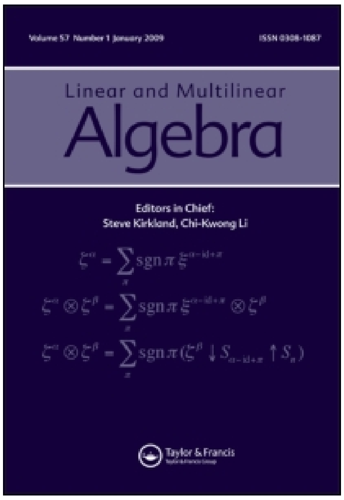

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Invariance of convex sets under linear transformations

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Invariance of Convex Sets Under Linear Transformations

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This paper deals with the space $\pi(X,X)$ of all linear transformations L that leave convex sets invariant; for convex sets X and Y in $\mathbb{R}^d$, $\pi(X,Y) = \{L \mid LX \subset Y\}$. If $(X = Y =)K$ is a convex body then faces of K invariant under L are determined in case $0 \notin \operatorname{int} K$. Moreover, invariant supporting hyperplanes of K are determined in case K is a simplex in general position. It is shown that $\pi(P_1,P_2)$ is polyhedral if P_1 and P_2 are polyhedral. Finally, it is shown that for any polyhedral set P, $\pi(P,P)$ is a polytope iff P is a polytope with $\operatorname{lin} P = \mathbb{R}^d$.

1. INTRODUCTION

Operators which leave invariant a cone in infinite-dimensional spaces have been studied extensively, especially in the context of generalizations of the Perron-Frobenius Theorem; see e.g. Barker, Schneider [2]. In Berman, Plemmons [3] nonnegative matrices of order n are studied as operators that map the nonnegative othant in $\mathbb{R}^d$ onto itself. In Adin [1] cone-preserving operators between d-dimensional polyhedral cones are investigated; using Gale diagrams, the extreme cone-preserving operators are determined in case the cone has either $d+1$ or $d+2$ extreme rays. In Elsner [5], spectral properties are determined for real square matrices that leave invariant a nontrivial convex set. In Sierksma, De Vos [10], the extreme simplex-preserving operators are determined.

In this paper we focus on convex sets and polytopes in $\mathbb{R}^d$.

For any sets $X,Y \subset \mathbb{R}^d$, we denote by $\pi(X,Y)$ the family of all linear transformations L from $\mathbb{R}^d$ into $\mathbb{R}^d$ such that $LX \subset Y$. If $X = Y$, we write $\pi(X)$ instead of $\pi(X,X)$ and say that X is *invariant* under L.

In this paper we consider linear transformations that leave convex sets invariant. In Section 2 we consider invariant faces, and in Section 3 invariant hyperplanes. In

*The research for this paper was done during V. Soltan's stay and T. Zamfirescu's visiting professorship at the University of Groningen in 1990.

Section 4 a representation of $\pi(M_1, M_2)$ is given in case M_1 and M_2 are closed convex sets and this is used in Section 5 to prove, among others, that $\pi(P)$ is polytopal iff P is polytopal with $\operatorname{lin} P = \mathbb{R}^d$.

A compact convex set with nonempty interior is called a *convex body*. A convex set is called *polyhedral* if it is the intersection of finitely many closed halfspaces. A bounded polyhedral set is called *polytopal* (or a *polytope*).

For any set $X \subset \mathbb{R}^d$, $\operatorname{conv} X$ denotes the *convex hull*, $\operatorname{aff} X$ the *affine hull* and $\operatorname{lin} X$ the *linear hull* of X. For a closed convex set $M \subset \mathbb{R}^d$, $\dim M$ denotes the *dimension*, $\operatorname{int} M$ the *interior*, $\operatorname{rint} M$ the *relative interior*, $\operatorname{bd} M$ the *boundary*, $\operatorname{ext} M$ the set of extreme points, and $\operatorname{extr} M$ the set of extreme rays of M; the intersection of M with a supporting hyperplane is called a *face* of M.

2. INVARIANT FACES

Throughout, $K \subset \mathbb{R}^d$ is a convex body ($d \geq 2$) and $L \in \pi(K)$. In the following two theorems, invariant faces of K are determined for the cases $0 \in \operatorname{bd} K$ and $0 \notin K$. The case that $0 \in \operatorname{int} K$ is not studied in this paper

THEOREM 1 *If $0 \in \operatorname{bd} K$, then the face of K of smallest dimension containing 0 is invariant under L.*

Proof Let F be the face of K of smallest dimension with $0 \in F$. Let $k = \dim F$ and consider the linear subspace G generated by F.

Clearly, if $k = 0$ then $F = \{0\}$ is invariant. So, assume $k \geq 1$. Let H be a hyperplane such that $F = K \cap H$. Suppose there is a point $b \in F$ such that $Lb \notin H$. Let $b^* \in \operatorname{int} K$. Denote by H_+ and H_- the two halfspaces bounded by H such that b^*, $Lb \in H_+$. From $L(-b) \in H_-$ it follows that

$$L(\lambda(b^*) + (1 - \lambda)(-b)) \in H_-$$

for $\lambda > 0$ small enough. Consider such a λ and define

$$c_\lambda = \lambda(b^*) + (1 - \lambda)(-b).$$

For any μ with $0 < \mu < 1$ there is a hyperplane separating K from μc_λ, because $L(\mu c_\lambda) \in H_-$ yields $\mu c_\lambda \notin K$. Thus (by taking $\mu \to 0$) there is a supporting hyperplane H' at 0 with $c_\lambda \in H'$ or separating c_λ from $\operatorname{int} K$. Since $c_\lambda \in H_+$, we have $H' \neq H$. Assume $H' \supset G$. Because $-b \in G$, either both b^* and c_λ lie on the same side of H', or both belong to H'. The first case contradicts the fact that H' contains c_λ or separates it from $\operatorname{int} K$. The second contradicts $b^* \in \operatorname{int} K$. Hence, $\dim G \cap H' < k$. Consider now any hyperplane H^* including $H \cap H'$, distinct from both H, H', and supporting K. Clearly,

$$H^* \cap K = H \cap H' \cap K = F \cap H' \subset G \cap H',$$

whence $\dim H^* \cap K \leq \dim G \cap H' < k$, which contradicts the assumption that F has smallest dimension.

Hence, for any point $b \in F$, $Lb \in H \cap K = F$. So, F is an invariant face. ■

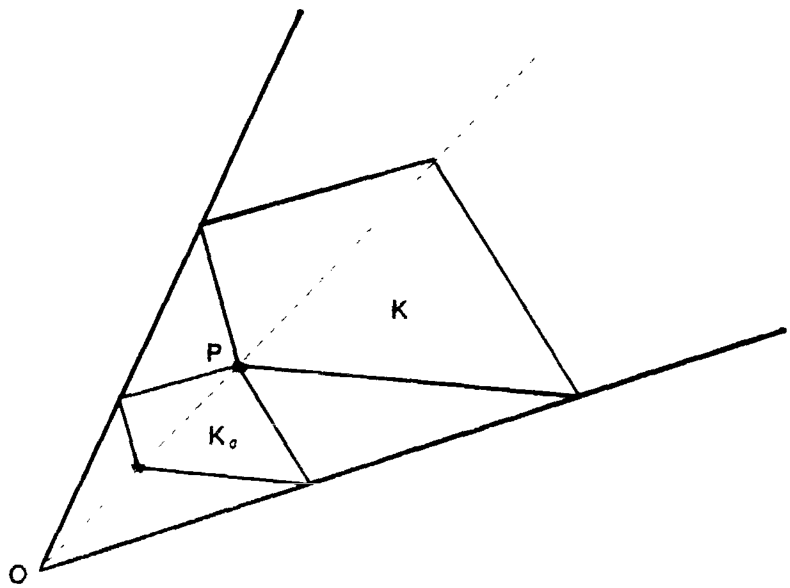

FIGURE 1.

The following example shows that faces of K that are not of smallest dimension, need not be invariant under L. Let K be the triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$. Then

$$L = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

projects all points of the triangle on the line through $(0,0)$ and $(1,1)$. Clearly, the face $((0,0),(2,0))$ is not invariant under L.

In the following theorem the concept of an *affine diameter* is used; it is a chord of a convex body such that there exist distinct parallel supporting hyperplanes through both endpoints of the chord.

For each K not containing the origin the *greatest lower homothetic copy* (glh-copy) K_0 of K is defined by

$$K_0 = \lambda_0 K$$

with

$$\lambda_0 = \inf\{\lambda \mid \lambda K \cap K \neq 0\};$$

see Figure 1.

THEOREM 2 *Let $0 \notin K$. Then K has an affine diameter (collinear with 0) such that each point of it is fixed under L. Moreover there exist two faces F_1 and F_2 of K with $\operatorname{aff} F_1 \cap \operatorname{aff} F_2 = \emptyset$, each invariant under L.*

Proof Let $0 \notin K$ and let K_0 be the glh-copy of K. Clearly, K_0 and K meet, but $\operatorname{int} K_0 \cap \operatorname{int} K = \emptyset$. Consider a hyperplane H separating $\operatorname{int} K_0$ from $\operatorname{int} K$ and the faces $F_0 = K_0 \cap H$ of K_0 and $F = K \cap H$ of K. Since both K_0 and K are invariant under L, the nonempty set

$$F_0 \cap F = K_0 \cap K$$

is also invariant under L.

By Brouwer's well-known theorem, L has a fixed point p in $F_0 \cap F$. Then p and $\lambda_0^{-1}p$ are the endpoints of an affine diameter of K, each point of which is fixed under L.

For the translated convex body $K - p$, $0 \in \mathrm{bd}(K - p)$; therefore, by Theorem 1, the face Φ of $K - p$ of smallest dimension containing the origin is invariant under L. Hence,

$$L(\Phi + p) = L(\Phi) + L(p) = L(\Phi) + p \subset \Phi + p,$$

and the face $F_1 = \Phi + p$ of K is invariant under L. Analogously, the face F_2 of K of smallest dimension containing $\lambda_0^{-1}p$ is invariant under L.

It remains to be shown that $\mathrm{aff}F_1$ and $\mathrm{aff}F_2$ are disjoint. Suppose, on the contrary, that $z \in \mathrm{aff}F_1 \cap \mathrm{aff}F_2$.

Consider the hyperplane H_1 such that $F_1 = K \cap H_1$. But either $z \notin H$ or $z \notin \lambda_0^{-1}H$, say $z \notin H$. Thus, $H_1 \neq H$. Let H^* be a hyperplane including $H_1 \cap H$, supporting K, and different from H_1 and H. Then the face $H^* \cap K$ contains p and is strictly included in F_1. Indeed, $F_1 \subset H^* \cap K$ would imply $\mathrm{aff}F_1 \subset H$, contradicting $z \notin H$. Thus

$$\dim H^* \cap K < \dim F_1,$$

which contradicts the smallest dimension of F_1 as a face of K containing p. ■

Note that if F_1 and F_2 are invariant faces of K (such faces exist according to Theorem 2), $p_1 \in F_1$, $p_2 \in F_2$, then for all fixed points $f \in K$ of L, the sections

$$(\mathrm{aff}F_1 + \mathrm{aff}F_2 + f - p_1 - p_2) \cap K$$

are invariant under L. The section ab in Figure 2 is such an invariant section. Note that in general for fixed points $f \in K$, $\bigcup_{f \in K}(\mathrm{aff}F_1 + \mathrm{aff}F_2 \quad f - p_1 - p_2) \neq K$; for instance this is the case when $F_1 = \{p_1\}$ and $F_2 = \{p_2\}$.

3. INVARIANT HYPERPLANES

Throughout this section $K \subset \mathbb{R}^d$ will be a convex body, not containing the origin 0. At the beginning of the proof of Theorem 2 we defined the greatest lower homothetic copy K_0 associated with the given convex body $K \subset \mathbb{R}^d$. The separating hyperplane of K and K_0 is in general not unique; see e.g. the case of Figure 1. However, if K is a smooth convex body then there is only one separating hyperplane H. In Klee [7], it has been shown that most (in the sense of Bäire categories) convex bodies are smooth. Therefore, the restriction to smooth convex bodies is justified. With the assumption of the uniqueness of H, we are able to establish the invariance of a subspace of codimension 1 under any map from $\pi(K)$. Let $L \in \pi(K)$ again.

THEOREM 3 *If there is a unique hyperplane H separating* $\mathrm{int}K$ *from* $\mathrm{int}K_0$, *then H is invariant under L.*

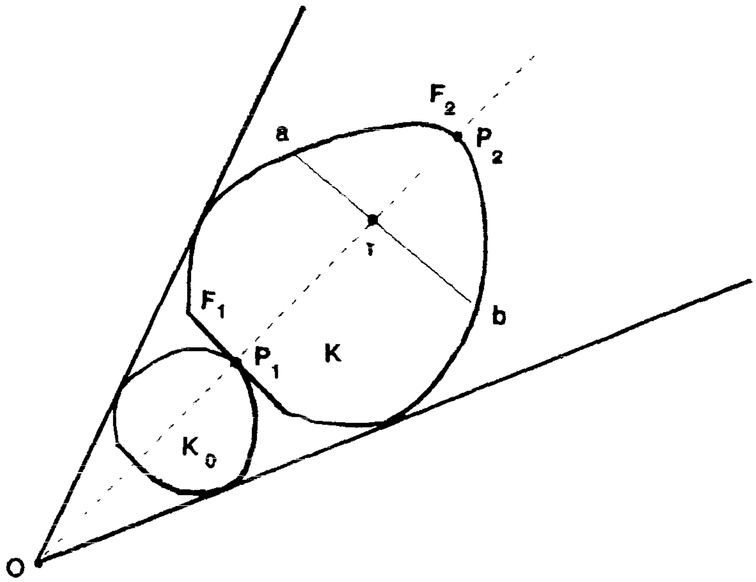

FIGURE 2.

Proof In the proof of Theorem 2 a fixed point $p \in K \cap K_0$ of L is found. Clearly, $p \in H$. Suppose $La \notin H$ for some $a \in H$. Let H_+ be the open halfspace determined by H and containing $\operatorname{int} K$, and H_- the one containing $\operatorname{int} K_0$. Then La belongs to either H_+ or to H_-, say, $La \in H_+$. Then $L(2p-a) \in H_-$ and in a whole neighborhood of $2p-a$ every point is mapped into H_-, so take $b \in H_+$ in such a neighborhood. Then $L(\lambda b+(1-\lambda)p) \in H_-$, and therefore $\lambda b \notin K$, for any $\lambda > 0$. This implies the existence of a supporting hyperplane H' at p which contains b or separates b from $\operatorname{int} K$. It follows that $H' \neq H$, which contradicts the hypothesis. ■

COROLLARY 1 *If either K or K_0 is smooth at all points of $K \cap K_0$, then the supporting hyperplane H of K or K_0, respectively, at any point of $K \cap K_0$ is invariant under L. Moreover, the section $H' \cap K$, for any hyperplane H' parallel to H and meeting K, is invariant under L.*

Proof The proof of the first part of the corollary follows directly from Theorem 3, because the separating hyperplane H of $\operatorname{int} K$ and $\operatorname{int} K_0$ is unique and supports both K and K_0 in each point of $K \cap K_0$. To prove the second statement of the theorem, take any $x' \in H' \cap K$. Then $x' = x + f$ with f a fixed point under L in K and $x \in H$. Clearly, $Lx' = L(x+f) = Lx + Lf = Lx + f \in H'$. ■

A simplex is said to be *in general position* if each proper subset of its vertices is linearly independent.

THEOREM 4 *Suppose $K \subset \mathbb{R}^d$ is a simplex in general position with $0 \notin K$ and $L \in \pi(K)$. Then there exists a unique hyperplane that separates* $\operatorname{int} K$ *from* $\operatorname{int} K_0$ *(and is therefore invariant under L).*

Proof Consider the rays with endpoints at 0 through the vertices of K. Because of the general position of K, there is a unique Radon partition of these rays, i.e. a partition $\{L_1, L_2\}$, with

$$\operatorname{conv} \cup L_1 \cap \operatorname{conv} \cup L_2 = l,$$

where l is a ray with endpoint 0. Define

$$V_i = (\cup L_i) \cap \operatorname{vert} K \qquad (i = 1, 2).$$

Clearly,

$$V_1 \cup V_2 = \operatorname{vert} K$$

and

$$\operatorname{conv} V_1 \cap \operatorname{conv} V_2 = \emptyset.$$

Let us denote

$$\{a_i\} = l \cap \operatorname{conv} V_i.$$

Of course $a_1 \neq a_2$ and we may assume that $\|a_1\| < \|a_2\|$. Now it is easily checked that

$$\lambda_0 = \|a_1\| \cdot \|a_2\|^{-1}$$

and that

$$\dim \operatorname{aff}(V_1 \cup \lambda_0 V_2) = d - 1.$$

Clearly, the hyerplane

$$H = \operatorname{aff}(V_1 \cup \lambda_0 V_2)$$

separates $\operatorname{int} K_0$ from $\operatorname{int} K$. The uniqueness of H follows directly from the fact that

$$\operatorname{conv} V_1 \cup \operatorname{conv}(\lambda_0 V_2) \subset H'$$

for any hyperplane H' separating $\operatorname{int} K_0$ from $\operatorname{int} K$. By Theorem 3, H is invariant under L. ■

For each hyperplane H separating $\operatorname{int} K$ and $\operatorname{int} K_0$, there can be constructed a linear transformation L that leaves K invariant: Some point $p \in K \cap K_0 \cap H$ has to be a fixed point of L. Hence, L has to have eigenvalue 1 with eigenvector p. Suppose L maps each point of K into the direction of the line l through 0 and p with a factor α $(0 < \alpha < 1)$ and parallel to H; i.e. for each $x \in H$, $Lx = \alpha x + (1 - \alpha)p$. H is left invariant by L if α is taken as eigenvalue of L with algebraic multiplicity $d - 1$ and with corresponding eigenspace E the $d - 1$ dimensional linear subspace parallel to H.

Each linear transformation L with eigenvalues 1 and α with $0 < \alpha < 1$ and corresponding eigenspaces the line l and the hyperplane E, leaves K invariant. Moreover, such L's leave H invariant (see Theorem 3), and keep each point of l fixed.

4. LINEAR MAPS FROM A CONVEX SET INTO ANOTHER ONE

The following lemma is easy but useful. The proof is left to the reader.

LEMMA 1 *If $X, Y \subset \mathbb{R}^d$ are linear subspaces (convex sets, convex cones), then $\pi(X,Y)$ is a linear subspace (convex set, convex cone, respectively) in $\mathbb{R}^{d\times d}$.*

Let M be any closed convex set in $\mathbb{R}^d$. It is well-known that M can be represented as a direct sum $M = N \oplus Q$, where N is a linear subspace and Q is a line-free closed convex set. Note that N is uniquely determined in this representation, while Q is not. Observe that N is a maximal subspace of $\mathbb{R}^d$ with the property that $M + N = M$. It is also known (see [6] and [8]) that a line-free closed convex set $Q \subset \mathbb{R}^d$ can be represented as

$$Q = \operatorname{conv}(\operatorname{ext} Q \cup \operatorname{extr} Q).$$

For any extreme ray l of Q, consider the ray $r(l) = l - e$, where e is the endpoint of l. Then

$$K_Q = \operatorname{conv}[\cup\{r(l) \mid l \in \operatorname{extr} Q\}]$$

is the characteristic (recession) cone of Q, and

$$Q = \operatorname{conv}(\operatorname{ext} Q) + K_Q.$$

We return to the representation of M as a direct sum, namely

$$M = N \oplus Q = N \oplus (\operatorname{conv}(\operatorname{ext} Q) + K_Q) = (N \oplus K_Q) + \operatorname{conv}(\operatorname{ext} Q)$$

$$= K_M + \operatorname{conv}(\operatorname{ext} Q).$$

Let M_1, M_2 be closed convex sets in $\mathbb{R}^d$, and

$$M_1 = N_1 \oplus Q_1 = K_{M_1} + \operatorname{conv}(\operatorname{ext} Q_1),$$

$$M_2 = N_2 \oplus Q_2 = K_{M_2} + \operatorname{conv}(\operatorname{ext} Q_2)$$

be their representations.

THEOREM 5 (Representation theorem) *In $\mathbb{R}^d$, let M_i be a closed convex set, N_i a linear subspace and Q_i a line-free closed convex set such that $M_i = N_i \oplus Q_i$ ($i = 1,2$). Then the following holds.*

$$\pi(M_1, M_2) = \pi(N_1, N_2) \cap \pi(Q_1, M_2)$$

$$= \pi(K_{M_1}, K_{M_2}) \cap \pi(\operatorname{ext} Q_1, M_2)$$

$$= \pi(N_1, N_2) \cap \pi(K_{Q_1}, K_{M_2}) \cap \pi(\operatorname{ext} Q_1, M_2).$$

For a proof, the following lemma is needed.

LEMMA 2 *For M_i and N_i defined as in Theorem 5 ($i = 1,2$), the following holds.*

$$\pi(M_1, M_2) \subset \pi(N_1, N_2) \cap \pi(K_{M_1}, K_{M_2}).$$

Proof Take any $A \in \pi(M_1, M_2)$ and suppose to the contrary that $A \notin \pi(N_1, N_2)$, i.e. $AN_1 \not\subset N_2$. This implies that there is an $x \in N_1$ such that $Ax \notin N_2$. Obviously, $x \neq 0$. Let $q \in Q_1$. As $M_1 = N_1 \oplus Q_1$, it follows that $x + q \in M_1$. On the other hand, $A(x+q) = Ax + Aq \notin M_2$, because $Ax \notin N_2$ and N_2 is uniquely determined in $M_2 = N_2 \oplus Q_2$. Hence, $A \notin \pi(M_1, M_2)$, which is a contradiction. Therefore, $A \in \pi(N_1, N_2)$.

The inclusion $\pi(M_1, M_2) \subset \pi(K_{M_1}, K_{M_2})$ can be proved similarly, by taking rays instead of lines. ■

Proof of Theorem 5 Since $Q_1 \subset M_1$, we have $\pi(M_1, M_2) \subset \pi(Q_1, M_2)$. This together with Lemma 2 implies that

$$\pi(M_1, M_2) \subset \pi(N_1, N_2) \cap \pi(Q_1, M_2).$$

Take any $A \in \pi(N_1, N_2) \cap \pi(Q_1, M_2)$ and choose any point $x \in M_1$.

Because $M_1 = N_1 \oplus Q_1$, there are points $y \in N_1$ and $z \in Q_1$ such that $x = y + z$. Hence,

$$Ax = A(y + z) = Ay + Az \subset N_2 + Q_2 = M_2,$$

so that $A \in \pi(M_1, M_2)$, and therefore we have in fact that

$$\pi(M_1, M_2) = \pi(N_1, N_2) \cap \pi(Q_1, M_2).$$

The second equality is verified analogously. It remains to prove the third one. By the second equality,

$$\pi(Q_1, M_2) = \pi(K_{Q_1}, K_{M_2}) \cap \pi(\operatorname{ext} Q_1, M_2).$$

Hence, using the first one,

$$\begin{aligned}\pi(M_1, M_2) &= \pi(N_1, N_2) \cap \pi(Q_1, M_2)\\ &= \pi(N_1, N_2) \cap \pi(K_{Q_1}, K_{M_2}) \cap \pi(\operatorname{ext} Q_1, M_2).\end{aligned}$$

This proves the theorem. ■

5. POLYHEDRAL SETS OF LINEAR MAPS

The following question is raised in [10]:

Is the set $\pi(X)$ a convex polytope in case X is a convex polytope in $\mathbb{R}^d$ with $\operatorname{lin} X = \mathbb{R}^d$?

It will be shown here that this question has an affirmative answer. We first treat the polyhedral case. Let P be any polyhedral set in $\mathbb{R}^d$. Then P can be represented as $P = N \oplus Q$ where N is a linear subspace and Q is a line-free polyhedral set. Furthermore,

$$Q = \operatorname{conv}(\operatorname{ext} Q) + K_Q,$$

where K_Q is a polyhedral cone pointed in 0. Moreover,

$$P = N \oplus Q = (N \oplus K_Q) + \operatorname{conv}(\operatorname{ext} Q) = K_P + \operatorname{conv}(\operatorname{ext} Q),$$

where $K_P = N \oplus K_Q$ is a polyhedral cone and $\operatorname{conv}(\operatorname{ext} Q)$ is a polytope.

THEOREM 6 *For any polyhedral sets $P_1, P_2 \subset \mathbb{R}^d$, the set $\pi(P_1, P_2)$ is polyhedral.*

Proof Claim 1: For any line-free polyhedral cone $K_1 \subset \mathbb{R}^d$ and for any polyhedral cone $K_2 \subset \mathbb{R}^d$, the set $\pi(K_1, K_2)$ is a polyhedral cone.

Claim 2: For any finite set $X \subset \mathbb{R}^d$ and for any polyhedral set $P \subset \mathbb{R}^d$, the set $\pi(X, P)$ is polyhedral.

The proof of Claim 1 can be found in [9]. The proof of Claim 2 is as follows. Let $X = \{x_1, \ldots, x_n\}$ with $x_\alpha = \{x_{\alpha 1}, \ldots, x_{\alpha d})$ for each $\alpha = 1, \ldots, r$. Moreover, let P be a polyhedral set in $\mathbb{R}^d$, represented by the closed halfspaces $H_1, \ldots, H_m$ with

$$H_i = \left\{ (z_1, \ldots, z_d) \in \mathbb{R}^d \,\middle|\, \sum_{j=1}^{d} b_{ij} z_j \le \lambda_i \right\}$$

for suitable real numbers $b_{i1}, \ldots, b_{id}$, and λ_i $(i = 1, \ldots, m)$. Let $A = \{a_{ij}\} \in \pi(X, P)$, i.e. $AX \subset P$. Hence $\{Ax_1, \ldots, Ax_r\} \subset P$, and this is equivalent to

$$\sum_{j,k=1}^{d} (b_{ij} a_{jk}) x_{\alpha k} \le \lambda_i$$

for each $i = 1, \ldots, m$ and $\alpha = 1, \ldots, r$. Each of these mr inequalities determines a closed halfspace $G_{i\alpha} \subset \mathbb{R}^{d \times d}$. As

$$\pi(X, P) = \bigcap_{i=1}^{m} \bigcap_{\alpha=1}^{r} G_{i\alpha},$$

it follows that in fact $\pi(X, P)$ is polyhedral. This proves Claim 2.

According to Theorem 5, we have

$$\pi(P_1, P_2) = \pi(N_1, N_2) \cap \pi(K_{Q_1}, K_{P_2}) \cap \pi(\operatorname{ext} Q_1, P_2),$$

with the usual notations. Lemma 1 implies that $\pi(N_1, N_2)$ is a linear subspace, Claim 1 implies that $\pi(K_{Q_1}, K_{P_2})$ is a polyhedral cone, and Claim 2 implies that $\pi(\operatorname{ext} Q_1, P_2)$ is polyhedral. Hence, $\pi(P_1, P_2)$ is in fact a polyhedral set. ∎

COROLLARY 2 *For any polyhedral set $P \subset \mathbb{R}^d$, the set $\pi(P)$ is polyhedral.*

We now turn our attention to polytopes.

THEOREM 7 *Let $P \subset \mathbb{R}^d$ be a polyhedral set. The set $\pi(P)$ is a polytope if and only if P is a polytope with* $\operatorname{lin} P = \mathbb{R}^d$.

Proof Let P be a polyhedral set in $\mathbb{R}^d$.

Claim 1: Let X be a bounded set in $\mathbb{R}^d$. Then the set $\pi(X)$ is bounded if and only if $\operatorname{lin} X = \mathbb{R}^d$.

The proof of Claim 1 is as follows. It is shown in [10] that $\pi(X)$ is bounded if $\operatorname{lin} X = \mathbb{R}^d$. Suppose that $\operatorname{lin} X \neq \mathbb{R}^d$, and denote by S a linear subspace of $\mathbb{R}^d$ satisfying

$$(\operatorname{lin} X) + S = \mathbb{R}^d, \qquad \text{and} \qquad (\operatorname{lin} X) \cap S = \{0\}.$$

For any $\lambda \geq 0$, define the linear transformation A_λ on $\mathbb{R}^d$ by

$$A_\lambda \mid_{\text{lin}\, X} = I \mid_{\text{lin}\, X}, \qquad \text{and} \qquad A_\lambda \mid_S = \lambda I \mid_S;$$

($B \mid_N$ denotes the restriction of the transformation B to the subspace N of $\mathbb{R}^d$, and $I \mid_N$ is the identity in N.) Obviously, $A_\lambda \in \pi(X)$. Since the set $\{A_\lambda \mid \lambda > 0\}$ is a ray, $\pi(X)$ is unbounded.

Claim 2: If a closed convex set $M \subset \mathbb{R}^d$ is unbounded, then $\pi(M)$ is also unbounded.

The proof of Claim 2 is as follows. Since M is unbounded, the characteristic cone K_M of M is nontrivial, i.e. $K_M \neq \{0\}$ (see for example [8], p. 64). Let l be any ray in K_M with the apex 0. Then for any point $x \in M$ we have $x + l \subset M$. Choose any point e_1 in $l \backslash \{0\}$ and let $e_2, \ldots, e_d$ be such that $e_1, \ldots, e_d$ form a basis of $\mathbb{R}^d$. Define, for $\lambda \geq 1$, $B_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ by

$$B_\lambda(x) = B_\lambda\left(\sum_{i=1}^{d} x_i e_i\right) = (\lambda - 1)x_1 e_1 + \sum_{i=1}^{d} x_i e_i = (\lambda - 1)x_1 e_1 + x$$

for each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

So, if $x = \sum_{i=1}^{d} x_i e_i \in M$, then $B_\lambda(x) = (\lambda - 1)x_1 e_1 + x \in l + x \subset M$.

Hence, $B_\lambda \in \pi(M)$. Since $\{B_\lambda \mid \lambda \geq 1\}$ is a ray, the set $\pi(M)$ is unbounded. This proves Claim 2.

Now suppose $\pi(P)$ is a polytope. By Claim 1, $\text{lin}\, P = \mathbb{R}^d$ and, by Claim 2, the polyhedral set P must be bounded, i.e. P must be a polytope.

Conversely, let P be a polytope with $\text{lin}\, P = \mathbb{R}^d$. By Theorem 6, $\pi(P)$ is a polyhedral set. According to Claim 1, $\pi(P)$ is bounded, i.e. $\pi(P)$ is a polytope. ∎

COROLLARY 3 *For any polytope P in $\mathbb{R}^d$ with $\text{lin}\, P \neq \mathbb{R}^d$, $\pi(P)$ is an unbounded polyhedral set.*

Proof Let P be any polytope in $\mathbb{R}^d$ with $\text{lin}\, P \neq \mathbb{R}^d$. Then, trivially, P is polyhedral, and therefore Corollary 2 implies that $\pi(P)$ is polyhedral as well. Theorem 7 implies that $\pi(P)$ has to be unbounded. ∎

Let K_1 be the unit ball in $\mathbb{R}^d$. It is easily seen that $\pi(K_1)$ is the unit ball in $\mathbb{R}^{d \times d}$. On the other hand, consider

$$K_2 = \{(x, y) \in \mathbb{R}^2 \mid (x - 2)^2 + y^2 \leq 1\}.$$

From Theorem 2 and Corollary 1, it follows that $\pi(K_2)$ is a line segment in $\mathbb{R}^{2 \times 2}$.

Hence, for a nonpolyhedral convex body $K \subset \mathbb{R}^d$, the set $\pi(K)$ may be polyhedral or not.

OPEN PROBLEM. Characterize the closed convex sets $M \subset \mathbb{R}^d$ for which the set $\pi(M)$ is (i) polyhedral, (ii) polytopal.

References

1. R. Adin, Extreme positive operators on minimal and almost minimal cones, *Linear Algebra Appl.* **44** (1982), 61–86.
2. G. P. Barker and H. Schneider, Algebraic Perron-Frobenius theory, *Linear Algebra Appl.* **11** (1975), 219–233.
3. A. Berman and R. J. Plemmons, Nonnegative matrices in the mathematical sciences, Academic Press, New York, 1979.
4. J. Eckhoff, Radon's theorem revisited, Contributions to geometry, *Proc. Geom. Symp. Siegen* 1978, J. Tölke and J. M. Wills, Ed., 1979, 164–185.
5. L. Elsner, On matrices leaving invariant a nontrivial convex set, *Linear Algebra Appl.* **42** (1982), 103–107.
6. V. Klee, Extremal structure of convex sets, *Arch. Math.* **8** (1957), 234–240.
7. V. Klee, Some new results on smoothness and rotundity in normed linear spaces, *Math. Ann.* **139** (1959), 51–63.
8. R. T. Rockafellar, *Convex Analysis*, Princeton, 1970.
9. H. Schneider and M. Vidyasagar, Cross-positive matrices, *SIAM J. Numer. Math.* **7** (1970), 508–519.
10. G. Sierksma and K. de Vos, Convex polytopes as matrix invariants, *Compositio Math.* **52** (1984), 203–210.