Atti Sem. Mat. Fis. Univ. Modena, XLII, 253-256 (1994)

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On the Curvatures of Convex Curves of Constant Width (**).

In his paper [8] included in this volume and presented in 1992 at the conference on Analisi Reale e Teoria della Misura on the marvellous island of Capri, O. Stefani proved the following result.

THEOREM. *If xy is a diameter of a convex curve of constant width w and if the radius of curvature $\rho(x)$ at x exists, then $0 \leq \rho(x) \leq w$, the radius of curvature $\rho(y)$ at y exists and $\rho(x) + \rho(y) = w$.*

We shall establish here a refinement of this result, which gives the exact relationship between the lower and upper curvatures at opposite points of an arbitrary planar convex body of constant width. Our result obviously implies the above theorem. Furthermore we shall describe the generic aspect of planar convex bodies of constant width with respect to their curvatures.

Let $\rho_i^+(x)$ and $\rho_s^+(x)$ denote the right lower and upper radius of curvature of the convex curve of constant width C, at the point $x \in C$; let $\rho_i^-(x)$ and $\rho_s^-(x)$ be the corresponding left radii (for a definition see [3], p. 14).

The space $\mathcal{C}$ of all planar convex curves of constant width w, endowed with the Pompeiu-Hausdorff distance, is a complete metric space (being closed in the space of all compact subsets of the plane).

Properties of «most» elements of a Baire space, i.e. shared by all

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(**) Nota giunta in Redazione il 20-I-1993.

This article was presented at the «Convegno di Analisi Reale e Teoria della Misura» held in Capri during September 7-11, 1992, and was supported by M.U.R.S.T.

elements except those in a set of first category, are called *generic*. For a survey of generic properties of convex bodies see [10].

THEOREM 1. *Let $C \in \mathcal{C}$ and consider a diameter of C with endpoints $x, y \in C$. Then*

$$\rho_i^+(x) + \rho_s^+(y) = w .$$

Before giving a proof let us remark that the theorem implies

$$\rho_s^+(x) + \rho_i^+(y) = \rho_i^-(x) + \rho_s^-(y) = \rho_s^-(x) + \rho_i^-(y) = w$$

too.

PROOF. First of all notice that $\rho_s^+(y) \leq w$. Indeed, if it were not so there would be a point $y' \in C$ arbitrarily close to y such that $\|x - - y'\| > w$, and C would not belong to $\mathcal{C}$. Hence the inequality $\rho_s^+(y) \leq \leq w - \rho_i^+(x)$ is verified for $\rho_i^+(x) = 0$.

In case $\rho_i^+(x) > 0$, let c be the point of the diameter xy such that

$$\|x - c\| = \rho_i^+(x)$$

and take $c^* \in cx$ different from c and x. Consider the concentric circles C', C'' with centre c^* and radii $\|c^* - x\|$, $\|c^* - y\|$. Then a small arc $A' \subset C$ starting at x and lying on the right side of x is outside C' except for the point x. Take an arbitrary point $y'' \in C''$ collinear with c^* and with some point of $A' \setminus \{x\}$. The orthogonal projection of C onto the line L through c^* and y'' has length w and contains $L \cap C'$ in its interior. So y'' lies in its exterior and therefore $y'' \notin C$. This proves that $\rho_s^+(y) \leq \|c^* - y\|$ and, since this is true for every $c^* \in cx \setminus \{c, x\}$,

$$\rho_s^+(y) \leq \|c - y\| = w - \rho_i^+(x).$$

If C includes an entire circular arc starting at x and lying on the right side of x then C must also include a circular arc starting at y and lying on the right side of y. So, trivially,

$$\rho_s^+(y) = w - \rho_i^+(x).$$

If C includes no such circular arc and $\rho_i^+(x) < w$, then there is a sequence of points $x_n \in C$ converging to x from the right, such that the circle through x_n tangent at x to C has its centre $c_n \in cy \setminus \{c, y\}$. Now consider the concentric circles C_n', C_n'' with centre $c_n' \in cc_n$ and radii $\|c_n - x\|$, $\|c_n' - y\|$ such that $c_n' \to c$. Let $A_n \subset C_n'$ be the circular arc on the right side of x from x to $\{x_n'\} = C_n' \cap c_n' x_n$. Then the open connected bounded set D_n with boundary $c_n' x \cup A_n \cup x_n' c_n'$ must intersect C. Thus any point of $D_n \cap C$ at minimal distance from c_n' is an endpoint of a diameter of C with the other endpoint outside C_n''. This proves that

$$\rho_s^+(y) \geq \lim_{n \to \infty} \|c_n' - y\| = \|c - y\| = w - \rho_i^+(x).$$

For $\rho_i^+(x) = w$, the inequality reduces to $\rho_s^+(y) \geq 0$ and follows from the convexity of C.

Hence

$$\rho_s^+(y) = w - \rho_i^+(x)$$

and the theorem is proved.

THEOREM 2. *For most convex curves $C \in \mathcal{C}$, at any point $x \in C$,*

$$\rho_i^+(x) = 0 \quad \text{or} \quad \rho_s^+(x) = w.$$

PROOF. The proof parallels that of Theorem 1 in [9]. Let $D_r(x)$ denote the half-disc with the boundary line-segment of length $2r$ on the inner normal at x to C, with x as a boundary point and with the circular boundary arc on the right side of x. Let $\mathcal{C}_n$ be the family of all convex curves $C \in \mathcal{C}$ with a point $x \in C$ such that

$$D_{n^{-1}}(x) \subset \operatorname{conv} C$$

and

$$C \subset D_{w - n^{-1}}(x).$$

The set $\mathcal{C}_n$ is nowhere dense in $\mathcal{C}$. Indeed $\mathcal{C}_n$ is clearly closed in $\mathcal{C}$ and, moreover, the family of all Reuleaux polygons of width w is dense in $\mathcal{C}$ (see [2], p. 510-511) and obviously disjoint from $\mathcal{C}_n$. Thus $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ is of first Baire category in $\mathcal{C}$.

It is an easy exercise to prove that $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ is precisely the set of all convex curves $C \in \mathcal{C}$ with

$$\rho_i^+(x) > 0 \quad \text{and} \quad \rho_s^+(x) < w$$

at some point $x \in C$.

Hence, for most $C \in \mathcal{C}$,

$$\rho_i^+(x) = 0 \quad \text{or} \quad \rho_s^+(x) = w$$

at each point $x \in C$.

COROLLARY. *For most convex curves $C \in \mathcal{C}$, the curvature exists and equals $1/w$ a.e.*

PROOF. By a classical theorem of Aleksandrov [1], the curvature of a convex curve exists and is finite a.e. This combined with Theorem 2 implies the corollary.

In 1975, M. Kallay [4] proved that the boundaries of the indecomposable convex bodies relative to $\mathcal{C}$ are precisely the convex curves described in Theorem 2. Here an element of $\mathcal{C}$ is *indecomposable relative to* $\mathcal{C}$ if it is not a convex combination of any two other elements of $\mathcal{C}$ (see [6], p. 156/7, but replace «compact» by «closed, bounded»). It is interesting to note that the situation is quite different in the space $\mathcal{K}_2$ of all convex bodies in the plane, where only the triangles are indocomposable (see [5]). However, in the space $\mathcal{K}_n$ of all convex bodies in the Euclidean n-space as well as in the space of all closed bounded convex sets whose elemets belong to $\mathcal{K}_n$, indecomposability is a generic property ([6], p. 152, and [7]).

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