A generic view on the theorems of Brouwer and Schauder

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Several generic fixed point theorems have been established by De Blasi [5, 6], Myjak [9], De Blasi and Myjak [7], Myjak and Sampalmieri [10], Butler [4], Dominguez Benavides [8], Vidossich [12], to quote just a few. However it seems that the well-known and important theorem of Schauder [11] and its finite-dimensional version, Brouwer's theorem [3], have not yet been generically investigated. We now fill this gap.

For a survey of generic results on convex bodies in Euclidean spaces, see [15].

A large set of fixed points

Let E be a Banach space and consider a compact convex set $K \subset E$ with more than one point. By the Schauder theorem, every continuous function $f: K \to K$ has at least one fixed point. We shall show here that, in the sense of Baire categories, most continuous functions have uncountably many fixed points. More precisely, the set of all their fixed points is homeomorphic to the Cantor set. This sharply contrasts with the generic finiteness of the fixed point set found by De Blasi [6] in another space of functions and with the behaviour of the non-expansive mappings which have, generically, a single fixed point ([12], see also [9, p. 29]). Of course, the space $\mathscr{C}(K)$ of all continuous functions $f: K \to K$, equipped with the metric of uniform convergence, is complete, hence a Baire space. "Most" means "all, except those in a first category set" (so, a property is generic if it is shared by most elements).

For $f \in \mathscr{C}(K)$, let F_f be the set of all fixed points of f. For $x \in E$, $B(x, r)$ denotes the open ball of centre x and radius r.

The proof of Theorem A below will make use of the following recent result.

Lemma. *Let C be a convex closed set in E, $f: C \to C$ be a map with $\overline{f(C)}$ compact and $\varepsilon > 0$ be given. Then there exists a map $g: C \to C$ such that F_g is finite and $\|f - g\| < \varepsilon$.*

This result was attributed in [2] to Brown and extended to finite unions of convex sets by Baillon and Rallis [2].

Theorem A. *For most functions in $\mathscr{C}(K)$ the set of fixed points is homeomorphic to the Cantor set.*

Proof. Clearly, F_f is closed for any $f \in \mathscr{C}(K)$. Thus we only have to show that, for most $f \in \mathscr{C}(K)$, F_f has no isolated point and is totally disconnected. We first prove that F_f has no isolated point. Let $\mathscr{C}_n$ be the set of all functions $f \in \mathscr{C}(K)$ admitting a fixed point $y \in K$ such that

$$F_f \cap B(y, n^{-1}) = \{y\} \quad (n \in \mathbb{N}) \ .$$

Clearly, $\bigcup_n \mathscr{C}_n$ is precisely the set of all $f \in \mathscr{C}(K)$ admitting isolated fixed points, so it suffices to prove that $\mathscr{C}_n$ is nowhere dense, for arbitrary n.

Let $\mathcal{O} \subset \mathscr{C}(K)$ be open and, if $\mathcal{O} \cap \mathscr{C}_n \neq \emptyset$, take $f \in \mathcal{O} \cap \mathscr{C}_n$. We shall find an open set in $\mathcal{O} \setminus \mathscr{C}_n$.

From the covering $\{B(y, (2n)^{-1}): y \in F_f\}$ of F_f we select a finite subcovering $\{B(y, (2n)^{-1}): y \in Y\}$. Clearly, for some $v \in (0, (2n)^{-1})$, if y and y' are distinct points of Y, then

$$B(y, v) \cap B(y', v) = \emptyset \ .$$

Let $\varepsilon \in (0, v)$ be such that $\|f - g\| < \varepsilon$ implies $g \in \mathcal{O}$. The set K being compact, we can find a number $\delta \in (0, \varepsilon/6)$ such that $x, x' \in K$ and $\|x - x'\| < 4\delta$ imply $\|f(x) - f(x')\| < \varepsilon/3$.

Since $L = K \setminus \bigcup_{y \in Y} B(y, (2n)^{-1})$ is compact and disjoint from F_f if it is nonempty, for some $\omega > 0$ the inequality $\|f(x) - x\| \geq \omega$ holds for any $x \in L$.

For every point $y \in Y$, consider a point $z_y \in K$ and a number α such that $0 < \alpha < \omega$, if $L \neq \phi$,

$$B(z_y, 2\alpha) \subset B(y, \delta) \quad \text{and} \quad B(y, 2\alpha) \cap B(z_y, 2\alpha) = \emptyset \ .$$

We define a function $f^*: K \to K$ as follows:

$$f^*(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{y \in Y} B(y, 4\delta), \\ f(y + (2\lambda - 4\delta)v) & \text{if } x = y + \lambda v \text{ and } 2\delta \leq \lambda < 4\delta, \\ y + (2\delta - \lambda)v & \text{if } x = y + \lambda v \text{ and } \delta \leq \lambda < 2\delta, \\ x & \text{if } x = y + \lambda v, \quad 2\alpha \leq \lambda < \delta \\ & \text{and } x \notin B(z_y, 2\alpha), \\ y + (2\lambda - 2\alpha)v & \text{if } x = y + \lambda v \text{ and } \alpha \leq \lambda < 2\alpha, \\ y & \text{if } x = y + \lambda v \text{ and } 0 \leq \lambda < \alpha, \\ z_y + (2\lambda - 2\alpha)v & \text{if } x = z_y + \lambda v \text{ and } \alpha \leq \lambda < 2\alpha, \\ z_y & \text{if } x = z_y + \lambda v \text{ and } 0 \leq \lambda < \alpha, \end{cases}$$

where, each time, y is some point in Y and v is some unit vector. It is easily checked that f^* is continuous and $\|f^* - f\| < 2\varepsilon/3$.

Thus, for every $y \in Y$, $f^*(\overline{B(y, \alpha)} \cap K) = \{y\}$ and $f^*(\overline{B(z_y, \alpha)} \cap K) = \{z_y\}$. Therefore, for every function $g \in \mathscr{C}(K)$ with $\|f^* - g\| < \alpha$,

$$g(\overline{B(y, \alpha)} \cap K) \subset \overline{B(y, \alpha)} \cap K$$

and

$$g(\overline{B(z_y, \alpha)} \cap K) \subset \overline{B(z_y, \alpha)} \cap K \ .$$

By Schauder's theorem, g has a fixed point in each of the sets $\overline{B(y, \alpha)}$ and $\overline{B(z_y, \alpha)}$, where $y \in Y$.

We investigate now the fixed points of g. Since $\alpha < \omega$, g has no fixed points in L. To any possibly existing fixed point in $K \backslash L$ there is a point $y \in Y$ at distance less than $(2n)^{-1}$ and a pair of fixed points, the one in $\overline{B(y, \alpha)}$ and the other in $\overline{B(z_y, \alpha)}$. These three fixed points of g have pairwise distances less than n^{-1}, and the last two are certainly distinct. Therefore $g \notin \mathscr{C}_n$. On the other hand $g \in \mathcal{O}$ because $\|f - g\| < \varepsilon$. Hence, for most $f \in \mathscr{C}(K)$, F_f has no isolated point.

Now we show that F_f is totally disconnected for most $f \in \mathscr{C}(K)$. Let $\mathscr{C}'_n$ be the set of all functions $f \in \mathscr{C}(K)$ such that F_f has a component of diameter at least n^{-1}. It suffices to prove that $\mathscr{C}'_n$ is nowhere dense.

Let $\mathcal{O} \subset \mathscr{C}(K)$ be open. By the lemma, there is some $f \in \mathcal{O}$ with finite F_f. Let α be the minimum distance between distinct points in F_f, or equal to 1 if F_f consists of a single point. Let $\beta = \min\{\alpha/3, (3n)^{-1}\}$. The set

$$L' = K \backslash \bigcup_{x \in F_f} B(x, \beta)$$

being compact and disjoint from F_f (or empty), for some $\omega > 0$ the inequality $\|f(x) - x\| \geqq \omega$ holds for all $x \in L'$.

For every function $g \in \mathscr{C}(K)$ with $\|f - g\| < \omega$ we have $F_g \cap L' = \emptyset$ because $\|g(x) - x\| \geqq \|f(x) - x\| - \|f(x) - g(x)\| > 0$ for all $x \in L'$. So each component of F_g lies in some open ball $B(x, \beta)$ with $x \in F_f$ and therefore has a diameter less than n^{-1}. This shows that $\mathscr{C}'_n$ is nowhere dense and the proof is finished.

A small set of fixed points

Let now $K \subset \mathbb{R}^d$ be compact, convex and with nonempty interior. By the well-known Brouwer fixed point theorem [3], for any $f \in \mathscr{C}(K)$, the set F_f of all fixed points of f is nonempty. By Theorem A, for most $f \in \mathscr{C}(K)$, F_f is uncountable. But, otherwise, how large is F_f for most f? In particular, is it of positive measure? We answer here this question.

As a matter of notation, for any sets $A, B \subset \mathbb{R}^d$ and point $x \in \mathbb{R}^d$,

$$\Delta(x, A) = \inf_{y \in A} \|x - y\| \quad \text{and} \quad D(A, B) = \inf_{x \in A} \Delta(x, B).$$

Let $M \subset \mathbb{R}^d$ and $x \in M$. For any $\varepsilon > 0$, let $\gamma_\varepsilon(x)$ be the radius of the largest open ball with centre in $\overline{B(x, \varepsilon)}$ and disjoint from M. Then

$$\rho_M = \inf_{x \in M} \limsup_{\varepsilon \to 0} \gamma_\varepsilon(x)/\varepsilon$$

is called the *porosity* of M. The set M is called *porous* if $\rho_M > 0$ and *strongly porous* if $\rho_M = 1$.

Let again $M \subset \mathbb{R}^d$ and $x \in M$. For any $\varepsilon > 0$, let $\delta_\varepsilon(x)$ be the largest number r such that, for any closed halfspace $H \ni x$, there exists a point $y \in \overline{B(x, \varepsilon)}$ such that $B(y, r) \subset H \backslash M$. If $\tau(x, M) = \limsup_{\varepsilon \to 0} \delta_\varepsilon(x)/\varepsilon$ then

$$\tau_M = \inf_{x \in M} \tau(x, M)$$

is called the *total porosity* of M, and M is said to be *totally porous* if $\tau_M > 0$. The total porosity was introduced (in metric spaces) by Agronsky and Bruckner [1].

An excellent survey on porosity is Zajíček's paper [13]. About its applications to Convexity see [14].

Theorem B. *Most functions in $\mathscr{C}(K)$ admit a set of fixed points which is strongly and totally porous.*

Proof. Let $f \in \mathscr{C}(K)$, $0 < \alpha < 1$ and $0 < \beta < \tau_{\mathrm{bd}\,K}(4 + \tau_{\mathrm{bd}\,K}^2)^{-1/2}$. If

$$(*) \; \forall x \in \mathbb{R}^d, \; \forall \varepsilon > 0, \; \exists y \in \overline{B(x, \varepsilon)} \text{ s.t. } \overline{B(y, \alpha \| x - y \|)} \cap F_f = \varnothing$$

then the porosity of F_f is at least α. If

$$(**) \; \forall x \in \mathbb{R}^d, \; \forall \varepsilon > 0, \; \forall H \ni x, \; \exists y \in \overline{B(x, \varepsilon)} \text{ s.t. } \overline{B(y, \beta \| x - y \|)} \subset H \backslash F_f$$

then the total porosity of F_f is at least β.

We prove that (*) and (**) hold for most $f \in \mathscr{C}(K)$. Let

$$\mathscr{F}_n = \{ f : \exists x \text{ s.t. } \forall y \in \overline{B(x, n^{-1})}, \; \overline{B(y, \alpha \| x - y \|)} \cap F_f \neq \varnothing \}$$

and

$$\mathscr{G}_n = \{ f : \exists x, \; \exists H \ni x \text{ s.t. } \forall y \in \overline{B(x, n^{-1})}, \; \overline{B(y, \beta \| x - y \|)} \not\subset H \backslash F_f \} \; .$$

We now show that the complements of $\mathscr{F}_n$ and $\mathscr{G}_n$ are dense.

Let $\mathcal{O} \subset \mathscr{C}(K)$ be open. Take $f \in \mathcal{O}$ and $\varepsilon > 0$ such that $\| f - g \| < \varepsilon$ implies $g \in \mathcal{O}$. Let $v \in (0, \varepsilon)$ be such that $x, x' \in K$ and $\| x - x' \| < v$ imply $\| f(x) - f(x') \| < \varepsilon$. From the covering $\{ B(y, v) : y \in F_f \}$ of F_f select a finite subcovering $\{ B(y, v) : y \in Y \}$. Let $\xi \in (0, D(K \backslash L, F_f))$, where

$$L = \{ x \in \mathbb{R}^d : \Delta(x, Y) < v \} \; ,$$

and set

$$L^* = \{ x \in L : \Delta(x, \mathrm{bd}\, L) < \xi \} \; .$$

For ξ small enough, the projection (nearest point mapping) $p_{\mathrm{bd}\,L} : L^* \to \mathrm{bd}\, L$ is single-valued and we assume ξ to be so small. Then $\mathrm{bd}(L \backslash L^*)$ is a differentiable surface and $p_{\mathrm{bd}\,L}$ is continuous. Let v be a unit vector and construct

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in K \backslash L \\ x + (1 - \xi^{-1} \| x - p_{\mathrm{bd}\,L}(x) \|) \, (f(p_K(p_{\mathrm{bd}\,L}(x))) - p_{\mathrm{bd}\,L}(x)) & \\ & \text{if } x \in K \cap L^* \\ x + \min \{ \varepsilon, \Delta(x, L^* \cup \mathrm{bd}\, K) \} v & \text{if } x \in K \cap L \backslash L^* \; . \end{cases}$$

It is easily checked that f^* is continuous, that $\| f - f^* \| < \varepsilon$ and that the fixed points of f^* form the set $\mathrm{bd}(K \cap L \backslash L^*)$, which is obviously strongly porous. To verify the total porosity of $\mathrm{bd}(K \cap L \backslash L^*)$ we notice:·

At every point $y \in (\mathrm{bd}(K \cap L \backslash L^*)) \backslash (\mathrm{bd}(L \backslash L^*))$, $\tau(y, F_{f^*}) \geq \tau_{\mathrm{bd}\,K}$. At every point $y' \in (\mathrm{bd}(K \cap L \backslash L^*)) \backslash (\mathrm{bd}\, K)$, $\tau(y', F_{f^*}) = 2^{-1/2}$ (the total porosity of a hyperplane). At every point $y'' \in (\mathrm{bd}\, K) \cap (\mathrm{bd}(L \backslash L^*))$,

$$\tau(y'', F_{f^*}) \geq \tau_{\mathrm{bd}\,K}(4 + \tau_{\mathrm{bd}\,K}^2)^{-1/2} \; ,$$

because $\mathrm{bd}(L \backslash L^*)$ behaves locally like a hyperplane, and the addition of a hyperplane to a set M decreases its total porosity to $\tau_M(4 + \tau_M^2)^{-1/2}$, at worst. Therefore

$$\tau_{F_{f^*}} \geq \tau_{\mathrm{bd}\,K}(4 + \tau_{\mathrm{bd}\,K}^2)^{-1/2} > \beta \; .$$

Hence $f^* \in \mathcal{O} \backslash (\mathscr{F}_n \cup \mathscr{G}_n)$. This shows that the complements of $\mathscr{F}_n$ and $\mathscr{G}_n$ are dense in $\mathscr{C}(K)$.

It is easily seen that both $\mathscr{F}_n$ and $\mathscr{G}_n$ are closed and therefore nowhere dense. Hence (*) and (**) hold for most $f \in \mathscr{C}(K)$. It follows that, for most functions f, $\rho_{F_f} \geqq \alpha$ for any rational $\alpha \in (0, 1)$, and $\tau_{F_f} \geqq \beta$, which means that F_f is strongly and totally porous.

Theorem B and Lebesgue's density theorem imply the following.

Corollary. *For most functions in $\mathscr{C}(K)$, the set of fixed points has (d-dimensional) measure zero.*

This corollary suggests that the Hausdorff dimension of F_f might be lower than d. That it is indeed much lower is confirmed by the following result, in which K lives again in a Banach space E. I am indebted to David Preiss, who thought of considering the Hausdorff dimension.

Theorem C. *Most functions in $\mathscr{C}(K)$ admit a set of fixed points which has Hausdorff dimension 0.*

Proof. Let $M \subset E$ and $\alpha > 0$. If, for any $\varepsilon > 0$, there is a covering $\{M_i\}_{i=1}^m$ of M with diam $M_i < \varepsilon$ and $\sum_{i=1}^m (\mathrm{diam}\, M_i)^\alpha < \varepsilon$, then the Hausdorff dimension of M is at most α. So, let

$$\mathscr{F}_n = \{f \in \mathscr{C}(K): \text{if } F_f \subset \bigcup_{i=1}^m M_i \text{ for some } m \text{ then diam } \mathrm{M}_i \geqq 1/n$$

$$\text{for some } i \text{ or } \sum_{i=1}^m (\mathrm{diam}\, M_i)^\alpha \geqq 1/n\} \, .$$

We prove that $\mathscr{F}_n$ is nowhere dense in $\mathscr{C}(K)$. Indeed, let $\mathcal{O} \subset \mathscr{C}(K)$ be open. By the lemma in the preceding section, we may choose a function $g \in \mathcal{O}$ with F_g finite, say $F_g = \{x_1, \ldots, x_m\}$. Obviously, for any $\beta \in (0, \min\{(mn)^{-1/\alpha}, n^{-1}\})$, we have $\beta < 1/n$ and $m\beta^\alpha < 1/n$, whence $h \notin \mathscr{F}_n$ if $F_h \subset \bigcup_{i=1}^m B(x_i, \beta/2)$ and this happens in a whole neighbourhood of g.

Hence most $f \in \mathscr{C}(K)$ do not belong to $\bigcup_{n=1}^\infty \mathscr{F}_n$ and have therefore Hausdorff dimension at most α. Since this is true for every rational $\alpha > 0$, for most $f \in \mathscr{C}(K)$ the Hausdorff dimension of F_f is 0.

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