective plane and whose lines also have the topology of the projective plane. In this topological projective plane the given arrangement becomes an arrangement of "straight lines". This makes it possible to realize a "topological sweep of an arrangement" as the familiar sweep by a family of parallel lines. We then use this result to construct a universal topological plane, one in which every arrangement of pseudolines is stretchable. Both results had been conjectured by B. Grünbaum.

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1 Introduction

An *arrangement of pseudolines* in the real projective plane $\mathbf{RP}^2$ is a finite family of Jordan curves in $\mathbf{RP}^2$ that do not separate the plane, every two of which intersect at precisely one point, at which they cross. Such arrangements, which exhibit many of the properties of straight line arrangements but are more general, have been studied since the work of F. Levi [7]; see [4] for an extensive bibliography up to 1971.

$\mathbf{RP}^2$ is normally thought of as the Euclidean plane, completed by adding a "line at infinity". For our purposes it will be convenient to represent $\mathbf{RP}^2$ as the closed unit disk Δ in the plane, with antipodal points on the unit circle identified. We will also find it convenient to fix a particular homeomorphism of the Euclidean plane to the interior of Δ, obtained by projecting points through the center of a sphere tangent to the xy plane to the lower hemisphere, followed by orthogonal projection back into the xy plane. This homeomorphism takes the vertical lines, for example, to a family V of vertical semi-ellipses centered at $(0,0)$ and passing through $(0,1)$ and $(0,-1)$. Thoughout this paper we will take this as the model of the real projective plane.

Assuming that the line at infinity (the boundary of the unit disk) is one of the pseudolines in an arrangement, we can represent the arrangement of pseudolines in the projective plane as a finite family of simple arcs in Δ connecting antipodal points with the property that every two intersect at precisely one point, at which they cross. A *spread of pseudolines* in $\mathbf{RP}^2$ is then the continuous version of an arrangement: it is an infinite family of simple arcs in Δ connecting antipodal points, such that

1. Every two arcs intersect at precisely one point, at which they cross.

2. There is a mapping l from the unit circle $C = \partial\Delta$ to the family of arcs such that $l(p)$ has an endpoint at p and is a continuous function (in the Hausdorff metric) of $p \in C$.

Thus a spread is a one-parameter family of pseudolines that resembles a one-parameter family of lines in $\mathbf{RP}^2$. On the other hand, we can imagine a topological version of the projective plane endowed with not only the topological structure of $\mathbf{RP}^2$ but including also *all* the lines of the projective plane; i.e., this collection of points and "lines" should be a projective plane by virtue of satisfying the axioms of projective geometry. Following Grünbaum [4] we define a *topological projective plane* T as an infinite family of simple arcs in Δ connecting antipodal points, such that

1. Every two arcs intersect either at their common endpoints, or else at precisely one point, at which they cross.

2. Given any two points p, q of Δ, at least one of which belongs to the interior of Δ, there is a unique pseudoline $l(p, q)$ of the family joining them, and $l(p, q)$ is a continuous function (in the Hausdorff metric) of p, q. (It then follows that given four points p, q, p', q' in Δ, the point $l(p, q) \cap l(p', q')$ varies continuously with the four points.)

Thus a topological plane is homeomorphic to the real projective plane, as is its space of lines with the topology induced by the Hausdorff metric. We can obtain many topological planes in a trivial way by taking any homeomorphism of the projective plane and designating as lines the images of the lines in the real projective plane. We shall consider two topological planes that differ in this way to be isomorphic. Hilbert [6] constructed a (Euclidean) topological plane in which Desargues' theorem is false (his goal was to show that this theorem was independent of the axioms of incidence, order, parallels, completeness, and most of the congruence axioms), and which thus cannot be completed to the real projective plane. What we call a topological plane is identical to the real projective plane in both its topology and its incidence structure. It differs only in that the lines of a topological plane need not arise from a linear structure, in the sense that the topological plane may not be coordinatizable; i.e., its lines may not simply be the solution sets of linear equations (which in fact will always be the case if Desargues' theorem does not hold).

Motivated by the fact that any finite arrangement of straight lines can be extended to a spread of straight lines, B. Grünbaum conjectured in [4] that the same should hold for pseudolines, a conjecture that we proved in [3]. Again in [4], he made the much bolder conjecture that any arrangement of pseudolines can be extended to a topological plane. The truth of this conjecture would immediately provide a proof of the existence of a spread since then we can mimic the argument for straight lines that consists of ordering the lines by their slopes and rotating each to its successor about their point of intersection.

It would also follow from the truth of this conjecture that the "topological sweep" method in computational geometry [1, 8], which is actually a discrete process mimicking a continuous sweep of a pseudoline through an arrangement, can in fact be carried out continuously. Simply take the arrangement that is to be swept, adjoin pseudolines through a fixed point p on the line at infinity corresponding to the "cuts" of the topological sweep, and extend this enlarged arrangement to a topological plane. The family of parallels through p then sweeps the given arrangement in the desired manner.

Our Theorem 1 will establish the validity of this stronger conjecture. The proof, which is much simpler than the proof of the first conjecture in [3], is yet another illustration of the fact that it is frequently easier to prove the "right" stronger theorem than a weaker version of that theorem.

It has been known since the work of Levi [7] that not every arrangement of pseudolines in $\mathbf{RP}^2$ is *stretchable*, i.e., isomorphic to an arrangement of straight lines. Thus $\mathbf{RP}^2$, which contains isomorphic copies of all *straight line* arrangements, fails to contain an arrangement isomorphic to a non-Desargues configuration, for example. On the other hand, there is no *a priori* reason why a topological plane that does contain such an arrangement, and which exists by virtue of Theorem 1, should necessarily contain isomorphic copies of all pseudoline arrangements. Thus Grünbaum asked, in [4], whether there can exist a topological plane that is universal for *all* arrangements, i.e., that contains an isomorphic copy of each. In Section 3 we use Theorem 1 to show that such a plane, in which every arrangement is simultaneously "stretchable", does in fact exist.

2 Extending an Arrangement to a Topological Plane

Let us call an arrangement *monotonic* if each pseudoline of the arrangement is compatible with the pseudolines in V (which we call "vertical" lines), i.e., if each pseudoline meets each vertical line exactly once. It is not difficult to see that every arrangement of pseudolines is isomorphic to a monotonic arrangement, and that this isomorphism can in fact be achieved by a homeomorphism of the projective plane; see [2] for a proof. We shall prove the stronger theorem (which we need in order to construct a universal topological plane) that every monotonic arrangement, together with V, can be extended to a topological plane. The idea of the proof is to extend the given arrangement by adjoining each line of V that passes though a vertex of the arrangement. Each face of this enlarged arranement is either a triangle (with a side in V) or a trapezoid with two vertical edges. Then for each face of this extended arrangement we straighten the arrangement of (4 or 5) pseudolines that consists of $C = \partial\Delta$ together with the pseudolines that support the edges of the face (see [2]). We then use the straight lines meeting the straightened face to construct the portion of the topological plane meeting that face, and finally we connect appropriately the various pieces of each pseudoline. It will then remain to show that each pair of the resulting "pseudolines" crosses at most once and that these pseudolines vary continuously in the Hausdorff metric.

The proof that each pair crosses at most once is facilitated by the notion of a *proper crossing*. Let l and l' be any two curves in Δ connecting distinct antipodal endpoints p_1, p_3 and p_2, p_4, respectively. l and l' may intersect at more than one point. Let q be an isolated point of intersection of l and l' at which they cross. There is some small topological disk Δ' containing q and no other point of intersection of l and l'. l and l' intersect the boundary of disk Δ' in four points, p'_1, p'_2, p'_3, and p'_4, where p'_i lies between p_i and q on l or l'. We say that q is a *proper intersection point* of l and l' if l and l' cross at q and if p_1, p_2, p_3, and p_4 occur in the same order around Δ (clockwise or counterclockwise) as p'_1, p'_2, p'_3, and p'_4 do around Δ'. (See Figure 1.)

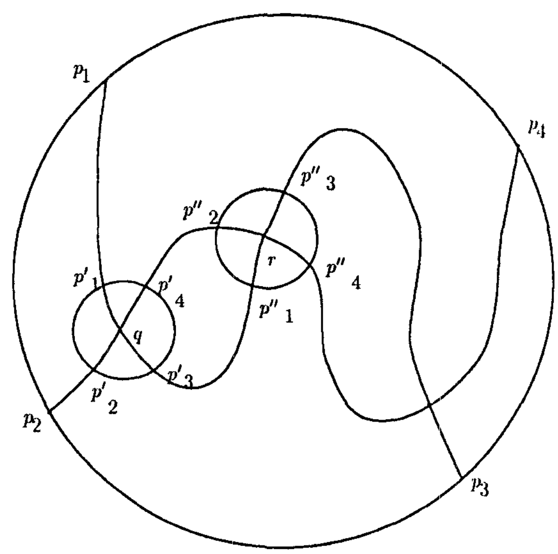

Figure 1: q is a proper intersection; r is not.

We can then replace the global condition that curves intersect at precisely one point, at which they cross, by the local condition that every point of intersection is proper.

Lemma 1 *Two pseudolines in the disk that have at most a finite number of intersections intersect at precisely one point, at which they cross, if and only if every point of intersection of the two curves is a proper intersection point.*

Proof: Since the endpoints of each curve are antipodal, the first curve separates the endpoints of the second and thus the curves must have at least one intersection point. If there is only one, that intersection must clearly be proper. On the other hand, if our two pseudolines l and l' intersect at more than one point, we can list the points of intersection in order along l. Let q and q' be two adjacent points of intersection. It is not hard to see that if q is proper then q' is not. □

Theorem 1 *Every monotonic arrangement of pseudolines in Δ, together with V, can be extended to a topological plane.*

Proof: First we describe the construction of the infinite arrangement of pseudolines that will constitute the lines of our topological plane.

Let $\mathcal{L}$ be the given arrangement of n monotonic pseudolines in the unit disk Δ with endpoints at antipodal vertices together with the pseudolines from V that pass through vertices of the given arrangement. The members of $\mathcal{L}$ partition Δ into a 2-dimensional cell complex, consisting of a set of open triangular and trapezoidal faces $F(\mathcal{L})$, relatively open edges $E(\mathcal{L})$, and vertices $V(\mathcal{L})$. Edges in C are considered as belonging to $E(\mathcal{L})$, and their endpoints as belonging to $V(\mathcal{L})$.

For each $f \in F(\mathcal{L})$ we fix a homeomorphism ϕ_f that straightens the arrangement consisting of C and the pseudolines bounding f, and maps straight vertical lines to straight vertical lines. For each pair of distinct points p and q lying on the boundary of f, there is a straight line segment s connecting $\phi_f(p)$ to $\phi_f(q)$. $\phi_f^{-1}(s)$ is an arc in Δ from p to q. These arcs form the "pieces" of the pseudolines we are constructing.

Thus for each point p lying on some pseudoline in $\mathcal{L}$, there is a set of arcs A_p having p as one endpoint. Eliminate from A_p any arc that is a subset of some other arc in A_p. (If $l \in \mathcal{L}$ contains p, then there initially will be many arcs with endpoint p contained in l; we eliminate all but the two longest.) Define a function ψ_p from A_p to C as follows. Each arc $a \in A_p$ is the inverse image of the straight line segment $\phi_f(a)$ under some transformation ϕ_f. The segment $\phi_f(a)$ lies on some ray r with endpoint $\phi_f(p)$ pointing toward a point $\phi_f(x) \in \phi_f(C)$. Let $\psi_p(a) = x$. Thus ψ_p maps each arc $a \in A_p$ to a point $x \in C$. It is not hard to see that this mapping is one-to-one and onto. It is also continuous in the Hausdorff metric.

We now link the pieces of a pseudoline that end at p. For each pair of antipodal points $x, \bar{x} \in C$, join $\psi_p^{-1}(x)$ to $\psi_p^{-1}(\bar{x})$. The linked pieces form curves connecting x to $\bar{x}$ which define the pseudolines in Δ.

Let l and l' be two curves in Δ connecting distinct antipodal endpoints p_1, p_3 and p_2, p_4, respectively. Let q be some intersection point of l and l' lying on the interior of a face f. l and l' intersect the boundary of f in four points, p'_1, p'_2, p'_3 and p'_4, where p'_i lies between p_i and q on l or l'. The order of p_1, p_2, p_3 and p_4 around Δ matches the order of $\phi_f(p_1), \phi_f(p_2), \phi_f(p_3)$, and $\phi_f(p_4)$ around $\phi_f(\Delta)$. The order of p'_1, p'_2, p'_3 and p'_4 around f matches the order of $\phi_f(p'_1), \phi_f(p'_2), \phi_f(p'_3)$, and $\phi_f(p'_4)$ around $\phi_f(f)$. By construction the intersection of $\phi_f(l)$ and $\phi_f(l')$ is proper, so the intersection of l and l' must be proper. A similar argument holds if q lies on the boundary of a face f. Thus the defined curves form pseudolines in Δ. It is equally clear that the resulting pseudolines vary continuously. □

Corollary 1 *Every arrangement of pseudolines in Δ can be extended to a spread* [3].

Proof: Take the extension to a topological plane provided by Theorem 1, and—the pseudolines of the arrangement being ordered by their intersection with C—take the union of the pencils found between each pseudoline and the next. It is easy to verify that the resulting family is a spread. □

3 Constructing a Universal Topological Plane

Theorem 2 *There exists a universal topological plane T which contains every arrangement up to isomorphism.*

$$\begin{array}{ccccccccc} M_1 & & & & M_2 & & & & \cdots \\ & & & & & & & & \\ A_1 & \subseteq & B_1 & \subseteq & A_2 & \subseteq & B_2 & \subseteq & \ldots \; A_\infty \\ \cap & & \cup & & \cap & & \cup & & \cdots \\ T_1 & \supseteq & C_1 & & T_2 & \supseteq & C_2 & & \cdots \; T_\infty \\ & & & & & & & & \\ \epsilon_1 & & & & \epsilon_2 & & & & \cdots \end{array}$$

Figure 2: Construction of tower of A_i's.

Proof: Let $M_1, M_2, M_3, \ldots$ be a listing of all arrangements up to isomorphism. It is easy to construct arrangements $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ such that A_i contains an arrangement isomorphic to M_i. If A_∞ is the union of all these A_i, then every arrangement is contained in A_∞. However, it is not clear how or even if A_∞ can be extended to a topological plane. (In fact in general it cannot, since continuity may already be violated in A_∞.)

We first outline a more specific construction of the tower of A_i's which will lead to a topological plane containing A_∞. For each A_i choose an $\epsilon_i > 0$ where $\lim_{i\to\infty} \epsilon_i = 0$. Assume we have already constructed a monotonic arrangement A_i containing an arrangement isomorphic to M_j for all $j \leq i$. Using Theorem 1, let T_i be a topological plane containing $A_i \cup V$ in its set $L(T_i)$ of pseudolines. We will choose a finite subset C_i of $L(T_i)$ which will form a "mesh" of "size" ϵ_i on T_i, in a sense to be made precise. Let $B_i = C_i \cup A_i$. Extend B_i to a monotonic arrangement containing an arrangement isomorphic to M_{i+1}. To extend B_i take a new member of $L(T_i)$ in general position with respect to the members of B_i, thicken it slightly to a strip S, and embed a "thin" copy of M_{i+1} into it, removing the rest of S. Let this arrangement be A_{i+1}. (See Figure 2.)

Let S be the set of sequences $l_1, l_2, l_3, \ldots$ where $l_i \in L(T_i)$. The pseudolines in our universal topological plane T_∞ will be the limits of the Cauchy sequences in S under the Hausdorff metric. (Since the space of non-empty closed subsets of Δ is compact in the Hausdorff metric [5, Theorem VI, p. 150], these Cauchy sequences converge.) We will show that these limiting sets are Jordan arcs connecting antipodal points which obey the axioms above. Since each curve $l \in A_\infty$ is the limit of a Cauchy sequence in S, the set $A_\infty \subseteq L(T_\infty)$, and thus $L(T_\infty)$ contains every finite arrangement of pseudolines up to isomorphism.

We now specify how the "meshes" C_i are constructed from the T_i. Let P_i' be the set of pairs of points $x, y \in T_i$ where $d(x,y) \geq \epsilon_i$. For each pair $(x,y) \in P_i'$, removing x, y and every pseudoline that passes through either x or y from T_i divides the remaining pseudolines in $L(T_i)$ that are within ϵ_i of $l_i(x,y)$ into four homotopy classes. Choose a representative from each class and let $C(x,y)$ be the resulting set of four pseudolines. Let N_x and N_y be neighborhoods of x and y in T_i that do not intersect any pseudoline in $C(x,y)$. $\{N_x \times N_y : x, y \in P_i'\}$ forms an open covering of P_i'. Since P_i' is compact, this covering has a finite subcovering Θ. Let

$$C_i = \bigcup_{N_x \times N_y \in \Theta} C(x,y).$$

Lemma 2 *Let T be any topological plane containing C_i. If $d(x,y) \geq \epsilon_i$, $x, y \in T$, then there exist neighborhoods N_x and N_y of x and y such that for all $l \in L(T)$ where $l \cap N_x \neq \emptyset$ and $l \cap N_y \neq \emptyset$, we have $d(l, l_i(x,y)) < 2\epsilon_i$.*

Proof: Omitted. □

We now claim that T_∞ is a topological plane. For this purpose we must show that $L(T_\infty)$, which is defined as the limit of Cauchy sequences of pseudolines, is in fact a set of curves connecting antipodal points. Curiously, this will follow from the fact that every two distinct points $x, y \in T_\infty$ are contained in a unique limiting set in $L(T_\infty)$.

Let x, y be any two distinct points in T_∞. Consider the sequence $l_1(x,y)$, $l_2(x,y)$, $l_3(x,y), \ldots$. For any $\epsilon > 0$ there exists an i such that $\epsilon_i < \min(\epsilon, d(x,y))$. By Lemma 2, $d(l_j(x,y), l_k(x,y)) < 2\epsilon_i$ for all $k, j \geq i$. Thus, the sequence $l_1(x,y), l_2(x,y), l_3(x,y), \ldots$ is Cauchy and its limit is in $L(T_\infty)$. Since x, y are points in every element of the sequence, they are also in the limit, hence there exists at least one set in $L(T_\infty)$ containing x, y.

Assume there were two distinct sets $s_1, s_2 \in L(T_\infty)$ containing both x and y. Without loss of generality, there must be some point $z \notin s_1$ with $z \in s_2$. The set s_1 is the limit of closed subsets of a compact set, so it must be compact. Let $\epsilon = d(z, s_1)$. There exists an i such that $\epsilon_i < \min(\epsilon/4, d(x,y))$.

By Lemma 2, there exist neighborhoods N_x and N_y of x and y such that any pseudoline $l \in T_k$, $k \geq i$ that intersects both N_x and N_y lies within distance $2\epsilon_i$ of $l_i(x,y)$. Since $x, y \in s_1$, there must be a Cauchy sequence converging to s_1 whose members eventually meet both N_x and N_y. By Lemma 2, each of these pseudolines lies within distance $2\epsilon_i$ of $l_i(x,y)$. Thus each of these pseudolines lies within distance $2\epsilon_i + d(l_i(x,y), z)$ of z, and $d(z, s_1) \leq 2\epsilon_i + d(l_i(x,y), z)$. Since $d(z, s_1) = \epsilon$ and $\epsilon_i < \epsilon/4$, we conclude that $d(l_i(x,y), z) > \epsilon/2$.

On the other hand, there must be a Cauchy sequence converging to s_2 whose members eventually meet both N_x and N_y. Again by Lemma 2, each of these pseudolines lies within distance $2\epsilon_i$ of $l_i(x,y)$. Since z lies in the limit of these pseudolines, $d(l_i(x,y), z) \leq 2\epsilon_i < \epsilon/2$, a contradiction. We conclude that $s_1 = s_2$, i.e., that every pair of points x, y is contained in a unique set $l_\infty(x,y) \in L(T_\infty)$.

We can now show that $L(T_\infty)$ is a set of curves. Since $V \subseteq T_i$ for every i, the vertical lines V all belong to $L(T_\infty)$. Let s be an element of $L(T_\infty)$ that is not a vertical line. s is the limit of a set of pseudolines each of which intersects every vertical line $l \in V$, so s must intersect each vertical line in V as well as the line at infinity. But s can only intersect each vertical line once, or else there would be a pair of points contained in two different sets of $L(T_\infty)$. Thus s can be parameterized by the vertical lines and can be expressed as the graph of a function on P^1. Since s is the limit of graphs of continuous functions *and is itself the graph of a function*, it also is continuous. Thus s is a curve connecting antipodal points.

Finally, the continuity of $l_\infty(x,y)$ as a function of x and y follows by yet another application of Lemma 2. □

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